Journal of Southeast University (English Edition) Vol. 28, No. 2, pp. 236 - 239 June 2012 ISSN 1003—7985

Dimension-down iterative algorithm for the mixed transportation network design problem

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Abstract: An optimal dimension-down iterative algorithm (DDIA) is proposed for solving a mixed (continuous/discrete) transportation network design problem (MNDP), which is generally expressed as a mathematical programming with equilibrium constraints (MPEC). The upper level of the MNDP aims to optimize the network performance via both the expansion of existing links and the addition of new candidate links, whereas the lower level is a traditional Wardrop user equilibrium (UE) model. The idea of the proposed DDIA is to reduce the dimensions of the problem. A group of variables (discrete/continuous) are fixed to alternately optimize another group of variables (continuous/discrete). Some continuous network design problems (CNDPs) and discrete network design problems (DNDPs) are solved repeatedly until the optimal solution is obtained. A numerical example is given to demonstrate the efficiency of the proposed algorithm.

Key words: mixed network design problem (MNDP); dimension-down iterative algorithm (DDIA); mathematical programming with equilibrium constraint (MPEC)

doi: 10.3969/j.issn.1003-7985.2012.02.018

The network design problem (NDP) is concerned with the modification of a transportation network configuration by adding new links or improving existing ones, so that certain social welfare objectives are maximized (e.g. total travel time over network). How to select the location of these new links and how much additional capacity is to be added to each of these existing links are motivating problems, which try to minimize the total system costs under limited budget and account for the route choice behavior of network users. The NDP can be roughly classified into three categories: the discrete network design problem (DNDP) that deals with the selection of the optimal locations (expressed by 0-1 integer decision variables) of new links to be added; the continuous network design problem (CNDP) that determines the optimal capacity enhancement (expressed by continuous decision variables) for a subset of existing links; and the mixed network design problem (MNDP) that combines both the CNDP and the DNDP in a network[1]. The NDP can be generally formulated as a mathematical programming with equilibrium constraints (MPEC). The determinate user equilibrium (UE) assignment model or the stochastic user equilibrium (SUE) assignment model is usually applied to describe the route choice behavior of network users. The CNDP has been widely studied[2-6] since its variables are continuous and easy for designing algorithms. There are also several studies on the DNDP[7-9]. The MNDP involves both the discrete and the continuous variables and can be generally expressed as a nonlinear mixed-integer bi-level programming which is normally difficult to implement[1].

This paper proposes an algorithm for solving the MNDP, and its idea is to reduce the dimensions of the problem by transforming it into an iterative solution of some CNDPs and DNDPs. A numerical example is given to demonstrate the efficiency of the proposed method.

1 Formulation of MNDP

The MNDP aims to find both the capacity expansions of existing links (continuous decision variables) and new link additions (0-1 decision variables) in order to minimize the total travel time of the network users subject to a budgetary constraint and the UE condition. The MNDP without budget constraint is formulated as

$$\min_{\boldsymbol{y},\boldsymbol{u}} Z(\boldsymbol{y},\boldsymbol{u},\boldsymbol{x}) = \sum_{a\in A_1} x_a t_a(x_a) + \sum_{a\in A_2} x_a t_a(x_a, y_a) + \sum_{a\in A_3} x_a t_a(x_a, y'_a) + \phi\sum_{a\in A_2} g_a(y_a) + \phi\sum_{a\in A_3} d_a u_a \tag{1}$$

$$\text{s.t.}\quad y_a^0 + y_a \leqslant \bar{y}_a \qquad \forall a\in A_2$$
$$u_a = 0 \text{ or } 1 \qquad \forall a\in A_3$$

where Z is the comprehensive expense of the travel time and construction cost; A_1 is the set of non-expanded links, A_2 is the set of expanded links, and A_3 is the set of new candidate links, $A = A_1\cup A_2\cup A_3$; x_a is the aggregate flow on link a, $a\in A$; $\boldsymbol{x}$ is a vector whose elements are x_a; y_a^0 is the original capacity on existing link a, $a\in A_1\cup A_2$; y_a is the incremental capacity on expanded link a, $a\in A_2$; y is a vector whose elements are y_a; y'_a is the fixed capacity on new candidate link a, $a\in A_3$; $\bar{y}_a$ is the upper bound of the capacity of link a, $a\in A_2$; t_a is the travel

Received 2012-01-06.

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Foundation items: The National Natural Science Foundation of China (No. 50908235), China Postdoctoral Science Foundation (No. 201003520).

Citation: Chen Qun, Yao Jialin. Dimension-down iterative algorithm for the mixed transportation network design problem[J]. Journal of Southeast University (English Edition), 2012, 28(2): 236 - 239. [doi: 10.3969/j.issn.1003-7985.2012.02.018]

time of link a, $a \in A$; $g_a(y_a)$ is the improvement cost function of expanded link a, $a \in A_2$; d_a is the construction cost per addition of new candidate link a, $a \in A_3$; u_a is the 0-1 decision variable, $u_a = 1$ if link $a(a \in A_3)$ is added; otherwise, $u_a = 0$; u is a vector whose elements are u_a; ϕ is the relative weight of construction cost and travel time.

$\boldsymbol{x}$ is the implicit function of $\boldsymbol{y}$ and $\boldsymbol{u}$ and it can be obtained by solving the lower-level problem [10].

$$\min T(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{x}) = \sum_{a \in A_1} \int_0^{x_a} t_a(w)\,\mathrm{d}w + \sum_{a \in A_2} \int_0^{x_a} t_a(w, y_a)\,\mathrm{d}w + \sum_{a \in A_3} \int_0^{x_a} t_a(w, y'_a)\,\mathrm{d}w \tag{2}$$

$$\text{s.t.} \quad \sum_{k \in L_{rs}} f_k^{rs} = q_{rs} \qquad \forall r \in R, s \in S$$

$$x_a = \sum_{r,s} \sum_k f_k^{rs} \delta_{a,k}^{rs} \qquad \forall a \in A$$

$$f_k^{rs} \geqslant 0 \qquad \forall r \in R, s \in S, k \in L_{rs}$$

where T is the integral function; R is the set of origin nodes; S is the set of destination nodes; r is the origin node index, $r \in R$; s is the destination node index, $s \in S$; q_{rs} is the travel demand between pair (r, s); f_k^{rs} is the flow of path k between pair (r, s); L_{rs} is the set of paths between pair (r, s); $\delta_{a,k}^{rs}$ is the path/link incidence variable, which equals 1 if link a is on path k between pair (r, s), otherwise 0.

The MNDP with the budget constraint is formulated as

$$\min_{\boldsymbol{y}, \boldsymbol{u}} Z(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{x}) = \sum_{a \in A_1} x_a t_a(x_a) + \sum_{a \in A_2} x_a t_a(x_a, y_a) + \sum_{a \in A_3} x_a t_a(x_a, y'_a) \tag{3}$$

$$\text{s.t.} \quad \sum_{a \in A_2} g_a(y_a) + \sum_{a \in A_3} d_a u_a \leqslant \text{budget}$$

$$y_a^0 + y_a \leqslant \bar{y}_a \qquad \forall a \in A_2$$

$$u_a = 0 \text{ or } 1 \qquad \forall a \in A_3$$

2 Dimension-Down Iterative Algorithm (DDIA) for Solving the MNDP

The idea of the DDIA is to reduce the dimensions of the problem. A group of variables (discrete/continuous) are fixed to alternately optimize another group of variables (continuous/discrete). Some CNDPs and DNDPs are solved repeatedly until the optimal solution is obtained.

Suppose that $\boldsymbol{u}^{(0)} = \{u_1^{(0)}, u_2^{(0)}, u_3^{(0)}, \dots\}$ is a feasible solution with the budget constraint; e. g., let $\boldsymbol{u}^{(0)} = \{0, 0, 0, \dots, 0\}$. $\boldsymbol{u}$ is fixed at $\boldsymbol{u}^{(0)}$ to optimize $\boldsymbol{y}$; therefore, the problem becomes a solution of a CNDP.

$$\min_{\boldsymbol{y}, \boldsymbol{u}^{(0)}} Z(\boldsymbol{y}, \boldsymbol{u}^{(0)}, \boldsymbol{x}) = \sum_{a \in A_1} x_a t_a(x_a) + \sum_{a \in A_2} x_a t_a(x_a, y_a) + \sum_{a \in A_3} x_a t_a(x_a, y'_a) + \phi \sum_{a \in A_2} g_a(y_a) + \phi \sum_{a \in A_3} d_a u_a^{(0)} \tag{4}$$

$$\text{s.t.} \quad y_a^0 + y_a \leqslant \bar{y}_a \qquad \forall a \in A_2$$

where $\boldsymbol{x}$ is the implicit function of $\boldsymbol{y}$ and it can be obtained by solving the lower-level problem.

$$\min T(\boldsymbol{y}, \boldsymbol{u}^{(0)}, \boldsymbol{x}) = \sum_{a \in A_1} \int_0^{x_a} t_a(w)\,\mathrm{d}w + \sum_{a \in A_2} \int_0^{x_a} t_a(w, y_a)\,\mathrm{d}w + \sum_{a \in A_3} \int_0^{x_a} t_a(w, y'_a)\,\mathrm{d}w \tag{5}$$

$$\text{s.t.} \quad \sum_{k \in L_{rs}} f_k^{rs} = q_{rs} \qquad \forall r \in R, s \in S$$

$$x_a = \sum_{r,s} \sum_k f_k^{rs} \delta_{a,k}^{rs} \qquad \forall a \in A$$

$$f_k^{rs} \geqslant 0 \qquad \forall r \in R, s \in S, k \in L_{rs}$$

If it is the case with the budget constraint, the problem to be solved is as follows:

$$\min_{\boldsymbol{y}, \boldsymbol{u}^{(0)}} Z(\boldsymbol{y}, \boldsymbol{u}^{(0)}, \boldsymbol{x}) = \sum_{a \in A_1} x_a t_a(x_a) + \sum_{a \in A_2} x_a t_a(x_a, y_a) + \sum_{a \in A_3} x_a t_a(x_a, y'_a) \tag{6}$$

$$\text{s.t.} \quad \sum_{a \in A_2} g_a(y_a) + \sum_{a \in A_3} d_a u_a^{(0)} \leqslant \text{budget}$$

$$y_a^0 + y_a \leqslant \bar{y}_a \qquad \forall a \in A_2$$

Solving the above CNDP and obtaining the solution $\boldsymbol{y}^{(0)} = \{y_1^{(0)}, y_2^{(0)}, y_3^{(0)}, \dots\}$, then $\boldsymbol{y}$ is fixed at $\boldsymbol{y}^{(0)}$ to optimize $\boldsymbol{u}$.

$$\min_{\boldsymbol{y}^{(0)}, \boldsymbol{u}} Z(\boldsymbol{y}^{(0)}, \boldsymbol{u}, \boldsymbol{x}) = \sum_{a \in A_1} x_a t_a(x_a) + \sum_{a \in A_2} x_a t_a(x_a, y_a^{(0)}) + \sum_{a \in A_3} x_a t_a(x_a, y'_a) + \phi \sum_{a \in A_2} g_a(y_a^{(0)}) + \phi \sum_{a \in A_3} d_a u_a \tag{7}$$

$$\text{s.t.} \quad u_a = 0 \text{ or } 1 \qquad \forall a \in A_3$$

where $\boldsymbol{x}$ is the implicit function of $\boldsymbol{u}$ and it can be obtained by solving the lower-level problem.

$$\min T(\boldsymbol{y}^{(0)}, \boldsymbol{u}, \boldsymbol{x}) = \sum_{a \in A_1} \int_0^{x_a} t_a(w)\,\mathrm{d}w + \sum_{a \in A_2} \int_0^{x_a} t_a(w, y_a^{(0)})\,\mathrm{d}w + \sum_{a \in A_3} \int_0^{x_a} t_a(w, y'_a)\,\mathrm{d}w \tag{8}$$

$$\text{s.t.} \quad \sum_{k \in L_{rs}} f_k^{rs} = q_{rs} \qquad \forall r \in R, s \in S$$

$$x_a = \sum_{r,s} \sum_k f_k^{rs} \delta_{a,k}^{rs} \qquad \forall a \in A$$

$$f_k^{rs} \geqslant 0 \qquad \forall r \in R, s \in S, k \in L_{rs}$$

If it is the case with the budget constraint, the problem to be solved is as follows:

$$\min_{\boldsymbol{y}^{(0)}, \boldsymbol{u}} Z(\boldsymbol{y}^{(0)}, \boldsymbol{u}, \boldsymbol{x}) = \sum_{a \in A_1} x_a t_a(x_a) + \sum_{a \in A_2} x_a t_a(x_a, y_a^{(0)}) + \sum_{a \in A_3} x_a t_a(x_a, y'_a) \tag{9}$$

$$\text{s.t.} \quad \sum_{a \in A_2} g_a(y_a^{(0)}) + \sum_{a \in A_3} d_a u_a \leqslant \text{budget}$$

$$u_a = 0 \text{ or } 1 \qquad \forall a \in A_3$$

The above problems (7) to (9) are used to solve a

DNDP. Suppose that the solution is $\boldsymbol{u}^{(1)}=\{u_1^{(1)},\ u_2^{(1)},\ u_3^{(1)},\ \cdots\}$, then $\boldsymbol{u}$ is fixed at $u^{(1)}$ to optimize $\boldsymbol{y}$. In this way, a series of solutions $\{\boldsymbol{u}^{(k)}\}$ and $\{\boldsymbol{y}^{(k)}\}$ $(k=0,\ 1,\ 2,\ \cdots)$ can be obtained.

Because

$$Z(\boldsymbol{y},\boldsymbol{u}^{(0)},\boldsymbol{x})\geqslant Z(\boldsymbol{y}^{(0)},\boldsymbol{u}^{(0)},\boldsymbol{x})\geqslant Z(\boldsymbol{y}^{(0)},\boldsymbol{u}^{(1)},\boldsymbol{x}) \quad (10)$$

the function value series $\{Z(\boldsymbol{y}^{(k)},\boldsymbol{u}^{(k)},\boldsymbol{x})\}$ are monotonically decreasing. They do not always converge to the global optimal solution, but they can converge to a local optimal solution. So, several initial values are taken for testing in order to select the best from some local solutions. Especially for the MNDP with the budget constraint, its solution usually depends on the selection of initial values. The method of selecting the initial value $\boldsymbol{u}^{(0)}$ can be conducted by randomly taking 0 link, 1 link, 2 links, $\cdots$with the budget constraint.

3 Numerical Example

A test network is given (see Fig. 1). Links 1 to 16 are expanded links while links 17, 18, 19 and 20 are new candidate links. The link parameters and the OD matrix are listed in Tab. 1 and Tab. 2, respectively. Let $\phi=1$ and the investment function $g_a(y_a)=c_a y_a$.

The objective function value of each iteration and the optimal solution of the MNDP obtained from the proposed solution algorithm are presented in Tab. 3. Note that here the DNDP is solved by the enumeration method while the Hooke-Jeeves algorithm[2] is applied for solving the CNDP.

To see whether the proposed solution algorithm has found the optimal solution of the MNDP for this test network, new candidate links 17, 18, 19 and 20 are combined to obtain 16 possible combinations in all. A corresponding CNDP is solved for each combination (see Tab. 4). In Tab. 4, the solution with a minimal objective function value is $\boldsymbol{u}=\{0,0,1,1\}$, $\boldsymbol{y}=\{1.5625, 1.1250, 3.6875, 0, 0, 0.7500, 0, 0, 0, 0, 0, 0, 0, 0, 0, 15.1875\}$, $Z=403.3460$, which is consistent with the solution found by the proposed solution algorithm.

It can be seen from the above results that only three iterations are required for finding the optimal solution through the proposed solution algorithm, while 16 iterations will be needed if analysis is done for all the possible combinations of new candidate links. And it can be forecasted that the saved work for calculation will be greater with the increase in the number of optimization variables.

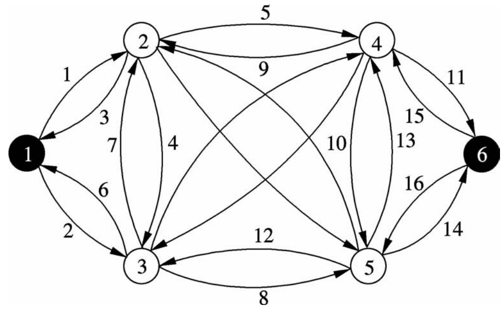

Fig. 1 The test network

Tab. 1 Link parameters for the test network

Link	Node i	Node j	y_a^0 or y_a'	α_a	β_a	c_a or d_a	$\boldsymbol{y}$ or $\boldsymbol{u}$
1	1	2	3	1	10	2	y_1
2	1	3	10	2	5	3	y_2
3	2	1	9	3	3	5	y_3
4	2	3	4	4	20	4	y_4
5	2	4	3	5	50	9	y_5
6	3	1	2	2	20	1	y_6
7	3	2	1	1	10	4	y_7
8	3	5	10	1	1	3	y_8
9	4	2	45	2	8	2	y_9
10	4	5	3	3	3	5	y_{10}
11	4	6	2	9	2	6	y_{11}
12	5	3	6	4	10	8	y_{12}
13	5	4	44	4	25	5	y_{13}
14	5	6	20	2	33	3	y_{14}
15	6	4	1	5	5	6	y_{15}
16	6	5	4.5	6	1	1	y_{16}
17	3	4	26	4	9	8	u_1
18	4	3	21	3	15	9	u_2
19	2	5	35	3	11	10	u_3
20	5	2	41	4	8	6	u_4

Tab. 2 OD matrix for the test network

Node	1	2	3	4	5	6
1	0	0	0	3	2	5
2	0	0	0	0	0	4
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	2	0	0	0	0	0
6	10	4	3	0	0	0

Tab. 3 Solution of the test network

Iteration		Fixed value	Solution	Z
1	Solving CNDP	$\boldsymbol{u}^{(0)}=\{0,0,0,0\}$	$\boldsymbol{y}^{(0)}=\{0,2.5625,4.0000,3.0000,0,0,0,0,3.3750,0,0,0,0,0,0,0.1250,16.3750\}$	474.9184
2	Solving DNDP	$\boldsymbol{y}^{(0)}=\{0,2.5625,4.0000,3.0000,0,0,0,0,3.3750,0,0,0,0,0,0,0.1250,16.3750\}$	$\boldsymbol{u}^{(1)}=\{0,0,1,1\}$	424.9987
3	Solving CNDP	$\boldsymbol{u}^{(1)}=\{0,0,1,1\}$	$\boldsymbol{y}^{(1)}=\{1.5625,1.1250,3.6875,0,0,0.7500,0,0,0,0,0,0,0,0,0,15.1875\}$	403.3460
4	Solving DNDP	$\boldsymbol{y}^{(1)}=\{1.5625,1.1250,3.6875,0,0,0.7500,0,0,0,0,0,0,0,0,0,15.1875\}$	$\boldsymbol{u}^{(2)}=\{0,0,1,1\}$	403.3460 Convergence

Tab. 4 Solution under the fixed $\boldsymbol{u}$

$\boldsymbol{u}$	$\boldsymbol{y}$	Z
{0,0,0,0}	{0,2.562 5,4.000 0,3.000 0,0,0,0,3.375 0,0,0,0,0,0,0,0.125 0,16.375 0}	474.918 4
{1,0,0,0}	{0,3.125 0,4.250 0,3.312 5,0,0,0,1.625 0,0,0,0,0,0,0,0.125 0,16.625 0}	471.242 4
{0,1,0,0}	{0,3.187 5,1.750 0,3.312 5,0,6.500 0,0,3.812 5,0,0,0,0,0,0,0,16.250 0}	473.019 5
{0,0,1,0}	{1.625 0,1.000 0,4.250 0,0,0,0,0,0,0,0,0,0,0,0,0,16.562 5}	437.450 0
{0,0,0,1}	{0,2.875 0,4.312 5,2.875 0,0,0,0,3.375 0,0,0,0,0,0,0,0,16.875 0}	440.614 7
{1,1,0,0}	{0,3.062 5,0.750 0,3.312 5,0,8.562 5,0,1.625 0,0,0,0,0,0,0,0,16.562 5}	465.360 6
{1,0,1,0}	{0,2.812 5,4.125 0,0,0,0,0,0,0,0,0,0,0,0,16.625 0}	439.010 0
{1,0,0,1}	{0,3.250 0,4.375 0,3.312 5,0,0.125 0,0,1.625 0,0,0,0,0,0,0,0,16.750 0}	439.945 1
{0,1,1,0}	{3.000 0,0.250 0,0.187 5,0,0,7.625 0,0,0,0,0,0,0,0,0,0.187 5,16.562 5}	431.005 8
{0,1,0,1}	{0,2.062 5,0.750 0,3.312 5,0,7.750 0,0,3.750 0,0,0,0,0,0,0,3.50 12.750}	450.241 6
{0,0,1,1}	{1.562 5,1.125 0,3.687 5,0,0,0.750 0,0,0,0,0,0,0,0,0,0,15.187 5}	403.346 0
{0,1,1,1}	{3.125 0,0.125 0,0.875 0,0,0,9.125 0,0,0,0,0,0,0,0,0,3.250 0,11.125 0}	414.805 7
{1,0,1,1}	{0.687 5,1.500 0,3.375 0,0,0,1.500 0,0,0,0,0,0,0,0,0,0,16.500 0}	406.130 8
{1,1,0,1}	{0.187 5,2.875 0,2.000 0,3.312 5,0,6.750 0,0,1.625 0,0,0,0,0,0,0,0,16.625 0}	444.456 7
{1,1,1,0}	{2.125 0,0.625 0,2.375 0,0,0,4.250 0,0,0,0,0,0,0,0,0,0,16.562 5}	437.623 2
{1,1,1,1}	{1.000 0,1.500 0,1.750 0,0,0,5.750 0,0,0,0,0,0,0,0,0,0,17.750 0}	410.800 2

4 Conclusion

An optimal DDIA for solving the MNDP is proposed, and its idea is to reduce the dimensions of the problem. A group of variables (discrete/continuous) are fixed to alternately optimize another group of variables (continuous/discrete). Some CNDPs and DNDPs are solved repeatedly until an optimal solution is obtained. The advantage of the proposed algorithm is that its calculation process is very simple and it can utilize some existing algorithms for the CNDP and DNDP in the solution process. A numerical example shows that the proposed algorithm is effective to solve the MNDP and it can find a good solution.

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混合交通网络设计问题的迭代降维算法

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摘要:提出了一种优化的迭代降维算法求解混合交通网络设计问题.混合(连续/离散)交通网络设计问题常表示为一个带均衡约束的数学规划问题,上层通过新建路段和改善已有路段来优化网络性能,下层是一个传统的 Wardrop 用户均衡模型.迭代降维算法的基本思想是降维,先保持一组变量(离散/连续)不变,交替地对另一组变量(连续/离散)实现最优化.以迭代的形式反复求解连续网络设计和离散网络设计问题,直至最后收敛到最优解.通过一个数值算例对算法的效果进行了验证.

关键词:混合网络设计问题;迭代降维算法;带均衡约束数学规划

中图分类号:U491