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A min-max optimization approach for weight determination in analytic hierarchy process

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Abstract: A min-max optimization method is proposed as a new approach to deal with the weight determination problem in the context of the analytic hierarchy process. The priority is obtained through minimizing the maximal absolute difference between the weight vector obtained from each column and the ideal weight vector. By transformation, the constrained min-max optimization problem is converted to a linear programming problem, which can be solved using either the simplex method or the interior method. The Karush-Kuhn-Tucker condition is also analytically provided. These control thresholds provide a straightforward indication of inconsistency of the pairwise comparison matrix. Numerical computations for several case studies are conducted to compare the performance of the proposed method with three existing methods. This observation illustrates that the min-max method controls maximum deviation and gives more weight to non-dominate factors.

Key words: analytic hierarchy process; min-max optimization; weight; linear programming

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As one of the most popular methods for multi-criteria decision-making, the analytic hierarchy process (AHP) has been successfully used in a variety of fields[1−3], including fuzzy membership determination[4], weapon system evaluation[5−8], comprehensive evaluation of naval tactical missile systems[9] and attack helicopters[10], and many other applications[11]. Forman and Peniwati[12] provided general guidelines and considerations for aggregate individual judgments and priorities in the AHP. Takeda and Yu[13] explored a framework for unifying existing weight determination methods. The interested reader may consult with the surveys reported in Refs. [14 − 15] for additional details.

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One of the key components in the AHP is the determination of weight or priority vector $\boldsymbol{w} = \{w_1, w_2, \ldots, w_n\}$ associated with different decision criteria from the so-called pairwise comparison matrix. Suppose that the relative importance of the i-th criterion with respect to the j-th criterion are defined by the ratios $A(x_i)/A(x_j)$, $i, j = 1, 2, \ldots, n$ and expressed in the form of a square matrix $\boldsymbol{A}$, namely,

$$\boldsymbol{A} = [a_{ij}] = \begin{bmatrix} \dfrac{A(x_1)}{A(x_1)} & \dfrac{A(x_1)}{A(x_2)} & \cdots & \dfrac{A(x_1)}{A(x_n)} \\ \dfrac{A(x_2)}{A(x_1)} & \dfrac{A(x_2)}{A(x_2)} & \cdots & \dfrac{A(x_2)}{A(x_n)} \\ \vdots & \vdots & & \vdots \\ \dfrac{A(x_n)}{A(x_1)} & \dfrac{A(x_n)}{A(x_2)} & \cdots & \dfrac{A(x_n)}{A(x_n)} \end{bmatrix} \tag{1}$$

$\boldsymbol{A}$ is a reciprocal matrix and satisfies the property of reciprocality; i.e., $a_{ij}a_{ji} = 1$ for $i, j = 1, 2, \ldots, n$ and $a_{ii} = 1$ ($i = 1, 2, \ldots, n$). An important concept involved in the pairwise comparison matrix is consistency. The matrix $\boldsymbol{A}$ is said to be consistent if $a_{ik}a_{kj} = a_{ij}$ or $a_{ij} = w_i/w_j$ for any $i, j, k = 1, 2, \ldots, n$, which means that the pairwise evaluation of relative importance has good agreement across the different decision criteria[16]. In reality, the entries of $\boldsymbol{A}$ are not known and need to be estimated through a series of pairwise comparisons. In designing the comparison matrix, reciprocality is automatically preserved. The level of priority of x_i over x_j is quantified numerically. The more x_i is preferred over x_j, the higher the numerical level associated with this pair. The level of the priority of x_i over x_j is always equal to 1, as shown by the diagonal elements of $\boldsymbol{A}$. If x_i is not preferred over x_j, then one considers the level of priority attached to the swapped pair of the elements. The number of necessary comparisons is, therefore, $n(n-1)/2$. It is expected that a unique weight vector can be obtained based on the pairwise comparison matrix.

The classical method originally proposed by Saaty[1] uses the principle eigenvector of the comparison matrix as the weight vector. The eigenvector method (EVM) is based on the fact that small perturbations of the eigenvalues of the elements a_{ij} from the perfect ratios w_i/w_j lead to small perturbations of the eigenvalues of the comparison matrix $\boldsymbol{A}$ around its eigenvalues. Mathematically, the

EVM requires solving the eigenvector that corresponds to the largest eigenvalue of $\boldsymbol{A}$; i. e. ,

$$\boldsymbol{A}\boldsymbol{w} = \lambda_{\max} w \tag{2}$$

The solution based on the EVM is not satisfactory when the inconsistency of the decision-makers' priority is large[16]. Also, the property of consistency implies that the ordering of the decision criteria should not be an influential factor in determining the weight vector. Clearly, the EVM does not satisfy this property.

Other methods for deriving the weight vector from the pairwise comparison matrix have also been studied in the literature. These methods are primarily optimization-based and the solution is not affected by the ordering of the decision criteria[17-19]. In an optimization based method, a rational objective function is established under proper constraints. Optimization techniques are then used to solve the mathematical programming problem resulting an optimum weight vector that minimizes or maximizes the objective function. Chu et al. [17] proposed the direct least squares method and the weighted least squares method (WLSM). Sun and Deng[19] also studied the performance and application of the WLSM. The WLSM method is based on the assumption that the elements of the pairwise comparison matrix should satisfy the property $a_{ij} \approx w_i / w_j$. Since this is equivalent to $a_{ij} w_j \approx w_i$, the priority assessment can be formulated as a constrained optimization problem; i. e. ,

$$\begin{aligned} &\min \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} w_j - w_i)^2 \\ &\text{s. t.} \quad \sum_{i=1}^{n} w_i = 1 \\ &w_i \geqslant 0 \quad i = 1, 2, \ldots, n \end{aligned} \tag{3}$$

It is shown that the solution of this optimization problem is equivalent to the solution of the simultaneous equation:

$$\boldsymbol{C}\boldsymbol{w} = \boldsymbol{\xi} \tag{4}$$

where $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}} - \eta$, $\eta = \mathrm{diag}(\eta_1, \eta_2, \ldots, \eta_n)$, $\eta_j = \sum_{i=1}^{n} (a_{ij}^2 + 1)$, $j = 1, 2, \ldots, n$ and $\boldsymbol{\xi} = \{\lambda, \lambda, \ldots, \lambda\}^{\mathrm{T}}$.

Crawford and Williams[20] developed the logarithmic least squares method (LLSM) for weight determination. The LLSM is also known as a geometric mean method and it minimizes the objective function as

$$\begin{aligned} &\min \sum_{i=1}^{n} \sum_{j=1}^{n} (\ln a_{ij} - \ln w_i + \ln w_j)^2 \\ &\text{s. t.} \quad \prod_{i=1}^{n} w_i = 1 \\ &w_i \geqslant 0 \quad i = 1, 2, \ldots, n \end{aligned} \tag{5}$$

It is proved that the solution of this optimization problem can be given as the geometric mean of the column of the comparison matrix $\boldsymbol{A}$[20-21].

$$w_i = \prod_{j=1}^{n} (a_{ij})^n \qquad i = 1, 2, \ldots, n \tag{6}$$

An excellent comparison analysis between the commonly used methods for deriving the weight vector can be found in Refs. [15, 17].

1 Modeling

It is known from the structure of the pairwise comparison matrix that when the pairwise comparison matrix is consistent, each column of $\boldsymbol{A}$ will produce a weight vector identical to the weight vector obtained from any other column. Furthermore, under consistency, all the existing methods should produce the same weight vector. Define the difference between the weight vector obtained from each column and the ideal weight vector as

$$\varepsilon_{ij} = a_{ij} w_j - w_i \qquad i, j = 1, 2, \ldots, n \tag{7}$$

When consistency holds, ε_{ij} should equal zero for all i and j. It is practically impossible, however, to maintain the transitivity of the entries of $\boldsymbol{A}$ because of the involvement of a large amount of subjective judgment in the process of quantifying the comparison matrix.

In reality, the comparison matrix is more likely to be inconsistent. Naturally, the difference ε_{ij} ($i, j = 1, 2, \ldots, n$) become a direct indicator reflecting such inconsistency. It is, therefore, rational to accept the maximum difference as an objective function to minimize. Since the minimized maximum difference is independent of the solution methods, and depends only on the inconsistency of the comparison matrix, it can be used as an ideal index to evaluate the inconsistency. For instance, denote a predefined threshold for accepting any difference ε_{ij} from a specified comparison matrix as $\boldsymbol{\vartheta}$. If the min-max difference ε_{ij} exceeds $\boldsymbol{\vartheta}$, one can suspect and reject the validity of the weight vector obtained from an inconsistent comparison matrix. Otherwise, the obtained weight vector can be accepted at a desired level of inconsistency. In this paper, an alternative method for determining the weight vector from the pairwise comparison matrix is proposed. This method is implemented through solving a min-max optimization problem. While the threshold $\boldsymbol{\vartheta}$ can be set up independently by decision makers and only affects the decision of accepting or rejecting the weight vector, the min-max difference ε_{ij} is obtained using mathematical programming methods.

2 Formulation

The method we propose here is to minimize the largest absolute difference $|\varepsilon_{ij}|$ for all $i, j = 1, 2, \ldots, n$, which can be formulated as a min-max optimization problem

$$\begin{aligned} &\min z = \max_{i,j=1,2,\ldots,n} \{ |\varepsilon_{ij}| = |a_{ij} w_j - w_i| \} \\ &\text{s. t.} \quad \sum_{i=1}^{n} w_i = 1 \end{aligned} \tag{8}$$

$$w_i \geqslant 0 \qquad i = 1, 2, \ldots, n$$

In some situations, instead of just minimizing the largest absolute difference $|\varepsilon_{ij}|$, it might be more preferable to impose control over the relative difference $|a_{ij}w_j - w_i| \leqslant \beta_{ij}w_i$, in which $\beta_{ij}(i, j = 1, 2, \ldots, n)$ are positive constant coefficients or thresholds, also predefined by the decision makers. This strategy may eliminate the scale or unit problem and it can be more reasonable. Following this consideration, the min-max optimization problem now becomes

$$\min \max \{ |a_{ij}w_j - w_i|, \ i, j = 1, 2, \ldots, n\} \tag{9}$$
$$\text{s. t.} \quad \sum_{i=1}^{n} w_i = 1$$
$$|a_{ij}w_j - w_i| \leqslant \beta_{ij}w_i \qquad i, j = 1, 2, \ldots, n$$
$$w_i \geqslant 0 \qquad i = 1, 2, \ldots, n$$

Clearly, the problem defined by Eq. (8) is only a special case of problem (9) because if one sets up β_{ij} as very large numbers, the second constraint in problem (9) becomes relaxed and problem (9) degrades to problem (8). Certainly, if β_{ij} are set up as very small numbers, problem (9) may end up with no feasible solutions.

In the following, only the problem formulated in (9) will be studied due to its generality. Define $d = \max\{ |a_{ij}w_j - w_i| \} \ (i, j = 1, 2, \ldots, n)$. It is straightforward to see that $d \geqslant |a_{ij}w_j - w_i|$, or equivalently, $d \geqslant (a_{ij}w_j - w_i)$ and $d \geqslant -(a_{ij}w_j - w_i)$. Using this extra variable d, the min-max problem given by (9) can now be reformulated as a linear programming (LP) problem,

$$\min z = d \tag{10}$$
$$\text{s. t.} \quad d + a_{ij}w_j - w_i \geqslant 0$$
$$d - a_{ij}w_j + w_i \geqslant 0$$
$$a_{ij}w_j + (\beta_{ij} - 1)w_i \geqslant 0$$
$$-a_{ij}w_j + (\beta_{ij} + 1)w_i \geqslant 0$$
$$\sum_{i=1}^{n} w_i = 1$$
$$d, w_i \geqslant 0 \qquad i, j = 1, 2, \ldots, n$$

3 Karush-Kuhn-Tucker Condition

The LP problem formulated in (10) can be solved by the simplex method. However, using the interior method can provide more insights in terms of the structure of this LP problem. Introduce slack variables δ_{ij}, δ'_{ij}, η_{ij} and η'_{ij} $(i, j = 1, 2, \ldots, n)$ to the constraints. The LP problem (10) is now equivalent to

$$\min z = d \tag{11}$$
$$\text{s. t.} \quad a_{ij}w_j - w_i - \delta_{ij} + d = 0$$
$$-a_{ij}w_j + w_i - \delta'_{ij} + d = 0$$
$$a_{ij}w_j + (\beta_{ij} - 1)w_i - \eta_{ij} = 0$$
$$-a_{ij}w_j + (\beta_{ij} + 1)w_i - \eta'_{ij} = 0$$
$$1 - \sum_{i=1}^{n} w_i = 0$$
$$d, \ w_i, \ \delta_{ij}, \ \delta'_{ij}, \ \eta_{ij}, \ \eta'_{ij} \geqslant 0 \qquad i, j = 1, 2, \ldots, n$$

Using the barrier method, the constrained LP problem (11) can be converted to an unconstrained LP problem whose objective function can be expressed as the Lagrange.

$$\begin{aligned} \min l(d, \lambda, w_i, \delta_{ij}, \delta'_{ij}, \pi_{ij}, \pi'_{ij}, \eta_{ij}, \eta'_{ij}, \theta_{ij}, \theta'_{ij}) = \\ d + \lambda\left(1 - \sum_{i=1}^{n} w_i\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_{ij}(a_{ij}w_j - w_i - \delta_{ij} + d) + \\ \sum_{i=1}^{n}\sum_{j=1}^{n} \pi'_{ij}(-a_{ij}w_j + w_i - \delta'_{ij} + d) + \\ \sum_{i=1}^{n}\sum_{j=1}^{n} \theta_{ij}[a_{ij}w_j + (\beta_{ij} - 1)w_i - \eta_{ij}] + \\ \sum_{i=1}^{n}\sum_{j=1}^{n} \theta'_{ij}[-a_{ij}w_j + (\beta_{ij} + 1)w_i - \eta'_{ij}] - \\ \mu\left[\ln d + \sum_{i=1}^{n} \ln w_i + \sum_{i=1}^{n}\sum_{j=1}^{n} (\ln \delta_{ij} + \ln \delta'_{ij} + \ln \eta_{ij} + \ln \eta'_{ij})\right] \end{aligned} \tag{12}$$

where λ, π_{ij} and π'_{ij} are the Lagrange multipliers and μ is the barrier coefficient. The condition for an unconstrained LP to be minimized requires

$$\nabla l(d, \lambda, w_k, \delta_{ij}, \delta'_{ij}, \pi_{ij}, \pi'_{ij}, \eta_{ij}, \eta'_{ij}, \theta_{ij}, \theta'_{ij}) = 0 \tag{13}$$

where ∇ represents the derivative of l with respect to all the variables; i. e.,

$$\nabla l = \left\{ \frac{\partial l}{\partial d}, \frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial w_k}, \frac{\partial l}{\partial \delta_{ij}}, \frac{\partial l}{\partial \delta'_{ij}}, \frac{\partial l}{\partial \pi_{ij}}, \frac{\partial l}{\partial \pi'_{ij}}, \frac{\partial l}{\partial \eta_{ij}}, \frac{\partial l}{\partial \eta'_{ij}}, \frac{\partial l}{\partial \theta_{ij}}, \frac{\partial l}{\partial \theta'_{ij}} \right\}^{\mathrm{T}}$$

Taking expansion of (13) gives the Karush-Kuhn-Tucker (KKT) conditions[22].

$$1 + \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} \pi'_{ij} - \frac{\mu}{d} = 0$$
$$\sum_{i=1}^{n} w_i = 1$$
$$n(\pi_{ii} - \pi'_{ii} + \theta_{ii} - \theta'_{ii}) + \sum_{j=1}^{n} (\pi'_{ij} - \pi_{ij}) + \sum_{j=1}^{n} [\theta_{ij}(\beta_{ij} - 1) + \theta'_{ij}(\beta_{ij} + 1)] - \frac{\mu}{w_i} = \lambda$$
$$\delta_{ij} = -\frac{\mu}{\pi_{ij}}, \ \delta'_{ij} = -\frac{\mu}{\pi'_{ij}}, \ \eta_{ij} = -\frac{\mu}{\theta_{ij}}, \ \eta'_{ij} = -\frac{\mu}{\theta'_{ij}}$$
$$a_{ij}w_j - w_i - \delta_{ij} + d = 0$$
$$-a_{ij}w_j + w_i - \delta'_{ij} + d = 0$$
$$a_{ij}w_j + (\beta_{ij} - 1)w_i - \eta_{ij} = 0$$
$$-a_{ij}w_j + (\beta_{ij} + 1)w_i - \eta'_{ij} = 0$$

The aforementioned nonlinear algebraic equation system has identical numbers of equations and unknowns. Therefore, the solution to the algebraic equation system is definite. One can symbolically express weight vector $\boldsymbol{w} = \{w_1, w_2, \ldots, w_n\}$, the objective function d and the Lagrange λ in terms of μ. According to the interior method, taking the limit when $\mu \to 0$, the solution of this nonlinear algebraic equation system gives the optimal weight vector.

4 Case Studies

In this section, numerical examples are presented to illustrate the method and to compare it with other commonly used ones. It should be pointed out that, in the following examples, β_{ij} are set up as very large positive numbers so that their role of restricting the relative difference is diminished.

Example 1 Saaty[1] reported that a fresh Ph. D. graduate made the following paired comparisons when he tried to assess the relative importance of six job selection criteria, namely research, growth, benefits, colleagues, location, and reputation (in that order). Tab. 1 gives such a pairwise comparison matrix.

The weight produced by the min-max method and three other methods are given in Tab. 2. The value of the objective function of the min-max optimization problem is 0.153. The numbers inside the parentheses are ranking information. The last two rows in Tab. 2 show the 1-norm and the 2-norm, which are used to measure the performances of these different methods.

$$S_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| a_{ij} - w_i/w_j \right|$$

$$S_2 = \left[\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - w_i/w_j)^2 \right]^{1/2}$$

In general, the solutions remain fairly similar since all the methods provide identical ranking for "colleagues", "location" and "reputation". But "research", "growth", and "benefits" are ranked differently by various methods. The WLSM gives the same ranking as the min-max method does. However, the EVM and LLSM give distinct ranking in terms of these three jobs. In this particular case, in terms of both the 1-norm and the 2-norm criteria, the LLSM gives the best solution, and the WLSM and the min-max method give the worst solutions.

Tab. 1 Relative importance of job selection matrix for Ph. D. graduates

Job	Research	Growth	Benefits	Colleagues	Location	Reputation
Research	1	1	1	4	1	1/2
Growth	1	1	2	4	1	1/2
Benefits	1	1/2	1	5	3	1/2
Colleagues	1/4	1/4	1/5	1	1/3	1/3
Location	1	1	1/3	3	1	1
Reputation	2	2	2	3	1	1

Tab. 2 Comparison of numerical results for job selection matrix

Job	EVM(rank)	WLSM (rank)	LLSM(rank)	Min-max(rank)
Research	0.159 (4)	0.174 (3)	0.169 (4)	0.178 (3)
Growth	0.189 (3)	0.190 (2)	0.189 (2)	0.217 (2)
Benefits	0.198 (2)	0.171 (4)	0.187 (3)	0.173 (4)
Colleagues	0.048 (6)	0.050 (6)	0.050 (6)	0.058 (6)
Location	0.150 (5)	0.130 (5)	0.150 (5)	0.112 (5)
Reputation	0.256 (1)	0.285 (1)	0.255 (1)	0.262 (1)
S_1	12.907	14.041	12.826	15.267
S_2	3.645	4.127	3.608	3.904

Example 2 The wealth-of-nations problem has been used by many authors for the evaluation of different prioritization methods[1]. The wealth-of-nations pairwise comparison matrix is given in Tab. 3. It represents the responses of an economic expert, who compresses the wealth of seven countries in 1972 using the pairwise comparisons within the Saaty scale 1/9 to 9.

The priorities generated by the four methods are represented in Tab. 4. Clearly, different methods give different priorities, but an identical final ranking of the wealth of nations. In this case all the methods preserve the same rank because the comparison matrix is rather consistent. Its consistency index is CI = 0.101 and the consistency ratio is CR = 0.077.

The last two rows in Tab. 4 list the performance indicators S_1 and S_2 for the different methods. Clearly, in this particular problem the LLSM and min-max methods have the best performance in terms of 1-norm and 2-norm, respectively. The EVM and WLSM are in the middle, similar to the previous example. The value of the objective function in the min-max optimization problem is 0.161. It means that the maximal difference between the estimated priority and the actual priority is $(\varepsilon_{ij})_{\max} = \varepsilon_{12} = 0.161$, while they are $(\varepsilon_{ij})_{\max} = \varepsilon_{12} = 0.500$ for the EVM; $(\varepsilon_{ij})_{\max} = \varepsilon_{13} = 0.224$ for the WLSM; and $(\varepsilon_{ij})_{\max} = \varepsilon_{12} = 0.514$ for the LLSM, respectively. Since the objective of the min-max optimization method is to control the maximum difference, it is clear that it is the best method in terms of reducing the individual differences between the column-based priority and the ideal priority. The comparisons between the min-max method and others gives an idea of the performance of this new method. It can be seen from these two classic, numerical examples that the min-max method can provide a reasonable solution for priority setting.

Tab. 3 1972 wealth-of-nations matrix

Country	US	USSR	China	France	UK	Japan	W. Germany
US	1	4	9	6	6	5	5
USSR	1/4	1	7	5	5	3	4
China	1/9	1/7	1	1/5	1/5	1/7	1/5
France	1/6	1/5	5	1	1	1/3	1/3
UK	1/6	1/5	5	1	1	1/3	1/3
Japan	1/5	1/3	7	3	3	1	2
W. Germany	1/5	1/4	5	3	3	1/2	1

Tab. 4 Comparison of numerical results for 1972 wealth-of-nations matrix

Country	EVM(rank)	WLSM(rank)	LLSM(rank)	Min-max(rank)
US	0.427 (1)	0.487 (1)	0.417 (1)	0.474 (1)
USSR	0.230 (2)	0.175 (2)	0.231 (2)	0.171 (2)
China	0.021 (6)	0.030 (6)	0.020 (6)	0.042 (6)
France	0.052 (5)	0.059 (5)	0.054 (5)	0.060 (5)
UK	0.052 (5)	0.059 (5)	0.054 (5)	0.060 (5)
Japan	0.123 (3)	0.104 (3)	0.128 (3)	0.114 (3)
W. Germany	0.094 (4)	0.085 (4)	0.096 (4)	0.079 (4)
S_1	44.146	40.779	44.996	
S_2	13.708	11.158	14.000	10.012

5 Conclusion

A min-max optimization method is proposed in this paper as an alternative to deal with weight determination problem in the context of the AHP. The priority is obtained through minimizing the maximal absolute difference between the weight vector obtained from each column and the ideal weight vector. A more general setting is achieved by introducing control thresholds over relative differences between the column-based weight vector and the ideal weight vector. By transformation, the constrained min-max optimization problem is converted to a linear programming problem, which can be solved using either the simplex method or the interior method. The KKT condition is also provided analytically. These control thresholds and the values of the objective function provide a straightforward indication of inconsistency of the pairwise comparison matrix. Based on two examples, it is shown that the sum of absolute value of deviation using the min-max method tends to be greater than that using the EVM, the WLSM and the LLSM, while the sum of square of absolute value of deviation using the min-max method tends to be less than that using the EVM, the WLSM and the LLSM. This observation illustrates that the min-max method controls the maximum deviation and gives more weight to non-dominant factors.

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层次分析法中用于确定权重的最小-最大优化方法

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摘要:提出了层次分析法中一种用于确定权重的最小-最大优化方法.其思路为通过最小化由两两比较矩阵中每列所得到的优先权和理想的权重向量之间的最大绝对差异来实现权重确定.通过适当的变换,问题转化为可以采用单纯形或内点法求解的线性优化问题.推导建立了解析的 Karush-Kuhn-Tucker 条件.所建立的临界阀值提供了关于两两比较矩阵不一致特性的一种直接的表征.给出了几种实例的数值算法,并比较了所提方法和 3 种现有的权重确定方法的性能,观察结果发现最小-最大优化方法对于非主导因素有较多考虑.

关键词:层次分析法;最小-最大优化;权重;线性优化

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