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On reducibility of a class of nonlinear quasi-periodic systems with small perturbational parameters near equilibrium

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Abstract: Consider the reducibility of a class of nonlinear quasi-periodic systems with multiple eigenvalues under perturbational hypothesis in the neighborhood of equilibrium. That is, consider the following system $\dot{\boldsymbol{x}} = (\boldsymbol{A} + \varepsilon \boldsymbol{Q}(t))\boldsymbol{x} + \varepsilon \boldsymbol{g}(t) + \boldsymbol{h}(\boldsymbol{x}, t)$, where $\boldsymbol{A}$ is a constant matrix with multiple eigenvalues; $\boldsymbol{h} = O(\boldsymbol{x}^2)(\boldsymbol{x}\to 0)$; and $\boldsymbol{h}(x, t)$, $\boldsymbol{Q}(t)$, and $\boldsymbol{g}(t)$ are analytic quasi-periodic with respect to t with the same frequencies. Under suitable hypotheses of non-resonance conditions and non-degeneracy conditions, for most sufficiently small ε, the system can be reducible to a nonlinear quasi-periodic system with an equilibrium point by means of a quasi-periodic transformation.

Key words: quasi-periodic; reducible; non-resonance condition; non-degeneracy condition; KAM iteration

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We consider the linear system $\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}$, $\boldsymbol{x} \in \mathbf{R}^n$, where $\boldsymbol{A}(t)$ is an $n \times n$ matrix. The well-known Floquet theorem tells us that if $\boldsymbol{A}(t)$ is a T-periodic matrix, then the linear differential equation $\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}$ is reducible to the constant coefficient differential equation $\dot{\boldsymbol{x}} = \boldsymbol{B}\boldsymbol{x}$ by a T-periodic change of variables. For the quasi-periodic coefficient system, Johnson and Sell [1] proved that if the quasi-periodic coefficients matrix $\boldsymbol{A}(t)$ satisfies full spectrum conditions, then $\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}$ is reducible. That is, there exists a quasi-periodic non-sigular transformation $\boldsymbol{x} = \boldsymbol{\phi}(t)\boldsymbol{y}$, where $\boldsymbol{\phi}(t)$ and $\boldsymbol{\phi}(t)^{-1}$ are quasi-periodic and bounded, such that $\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}$ is transformed to $\dot{\boldsymbol{y}} = \boldsymbol{B}\boldsymbol{y}$, where $\boldsymbol{B}$ is a constant matrix. In Ref. [2], Jorba and Simó considered the reducibility of the following linear quasi-periodic system $\dot{\boldsymbol{x}} = (\boldsymbol{A} + \varepsilon \boldsymbol{Q}(t))\boldsymbol{x}$, $\boldsymbol{x} \in \mathbf{R}^n$, where $\boldsymbol{A}$ is a constant matrix with different eigenvalues. They proved that under the non-resonance conditions and the non-degeneracy conditions, there exists a non-empty Cantor subset E, such that for $\varepsilon \in E$, the system is reducible. In Ref. [3], Xu considered the linear quasi-periodic system $\dot{\boldsymbol{x}} = (\boldsymbol{A} + \varepsilon \boldsymbol{Q}(t))\boldsymbol{x}$, $\boldsymbol{x} \in \mathbf{R}^n$, where $\boldsymbol{A}$ is a constant matrix with multiple eigenvalues. He proved that under the non-resonance conditions and the non-degeneracy conditions, there exists a non-empty Cantor subset E, such that the system is reducible for $\varepsilon \in E$.

In Ref. [4], Eliasson studied the one-dimensional linear Schrödinger equation $\dfrac{d^2 q}{dt^2} + Q(\boldsymbol{\omega} t)q = Eq$, where Q is an analytic quasi-periodic function and E is an energy parameter. The result in Ref. [4] implies that for almost every sufficiently large E, the quasi-periodic system is reducible. Recently, a similar problem was considered by Her and You [5].

In 1996, Jorba and Simó[6] extended the conclusion of the linear system to the nonlinear case. They considered the quasi-periodic system $\dot{\boldsymbol{x}} = (\boldsymbol{A} + \varepsilon \boldsymbol{Q}(t))\boldsymbol{x} + \varepsilon \boldsymbol{g}(t) + \boldsymbol{h}(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \mathbf{R}^n$, where $\boldsymbol{A}$ has n different nonzero eigenvalues λ_i, and proved that under some non-resonance conditions and some non-degeneracy conditions, there exists a non-empty Cantor subset $E \in (0, \varepsilon_0)$, such that the system is reducible for $\varepsilon \in E$.

Wang and Xu[7] considered the nonlinear quasi-periodic system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \mathbf{R}^2$. They proved that without any non-degeneracy condition, the system is reducible.

Motivated by Refs. [3, 6], in this paper we extend the results in Ref. [6] to the general case of multiple eigenvalues.

1 Preliminaries

Definition 1 A function f is called a quasi-periodic function with frequencies $\boldsymbol{\omega} = \{\omega_1, \omega_2, \dots, \omega_r\}$ if $f(t) = F(\omega_1 t, \omega_2 t, \dots, \omega_r t)$, where $F(\theta_1, \theta_2, \dots, \theta_r)$ is 2π periodic in all arguments and $\theta_i = \omega_i t$, $i = 1, 2, \dots, r$. If $F(\boldsymbol{\theta})$ ($\boldsymbol{\theta} = \{\theta_1, \theta_2, \dots, \theta_r\}$) is analytic on $D_\rho = \{\boldsymbol{\theta} \in C^r \mid |\mathrm{Im}\theta_i| \leqslant \rho, i = 1, 2, \dots, r\}$, we call $f(t)$ analytic quasi-periodic on D_ρ. Denote the sup-norm of f on D_ρ by $\| f \|_{D_\rho} = \sup\limits_{\theta \in D_\rho} |F(\boldsymbol{\theta})|$.

Definition 2 A matrix function $\boldsymbol{Q}(t) = (q_{ij}(t))_{1 \leqslant i,j \leqslant n}$ is called analytic quasi-periodic on D_ρ if all $q_{ij}(t)$ ($i, j = 1, 2, \dots, n$) are analytic quasi-periodic on D_ρ.

Define the norm of $\boldsymbol{Q}$ by

$$\| \boldsymbol{Q} \|_{D_\rho} = n \max_{1 \leqslant i,j \leqslant n} \| q_{ij} \|_{D_\rho}$$

Clearly, $\| \boldsymbol{Q}_1 \boldsymbol{Q}_2 \|_{D_\rho} \leqslant \| \boldsymbol{Q}_1 \|_{D_\rho} \| \boldsymbol{Q}_2 \|_{D_\rho}$. For convenience, if $\boldsymbol{Q}$ is a constant matrix, we denote $\| \boldsymbol{Q} \| = \| \boldsymbol{Q} \|_{D_\rho}$. The average of $\boldsymbol{Q}$ is denoted by $\overline{\boldsymbol{Q}} =$

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$(\bar{q}_{ij})_{1\leqslant i,j\leqslant n}$, where $\bar{q}_{ij}=\lim\limits_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}q_{ij}(t)\,\mathrm{d}t$. For the existence of its limit, see Ref. [8].

2 Main Results

Theorem 1 Consider the equation

$$\dot{\boldsymbol{x}}=(\boldsymbol{A}+\varepsilon\boldsymbol{Q}(t,\varepsilon))\boldsymbol{x}+\varepsilon\boldsymbol{g}(t,\varepsilon)+\boldsymbol{h}(\boldsymbol{x},t,\varepsilon)\qquad \boldsymbol{x}\in\mathbf{R}^n \tag{1}$$

Suppose that $\boldsymbol{A}=\mathrm{diag}(\lambda_1\boldsymbol{I}_{r_1},\lambda_2\boldsymbol{I}_{r_2},\ldots,\lambda_l\boldsymbol{I}_{r_l})$, where $\boldsymbol{I}_r$ is the identity matrix of r order, $r_1+r_2+\ldots+r_l=n$, $\lambda_i\neq 0$, and $\lambda_i\neq\lambda_j$, $i\neq j$. Assume that $\boldsymbol{Q}(t,\varepsilon)=(q_{ij}(t))_{1\leqslant i,j\leqslant n}$, $\boldsymbol{g}(t,\varepsilon)$, and $\boldsymbol{h}(\boldsymbol{x},t,\varepsilon)$ are analytic quasi-periodic on D_ρ with frequencies $\omega_1,\omega_2,\ldots,\omega_r$, and analytic with respect to ε. Moreover, $\boldsymbol{h}(\boldsymbol{x},t)$ is analytic with respect to $\boldsymbol{x}$ on $B_a(0)$, $\boldsymbol{h}(\boldsymbol{0},t,\varepsilon)=0$, and $D_x\boldsymbol{h}(\boldsymbol{0},t,\varepsilon)=0$. Here $B_a(0)$ is a ball (in the complex plane) centered in 0 with radius a; $\varepsilon\in(0,\varepsilon_0)$ is a parameter.

We give the following assumptions.

Assumption 1 (non-resonance conditions) $\boldsymbol{\lambda}=\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ and $\boldsymbol{\omega}=\{\omega_1,\omega_2,\ldots,\omega_r\}$ satisfy

$$|\langle\boldsymbol{k},\boldsymbol{\omega}\rangle\sqrt{-1}+\lambda_i|\geqslant\frac{\alpha_0}{|\boldsymbol{k}|^{\tau}} \tag{2}$$

$$|\langle\boldsymbol{k},\boldsymbol{\omega}\rangle\sqrt{-1}+\lambda_i-\lambda_j|\geqslant\frac{\alpha_0}{|\boldsymbol{k}|^{\tau}} \tag{3}$$

for all $\boldsymbol{k}\in\mathbf{Z}^r\setminus\{\boldsymbol{0}\}$, where α_0 is a small constant and $\tau>r-1$.

Assumption 2 (non-degeracy conditions) Denote by $\underline{\boldsymbol{x}}$ the unique solution of $\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\varepsilon\boldsymbol{g}(t)$. Let $\hat{\boldsymbol{Q}}=\boldsymbol{Q}(t)+\frac{1}{\varepsilon}D_x\boldsymbol{h}(\underline{\boldsymbol{x}}(t),t)$ and $\bar{\hat{\boldsymbol{Q}}}=(\bar{\hat{q}}_{ij})_{1\leqslant i,j\leqslant n}$. Denote $\bar{\hat{\boldsymbol{Q}}}=(\boldsymbol{R}_{ij})_{1\leqslant i,j\leqslant l}$, where $\boldsymbol{R}_{kk}$ is the r_k order matrix for $1\leqslant k\leqslant l$. Define $\boldsymbol{R}_i=\boldsymbol{R}_{ii}$ and $\boldsymbol{R}=\mathrm{diag}(\boldsymbol{R}_1,\boldsymbol{R}_2,\ldots,\boldsymbol{R}_l)$. Assume that the eigenvalues of $\boldsymbol{A}+\varepsilon\boldsymbol{R}$ are $\lambda_1^+,\lambda_2^+,\ldots,\lambda_n^+$ and the eigenvalues of $\boldsymbol{R}_i$ are $\delta_i^1,\delta_i^2,\ldots,\delta_i^{r_i}$, which satisfy $|\delta_i^j(\varepsilon)-\delta_i^{j'}(\varepsilon)|\geqslant\delta>0$, $\delta_i^j(0)\neq\delta_{i'}^{j'}(0)$, $\left|\frac{\mathrm{d}\delta_i^j(\varepsilon)}{\mathrm{d}\varepsilon}\right|\leqslant\delta_0$, $\delta_2\geqslant\left|\frac{\mathrm{d}(\lambda_k^+(\varepsilon))}{\mathrm{d}\varepsilon}\right|=\left|\frac{\mathrm{d}(\varepsilon\delta_i^j(\varepsilon))}{\mathrm{d}\varepsilon}\right|\geqslant\delta_1>0$, and $\delta_2\geqslant\left|\frac{\mathrm{d}(\lambda_{k_1}^+(\varepsilon)-\lambda_{k_2}^+(\varepsilon))}{\mathrm{d}\varepsilon}\right|=\left|\frac{\mathrm{d}(\varepsilon\delta_i^j(\varepsilon)-\varepsilon\delta_{i'}^{j'}(\varepsilon))}{\mathrm{d}\varepsilon}\right|\geqslant\delta_1>0$, where $j\neq j'$ or $i\neq i'$; $i,i'=1,2,\ldots,l$; $k,k_1,k_2=1,2,\ldots,n$, $k_1\neq k_2$; $\varepsilon\in(0,\varepsilon_0)$; δ,δ_0,δ_1 and δ_2 are constants.

Assumption 3 $\|D_{xx}\boldsymbol{h}(\boldsymbol{x},t,\varepsilon)\|\leqslant K$, where $\boldsymbol{x}\in B_a(0)$ and $\varepsilon\in(0,\varepsilon_0)$. Then there exists a non-empty Cantor subset $E^*\subset(0,\varepsilon_0)$ with a positive Lebesgue measure, such that for $\varepsilon\in E^*$, there exists a quasi-periodic transformation $\boldsymbol{x}=\boldsymbol{\psi}(t)\boldsymbol{y}+\boldsymbol{\varphi}(t)$ which changes Eq. (1) to $\dot{\boldsymbol{y}}=\boldsymbol{B}\boldsymbol{y}+\boldsymbol{h}_\infty(\boldsymbol{y},t)$, where $\boldsymbol{\psi}(t)$ and $\boldsymbol{\varphi}(t)$ are quasi-periodic with frequencies $\boldsymbol{\omega}$; $\boldsymbol{B}$ is a constant matrix and $\boldsymbol{h}_\infty(\boldsymbol{y},t)=O(\boldsymbol{y}^2)\,(\boldsymbol{y}\to 0)$. Moreover, meas $((0,\varepsilon_0)\setminus E^*)=o(\varepsilon_0)$ when $\varepsilon_0\to 0$.

Remark 1 In general, $\boldsymbol{Q}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ depend on ε. Here and below, for simplicity, we do not indicate this dependence explicitly. The subset $E^*\subset(0,\varepsilon_0)$ is a Cantor-like set and so the smoothness of the function with respect to ε on E^* should be understood in the sense of Whitney. We refer to Ref. [9].

3 Preliminary Step

In this section, we will give the first KAM step. For Eq. (1), the purpose of the first step is that $\boldsymbol{A}$ is changed from the case of multiple eigenvalues to the case of different eigenvalues.

If $|\langle\boldsymbol{k},\boldsymbol{\omega}\rangle\sqrt{-1}+\lambda_i|\geqslant\frac{\alpha_0}{|\boldsymbol{k}|^{3\tau}}$ for $\boldsymbol{k}\in\mathbf{Z}^r\setminus\{\boldsymbol{0}\}$, we denote by $\underline{\boldsymbol{x}}$ the solution of $\dot{\boldsymbol{x}}=\boldsymbol{A}x+\varepsilon\boldsymbol{g}(t)$ on $D_{\rho-s}$. Under the change of variables $\boldsymbol{x}=\underline{\boldsymbol{x}}+\boldsymbol{y}$, Eq. (1) is changed into

$$\dot{\boldsymbol{y}}=(\boldsymbol{A}+\varepsilon\hat{\boldsymbol{Q}})\boldsymbol{y}+\varepsilon^2\hat{\boldsymbol{g}}+\hat{\boldsymbol{h}}(\boldsymbol{y},t) \tag{4}$$

where $\hat{\boldsymbol{Q}}(t)=\boldsymbol{Q}(t)+\frac{1}{\varepsilon}D_x\boldsymbol{h}(\underline{\boldsymbol{x}},t)$, $\hat{\boldsymbol{g}}(t)=\frac{1}{\varepsilon^2}\boldsymbol{h}(\underline{\boldsymbol{x}},t)+\frac{1}{\varepsilon}\boldsymbol{Q}(t)\underline{\boldsymbol{x}}$, $\hat{\boldsymbol{h}}(\boldsymbol{y},t)=\boldsymbol{h}(\underline{\boldsymbol{x}}+\boldsymbol{y},t)-\boldsymbol{h}(\underline{\boldsymbol{x}},t)-D_x\boldsymbol{h}(\underline{\boldsymbol{x}},t)\boldsymbol{y}$.

We define the average of $\hat{\boldsymbol{Q}}$ by $\bar{\hat{\boldsymbol{Q}}}=(\bar{\hat{q}}_{ij})_{1\leqslant i,j\leqslant n}$. Denote $\bar{\hat{\boldsymbol{Q}}}=(\boldsymbol{R}_{ij})_{1\leqslant i,j\leqslant l}$, where $\boldsymbol{R}_{kk}$ is the r_k order matrix for $1\leqslant k\leqslant l$. Define $\boldsymbol{R}_i=\boldsymbol{R}_{ii}$ and $\boldsymbol{R}=\mathrm{diag}(\boldsymbol{R}_1,\boldsymbol{R}_2,\ldots,\boldsymbol{R}_l)$. Let eigenvalues of $\boldsymbol{R}_i$ be $\delta_i^1,\delta_i^2,\ldots,\delta_i^{r_i}$.

There exists a nonsingular matrix $\boldsymbol{S}$, such that

$$\boldsymbol{S}^{-1}(\boldsymbol{A}+\varepsilon\boldsymbol{R})\boldsymbol{S}=\boldsymbol{A}^+=\mathrm{diag}(\lambda_1^+,\lambda_2^+,\ldots,\lambda_n^+)$$

where the eigenvalues of $\boldsymbol{A}^+$ are $\{\lambda_i+\varepsilon\delta_i^j\}$ for $j=1,2,\ldots,r_i$ and $i=1,2,\ldots,l$. Denote $\boldsymbol{A}+\varepsilon\boldsymbol{R}=\mathrm{diag}(\boldsymbol{A}_1^1,\boldsymbol{A}_2^1,\ldots,\boldsymbol{A}_l^1)$, where $\boldsymbol{A}_k^1$ is the r_k order matrix for $1\leqslant k\leqslant l$.

Now we can make the change of variables $\boldsymbol{y}=\boldsymbol{S}\boldsymbol{z}$ to Eq. (4) and obtain

$$\dot{\boldsymbol{z}}=(\boldsymbol{A}^++\varepsilon\tilde{\boldsymbol{Q}})\boldsymbol{z}+\varepsilon^2\tilde{\boldsymbol{g}}(t)+\tilde{\boldsymbol{h}}(\boldsymbol{z},t) \tag{5}$$

where $\boldsymbol{S}^{-1}(\hat{\boldsymbol{Q}}-\boldsymbol{R})\boldsymbol{S}=\tilde{\boldsymbol{Q}}$, $\boldsymbol{S}^{-1}\hat{\boldsymbol{g}}=\tilde{\boldsymbol{g}}$, $\boldsymbol{S}^{-1}\hat{\boldsymbol{h}}(\boldsymbol{S}\boldsymbol{z},t)=\tilde{\boldsymbol{h}}(\boldsymbol{z},t)$. We denote

$$\tilde{\boldsymbol{Q}}=(\tilde{q}_{ij})_{1\leqslant i,j\leqslant n}\qquad \bar{\tilde{\boldsymbol{Q}}}=(\bar{\tilde{Q}}_{ij})_{1\leqslant i,j\leqslant n}$$

Making the change of variables $\boldsymbol{z}=(\boldsymbol{I}+\varepsilon\boldsymbol{P}(t))\boldsymbol{z}_1$ into Eq. (5), we obtain

$$\begin{aligned}\dot{\boldsymbol{z}}_1=&((\boldsymbol{I}+\varepsilon\boldsymbol{P})^{-1}(\boldsymbol{A}^++\varepsilon(\boldsymbol{A}^+\boldsymbol{P}-\dot{\boldsymbol{P}}+\tilde{\boldsymbol{Q}}))+\\&\varepsilon^2(\boldsymbol{I}+\varepsilon\boldsymbol{P})^{-1}\tilde{\boldsymbol{Q}}\boldsymbol{P})\boldsymbol{z}_1+\varepsilon^2(\boldsymbol{I}+\varepsilon\boldsymbol{P})^{-1}\tilde{\boldsymbol{g}}(t)+\\&(\boldsymbol{I}+\varepsilon\boldsymbol{P})^{-1}\tilde{\boldsymbol{h}}((\boldsymbol{I}+\varepsilon\boldsymbol{P})\boldsymbol{z}_1,t)\end{aligned} \tag{6}$$

We would like to have $(\boldsymbol{I}+\varepsilon\boldsymbol{P})^{-1}(\boldsymbol{A}^++\varepsilon(\boldsymbol{A}^+\boldsymbol{P}-\dot{\boldsymbol{P}}+\tilde{\boldsymbol{Q}}))=\boldsymbol{A}^+$, which is equivalent to

$$\dot{\boldsymbol{P}}=\boldsymbol{A}^+\boldsymbol{P}-\boldsymbol{P}\boldsymbol{A}^++\tilde{\boldsymbol{Q}} \tag{7}$$

Expand $\tilde{\boldsymbol{Q}}$ and $\boldsymbol{P}$ into the Fourier series $\tilde{\boldsymbol{Q}} = \sum_{\boldsymbol{k} \in Z^r} \tilde{\boldsymbol{Q}}_{\boldsymbol{k}} e^{i\langle \boldsymbol{k}, \boldsymbol{\omega} \rangle t}$, $\boldsymbol{P} = \sum_{\boldsymbol{k} \in \boldsymbol{Z}^r} \boldsymbol{P}_{\boldsymbol{k}} e^{i\langle \boldsymbol{k}, \boldsymbol{\omega} \rangle t}$, where $\tilde{\boldsymbol{Q}}_{\boldsymbol{k}} = (\tilde{q}_{ij}^k)_{1 \leqslant i,j \leqslant n}$, $\boldsymbol{P}_{\boldsymbol{k}} = (p_{ij}^k)_{1 \leqslant i,j \leqslant n}$. If $|\langle \boldsymbol{k}, \boldsymbol{\omega} \rangle \sqrt{-1} + \lambda_i^+ - \lambda_j^+| \geqslant \frac{\alpha_1}{|\boldsymbol{k}|^{3\tau}}$ for $\boldsymbol{k} \in \boldsymbol{Z}^r \setminus \{\boldsymbol{0}\}$, and $|\lambda_{i'}^+ - \lambda_{j'}^+| \geqslant \delta$ when $\lambda_{i'}^+$ are the eigenvalues of $\boldsymbol{A}_{i_1}^1$ and $\lambda_{j'}^+$ are the eigenvalues of $\boldsymbol{A}_{i_2}^1$ with $i_1 \neq i_2$, then by comparing the coefficients of Eq. (7), we obtain that $p_{ij}^0 = 0$, if i, j are the subscripts of the elements of corresponding blocks $\boldsymbol{R}_i$ in $\boldsymbol{R} = (r_{ij})_{1 \leqslant i, j \leqslant n}$ with $1 \leqslant i \leqslant l$; or else,

$$p_{ij}^k = \frac{\tilde{q}_{ij}^k}{\langle \boldsymbol{k}, \boldsymbol{\omega} \rangle \sqrt{-1} + \lambda_i^+ - \lambda_j^+}$$

Since $\tilde{\boldsymbol{Q}}$ is analytic on $D_{\rho - s}$, we have $|\tilde{\boldsymbol{Q}}_k| \leqslant \| \tilde{\boldsymbol{Q}} \|_{D_{\rho-s}} e^{-|\boldsymbol{k}|(\rho - s)}$. Hence, we can solve Eq. (7) for $\boldsymbol{P}$ in a smaller domain $D_{\rho - 2s}$ and

$$\| \boldsymbol{P} \|_{D_{\rho-2s}} \leqslant \sum_{k \in Z^r} |\boldsymbol{P}_k| e^{|\boldsymbol{k}|(\rho - 2s)} \leqslant \frac{c}{\alpha_1 s^v} \| \tilde{\boldsymbol{Q}} \|$$

By Assumption 2, we obtain that the eigenvalues of $\boldsymbol{R}(\varepsilon)|_{\varepsilon = 0}$ are different. Since $\boldsymbol{R}(\varepsilon) = \boldsymbol{R}(0) + O(\varepsilon)$ and the eigenvalues of $\boldsymbol{R}(0)$ are different, it follows that $\| \boldsymbol{S}^{-1} \|, \| \boldsymbol{S} \| \leqslant c'$. Then we have

$$\| \boldsymbol{P} \|_{D_{\rho_1}} \leqslant \frac{c''}{\alpha_1 s^v} \| \hat{\boldsymbol{Q}} \|_{D_{\rho - s}} \tag{8}$$

where c, c', and c'' are positive constants; $v = 3\tau + r$ and $0 < s \leqslant \rho/4$.

Using Eq. (6), we obtain

$$\dot{z}_1 = (\boldsymbol{A}^+ + \varepsilon^2 \boldsymbol{Q}_+) z_1 + \varepsilon^2 \boldsymbol{g}_+(t) + \boldsymbol{h}_+(z_1, t) \qquad z_1 \in B_{a_1}(0)$$

where $a_1 = \frac{a - \| \underline{\boldsymbol{x}} \|_{D_{\rho-s}}}{\| \boldsymbol{S} \| (\boldsymbol{I} + \varepsilon \| \boldsymbol{P}(t) \|_{D_{\rho_1}})}$, $\boldsymbol{Q}_+ = (\boldsymbol{I} + \varepsilon \boldsymbol{P})^{-1} \tilde{\boldsymbol{Q}} \boldsymbol{P}$, $\boldsymbol{g}_+ = (\boldsymbol{I} + \varepsilon \boldsymbol{P})^{-1} \tilde{\boldsymbol{g}}(t)$, $\boldsymbol{h}_+ = (\boldsymbol{I} + \varepsilon \boldsymbol{P})^{-1} \tilde{\boldsymbol{h}}((\boldsymbol{I} + \varepsilon \boldsymbol{P}) z_1, t)$. If $\| \varepsilon \boldsymbol{P} \| \leqslant \frac{1}{2}$, we have $\| (\boldsymbol{I} + \varepsilon \boldsymbol{P})^{-1} \|_{D_{\rho_1}} \leqslant 2$. By Eq. (8), we have

$$\| \boldsymbol{Q}_+ \|_{D_{\rho_1}} \leqslant \frac{c_0}{\alpha_1 s^v} \| \hat{\boldsymbol{Q}} \|_{D_{\rho-s}}^2 \leqslant \frac{c_0}{\alpha_1 s^v} (\| \boldsymbol{Q} \|_{D_\rho} + KL_1 \| \boldsymbol{g} \|_{D_\rho})^2 \tag{9}$$

This completes the first step.

4 Proof of Theorem 1

4.1 KAM-step

After the first step, $\boldsymbol{A}^+$ has n different eigenvalues and $\varepsilon^2 \boldsymbol{Q}_+$ and $\varepsilon^2 \boldsymbol{g}_+$ are smaller perturbations.

The next step is standard. Let us consider the equation:

$$\dot{\boldsymbol{x}}_m = (\boldsymbol{A}_m + \varepsilon^{2^m} \boldsymbol{Q}_m) \boldsymbol{x}_m + \varepsilon^{2^m} \boldsymbol{g}_m(t) + \boldsymbol{h}_m(\boldsymbol{x}_m, t) \qquad m \geqslant 1 \tag{10}$$

where $\boldsymbol{x}_m \in B_{a_m}(0)$, and λ_i^m are eigenvalues of $\boldsymbol{A}_m$ with $|\lambda_i^m - \lambda_j^m| \geqslant \delta\varepsilon$, $\forall\, i \neq j$.

If $|\langle \boldsymbol{k}, \boldsymbol{\omega} \rangle \sqrt{-1} + \lambda_i^m| \geqslant \frac{\alpha_m}{|\boldsymbol{k}|^{3\tau}}$ for $\boldsymbol{k} \in \boldsymbol{Z}^r \setminus \{\boldsymbol{0}\}$, under the change of variables $\boldsymbol{x}_m = \underline{\boldsymbol{x}_m} + \boldsymbol{y}$, where $\underline{\boldsymbol{x}_m}$ satisfies $\underline{\dot{\boldsymbol{x}}_m} = \boldsymbol{A}_m \underline{\boldsymbol{x}_m} + \varepsilon^{2^m} \boldsymbol{g}_m$ on $D_{\rho_m - s_m}$, Eq. (10) is changed into

$$\dot{\boldsymbol{y}} = (\boldsymbol{A}_m + \varepsilon^{2^m} \hat{\boldsymbol{Q}}_m) \boldsymbol{y} + \varepsilon^{2^{m+1}} \hat{\boldsymbol{g}}_m(t) + \hat{\boldsymbol{h}}_m(\boldsymbol{y}, t) \tag{11}$$

where $\hat{\boldsymbol{Q}}_m = \boldsymbol{Q}_m + \frac{1}{\varepsilon^{2^m}} D_{\boldsymbol{x}} \boldsymbol{h}_m(\underline{\boldsymbol{x}_m}, t)$, $\hat{\boldsymbol{g}}_m = \frac{1}{\varepsilon^{2^{m+1}}} \boldsymbol{h}_m(\underline{\boldsymbol{x}_m}, t) + \frac{1}{\varepsilon^{2^m}} \boldsymbol{Q}_m \underline{\boldsymbol{x}_m}$, $\hat{\boldsymbol{h}}_m(\boldsymbol{y}, t) = \boldsymbol{h}_m(\underline{\boldsymbol{x}_m} + \boldsymbol{y}, t) - \boldsymbol{h}_m(\underline{\boldsymbol{x}_m}, t) - D_{\boldsymbol{x}} \boldsymbol{h}_m(\underline{\boldsymbol{x}_m}, t) \boldsymbol{y}$. Let us define the average of $\hat{\boldsymbol{Q}}_m$ by $\bar{\hat{\boldsymbol{Q}}}_m = (\bar{\hat{q}}_{ij}^m)_{1 \leqslant i,j \leqslant n}$. Denote $\bar{\hat{\boldsymbol{Q}}}_m = (\boldsymbol{R}_{ij}^m)_{1 \leqslant i,j \leqslant l}$, where $\boldsymbol{R}_{kk}^m$ is the r_k order matrix with $1 \leqslant k \leqslant l$. Let us define $\boldsymbol{R}_i^m = \boldsymbol{R}_{ii}^m$ and $\boldsymbol{R}_m = \mathrm{diag}(\boldsymbol{R}_1^m, \boldsymbol{R}_2^m, \ldots, \boldsymbol{R}_l^m)$.

Eq. (11) is changed into

$$\dot{\boldsymbol{y}} = (\boldsymbol{A}_{m+1} + \varepsilon^{2^m} \tilde{\boldsymbol{Q}}_m) \boldsymbol{y} + \varepsilon^{2^{m+1}} \tilde{\boldsymbol{g}}_m + \tilde{\boldsymbol{h}}_m(\boldsymbol{y}, t) \tag{12}$$

where $\boldsymbol{A}_{m+1} = \boldsymbol{A}_m + \varepsilon^{2^m} \boldsymbol{R}_m$, $\hat{\boldsymbol{Q}}_m - \boldsymbol{R}_m = \tilde{\boldsymbol{Q}}_m$, $\hat{\boldsymbol{h}}_m(\boldsymbol{y}, t) = \tilde{\boldsymbol{h}}_m(\boldsymbol{y}, t)$. Denote $\boldsymbol{A}_{m+1} = \mathrm{diag}(\boldsymbol{A}_1^{m+1}, \boldsymbol{A}_2^{m+1}, \ldots, \boldsymbol{A}_l^{m+1})$, where $\boldsymbol{A}_k^{m+1}$ is the r_k order matrix for $1 \leqslant k \leqslant l$.

We make the change of variables $\boldsymbol{y} = (\boldsymbol{I} + \varepsilon^{2^m} \boldsymbol{P}_m) \boldsymbol{x}_{m+1}$ into Eq. (12) to obtain

$$\begin{aligned} \dot{\boldsymbol{x}}_{m+1} = {} & (\boldsymbol{A}_{m+1} + \varepsilon^{2^{m+1}} \boldsymbol{Q}_{m+1}) \boldsymbol{x}_{m+1} + \varepsilon^{2^{m+1}} \boldsymbol{g}_{m+1}(t) + \\ & \boldsymbol{h}_{m+1}(\boldsymbol{x}_{m+1}, t) \qquad \boldsymbol{x}_{m+1} \in B_{a_{m+1}}(0) \end{aligned}$$

where $\boldsymbol{Q}_{m+1} = (\boldsymbol{I} + \varepsilon^{2^m} \boldsymbol{P}_m)^{-1} \tilde{\boldsymbol{Q}}_m \boldsymbol{P}_m$, $\boldsymbol{g}_{m+1} = (\boldsymbol{I} + \varepsilon^{2^m} \boldsymbol{P}_m)^{-1} \tilde{\boldsymbol{g}}_m$, $\boldsymbol{h}_{m+1} = (\boldsymbol{I} + \varepsilon^{2^m} \boldsymbol{P}_m)^{-1} \tilde{\boldsymbol{h}}_m((\boldsymbol{I} + \varepsilon^{2^m} \boldsymbol{P}_m) \boldsymbol{x}_{m+1}, t)$. Let the eigenvalues of $\boldsymbol{A}_{m+1}$ be $\lambda_1^{m+1}, \lambda_2^{m+1}, \ldots, \lambda_n^{m+1}$. The same as the first step, we now need solve the equations $\dot{\boldsymbol{P}}_m = \boldsymbol{A}_{m+1} \boldsymbol{P}_m - \boldsymbol{P}_m \boldsymbol{A}_{m+1} + \tilde{\boldsymbol{Q}}_m$, where $\tilde{\boldsymbol{Q}}_m = \hat{\boldsymbol{Q}}_m - \boldsymbol{R}_m$. If $|\langle \boldsymbol{k}, \boldsymbol{\omega} \rangle \sqrt{-1} + \lambda_i^{m+1} - \lambda_j^{m+1}| \geqslant \frac{\alpha_{m+1}}{|\boldsymbol{k}|^{3\tau}}$ for all $\boldsymbol{k} \in \boldsymbol{Z}^r \setminus \{\boldsymbol{0}\}$, and $|\lambda_{i'}^{m+1} - \lambda_{j'}^{m+1}| \geqslant \delta$ when $\lambda_{i'}^{m+1}$ are the eigenvalues of $\boldsymbol{A}_{i_1}^{m+1}$ and $\lambda_{j'}^{m+1}$ are the eigenvalues of $\boldsymbol{A}_{i_2}^{m+1}$ with $i_1 \neq i_2$, then we can solve the above equations for $\boldsymbol{P}_m$ in a smaller domain $D_{\rho_m - 2s_m}$ and

$$\| \boldsymbol{P}_m \|_{D_{\rho_m - 2s_m}} \leqslant \frac{c}{\alpha_{m+1} s_m^v} \| \hat{\boldsymbol{Q}}_m \|_{D_{\rho_m - s_m}} \tag{13}$$

If $\| \varepsilon^{2^m} \boldsymbol{P}_m \| \leqslant \frac{1}{2}$, we have $\| (\boldsymbol{I} + \varepsilon^{2^m} \boldsymbol{P}_m)^{-1} \| \leqslant 2$. By (13), we have

$$\begin{aligned} \| \boldsymbol{Q}_{m+1} \|_{D_{\rho_{m+1}}} & \leqslant \frac{c_1}{\alpha_{m+1} s_m^v} \| \hat{\boldsymbol{Q}}_m \|_{D_{\rho_m - s_m}}^2 \leqslant \\ & \frac{c_1}{\alpha_{m+1} s_m^v} (\| \boldsymbol{Q}_m \|_{D_{\rho_m}} + K_m L_{1,m} \| \boldsymbol{g}_m \|_{D_{\rho_m}})^2 \end{aligned} \tag{14}$$

4.2 Iteration

Now we prove the convergence of the iteration as $m\to\infty$. In the m-th step, we choose

$$\alpha_m=\frac{\alpha_0}{(m+1)^2},\ \rho_m=\rho-2(s_0+s_1+\ldots+s_{m-1})$$

$$s_m=\frac{\rho}{4^{m+1}},\ D_m=D_{\rho_m},\ a_{m+1}=\frac{a_m-\|\underline{\boldsymbol{x}_m}\|_{D_{\rho_m-s_m}}}{1+\varepsilon^{2^m}\|\boldsymbol{P}_m\|_{D_{m+1}}}$$

Then for all $m\in\mathbf{N}$, there exists $b>0$, such that

$$\|\boldsymbol{g}_{m+1}\|_{D_{m+1}}\leqslant\|(\boldsymbol{I}+\varepsilon^{2^m}\boldsymbol{P}_m)^{-1}\|\|\tilde{\boldsymbol{g}}_m\|\leqslant b\frac{K_mL_{1,m}^2}{2}\|\boldsymbol{g}_m\|_{D_m}^2+bL_{1,m}\|\boldsymbol{Q}_m\|_{D_m}\|\boldsymbol{g}_m\|_{D_m}$$

where b is a constant. By (14), we obtain

$$\|\boldsymbol{Q}_{m+1}\|_{D_{m+1}}\leqslant\frac{c_1}{\alpha_{m+1}s_m^v}(\|\boldsymbol{Q}_m\|_{D_m}+K_mL_{1,m}\|\boldsymbol{g}_m\|_{D_m})^2$$

For simplicity, let us define $\nu_m=\|\boldsymbol{Q}_m\|_{D_m}$ and $\beta_m=\|\boldsymbol{g}_m\|_{D_m}$. Define $\eta_m=\max\{\beta_m,\nu_m\}$. Hence, $\eta_{m+1}\leqslant\frac{c_1}{\alpha_{m+1}s_m^v}\times2\times\left(\frac{9}{2}\right)^{2m}L_{1,m}^2\eta_m^2$. We have $\eta_m\leqslant M^{2^m}$, where M is a positive constant. Hence, $\|\boldsymbol{Q}_m\|_{D_m}\leqslant M^{2^m}$ and $\|\boldsymbol{g}_m\|_{D_m}\leqslant M^{2^m}$. If $0<M\varepsilon<1$, we have

$$\lim_{m\to\infty}\varepsilon^{2^m}\boldsymbol{Q}_m=0,\quad \lim_{m\to\infty}\varepsilon^{2^m}\boldsymbol{g}_m=0\tag{15}$$

Next we consider $\|\boldsymbol{P}_m\|$. By (13), we have

$$\|\boldsymbol{P}_m\|_{D_{m+1}}\leqslant\frac{c_1}{\alpha_{m+1}s_m^v}(\|\boldsymbol{Q}_m\|_{D_m}+K_mL_{1,m}\|\boldsymbol{g}_m\|_{D_m})\leqslant M^{2^m}$$

If $\varepsilon<\varepsilon_1=M^{-1}$, we have

$$\lim_{m\to\infty}\varepsilon^{2^m}\boldsymbol{P}_m=0\tag{16}$$

Besides, $\|\underline{\boldsymbol{x}_m}\|_{D_{\rho_m-s_m}}\leqslant\varepsilon^{2^m}L_{1,m}\|\boldsymbol{g}_m\|_{D_m}\leqslant\varepsilon^{2^m}M^{2^m}$. If $\varepsilon<\varepsilon_1=M^{-1}$, we have

$$\lim_{m\to\infty}\|\underline{\boldsymbol{x}_m}\|_{D_{\rho_m-s_m}}=0\tag{17}$$

We have $a_{m+1}=\frac{a_m-\|\underline{\boldsymbol{x}_m}\|_{D_{\rho_m-s_m}}}{1+\varepsilon^{2^m}\|\boldsymbol{P}_m\|_{D_{m+1}}}$ and we define

$$b_m=\frac{1}{1+\varepsilon^{2^m}\|\boldsymbol{P}_m\|_{D_{m+1}}},\ c_m=\frac{\|\underline{\boldsymbol{x}_m}\|_{D_{\rho_m-s_m}}}{1+\varepsilon^{2^m}\|\boldsymbol{P}_m\|_{D_{m+1}}}$$

It is easy to prove that $\prod_{m=0}^{\infty}b_m$ and $\sum_{m=0}^{\infty}c_m$ are convergent. So we have $a_\infty\geqslant ba_0-c$, which is positive when ε is small enough.

By $\|D_{yy}\boldsymbol{h}_m\|\leqslant K_m$, we obtain

$$K_{m+1}\leqslant\frac{(1+\varepsilon^{2^m}\|\boldsymbol{P}_m\|_{D_{m+1}})^2}{1-\varepsilon^{2^m}\|\boldsymbol{P}_m\|_{D_{m+1}}}K_m$$

So

$$K_{m+1}\leqslant(1+2\varepsilon^{2^m}\|\boldsymbol{P}_m\|_{D_{m+1}})^3K_m$$

Since $K_m\leqslant\left(\frac{9}{2}\right)^mK$, we obtain that K_m is convergent as $m\to\infty$. Let

$$\lim_{m\to\infty}K_m=K_\infty\tag{18}$$

Hence, $\boldsymbol{h}_m(\boldsymbol{x}_m,t)$ converges as $m\to\infty$.

Moreover,

$$\|\boldsymbol{A}_{m+1}-\boldsymbol{A}_m\|\leqslant\varepsilon^{2^m}\|\boldsymbol{R}_m\|_{D_m}\leqslant\varepsilon^{2^m}(\|\boldsymbol{Q}_m\|_{D_m}+K_mL_{1,m}\|\boldsymbol{g}_m\|_{D_m})$$

We can find a suitable constant $M>0$, such that $\|\boldsymbol{Q}_m\|_{D_m}+K_mL_{1,m}\|\boldsymbol{g}_m\|_{D_m}\leqslant M^{2^m}$. If $\varepsilon<\varepsilon_1=M^{-1}$, we obtain that $\boldsymbol{A}_m$ is convergent as $m\to\infty$. Let

$$\lim_{m\to\infty}\boldsymbol{A}_m=\boldsymbol{A}^*\tag{19}$$

In conclusion, $\lim_{m\to\infty}\boldsymbol{A}_m=\boldsymbol{A}^*$, $\lim_{m\to\infty}\varepsilon^{2^m}\boldsymbol{P}_m=0$, $\lim_{m\to\infty}\|\underline{\boldsymbol{x}_m}\|_{D_{\rho_m-s_m}}=0$. Let us define $D_*=\bigcap_{m=0}^{\infty}D_m$. By Eqs. (16) and (17), we know the existence of the change $\boldsymbol{x}=\boldsymbol{\psi}(t)\boldsymbol{y}+\boldsymbol{\varphi}(t)$. Under this change, the equation $\dot{\boldsymbol{x}}=(\boldsymbol{A}^++\varepsilon^2\boldsymbol{Q}_+)\boldsymbol{x}+\varepsilon^2\boldsymbol{g}_+(t)+\boldsymbol{h}_+(\boldsymbol{x},t)$ becomes $\dot{\boldsymbol{y}}=\boldsymbol{A}^*\boldsymbol{y}+\boldsymbol{h}_\infty(\boldsymbol{y},t)$.

4.3 Estimate of measure

Now we prove when ε_0 is small enough, the non-resonance conditions $|\langle\boldsymbol{k},\boldsymbol{\omega}\rangle\sqrt{-1}+\lambda_i^m|\geqslant\frac{\alpha_m}{|\boldsymbol{k}|^{3\tau}}$ and $|\langle\boldsymbol{k},\boldsymbol{\omega}\rangle\sqrt{-1}+\lambda_i^m-\lambda_j^m|\geqslant\frac{\alpha_m}{|\boldsymbol{k}|^{3\tau}}$ hold for most $\varepsilon\in(0,\varepsilon_0)$, where $i,j=1,2,\ldots,n$; $m=0,1,2,\ldots$; and $\boldsymbol{k}\in\mathbf{Z}^r\backslash\{\mathbf{0}\}$. Let $l(f(\varepsilon))$ be the Lipschitz constant from below of $f(\varepsilon)$ and $L(f(\varepsilon))$ be the Lipschitz constant from above of $f(\varepsilon)$. Without loss of generality, assume that $\lambda_i^m-\lambda_j^m$ are pure imaginary numbers. Then we consider

$$|\langle\boldsymbol{k},\boldsymbol{\omega}\rangle\sqrt{-1}+\lambda_i^m-\lambda_j^m|\geqslant\frac{\alpha_m}{|\boldsymbol{k}|^{3\tau}}\tag{20}$$

where $\lambda_i^m-\lambda_j^m$ satisfies $l(\lambda_i^m(\varepsilon)-\lambda_j^m(\varepsilon))\geqslant\delta>0$ for $i\neq j$.

By the condition (Assumption 1) of the theorem, (20) holds for $m=0$.

If $i=j$, by (3), we obtain that (20) holds.

Let $f(\varepsilon)=\langle\boldsymbol{k},\boldsymbol{\omega}\rangle\sqrt{-1}+\lambda_i^m-\lambda_j^m$ for $i\neq j$ and

$$O_{ijm}^k=\left\{\varepsilon\in(0,\varepsilon_0)\,\middle|\,|f(\varepsilon)|<\frac{\alpha_m}{|\boldsymbol{k}|^{3\tau}}\right\}$$

where ε_0 is small enough such that Eq. (10) is convergent for $\varepsilon\in(0,\varepsilon_0)$, and

$$l(\lambda_i^m(\varepsilon)-\lambda_j^m(\varepsilon))\geqslant\delta\tag{21}$$

Since $|f(\varepsilon)| \geqslant |\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle \sqrt{-1} + \lambda_i^0 - \lambda_j^0| - 2M\varepsilon \geqslant \frac{\alpha_0}{|\boldsymbol{k}|^\tau} - 2M\varepsilon_0$, we have that if $\frac{1}{|\boldsymbol{k}|^\tau} > \frac{4M\varepsilon_0}{\alpha_0}$, then $|f(\varepsilon)| \geqslant \frac{\alpha_0}{2|\boldsymbol{k}|^\tau} > \frac{\alpha_m}{|\boldsymbol{k}|^{3\tau}}$, and $O_{ijm}^k = \varnothing$. However, if $\frac{1}{|\boldsymbol{k}|^\tau} < \frac{4M\varepsilon_0}{\alpha_0}$, by (21), we have meas $(O_{ijm}^k) < \frac{2\alpha_m}{\delta|\boldsymbol{k}|^{3\tau}}$. Then,

$$\operatorname{meas}\Big(\bigcup_{i\neq j}\bigcup_{\mathbf{0}\neq \boldsymbol{k}\in \mathbf{Z}^r} O_{ijm}^k\Big) \leqslant \frac{2n^2\alpha_m}{\delta}\sum_{\frac{1}{|\boldsymbol{k}|^\tau}<4M\varepsilon_0/\alpha_0}\frac{1}{|\boldsymbol{k}|^{3\tau}} \leqslant c\varepsilon_0^2\,\alpha_m\sum_{\mathbf{0}\neq\boldsymbol{k}\in\mathbf{Z}^r}\frac{1}{|\boldsymbol{k}|^\tau} \leqslant \frac{c\varepsilon_0^2}{(m+1)^2}$$

where c is a constant depending on α_0.

Let $E_m = \Big\{\varepsilon \in (0, \varepsilon_0) \,\|\, \langle \boldsymbol{k}, \boldsymbol{\omega}\rangle \sqrt{-1} + \lambda_i^m - \lambda_j^m| \geqslant \frac{\alpha_m}{|\boldsymbol{k}|^{3\tau}}, \mathbf{0}\neq \boldsymbol{k}\in \mathbf{Z}^r, i\neq j\Big\}$. Then $(0, \varepsilon_0) - E_m = \bigcup_{i\neq j}\bigcup_{\mathbf{0}\neq\boldsymbol{k}\in\mathbf{Z}^r} O_{ijm}^k$. Thus meas $((0,\varepsilon_0) - E_m) \leqslant \frac{c\varepsilon_0^2}{(m+1)^2}$. Let $E^* = \bigcap_{m=0}^{\infty} E_m$. Hence, meas $((0,\varepsilon_0) - E^*) \leqslant c\varepsilon_0^2$. Then it follows that $\lim_{\varepsilon_0\to 0}\frac{\operatorname{meas}((0,\varepsilon_0)-E^*)}{\varepsilon_0} = 0$. So if ε_0 is sufficiently small, E^* is a nonempty subset of $(0, \varepsilon_0)$.

Similar to the above discussion, when ε_0 is small enough,

$$|\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle \sqrt{-1} + \lambda_i^m| \geqslant \frac{\alpha_m}{|\boldsymbol{k}|^{3\tau}}$$

holds for most $\varepsilon \in (0, \varepsilon_0)$.

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一类具有小扰动参数的非线性拟周期系统在平衡点附近的可约化性

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摘要:考虑一类有重特征值的非线性拟周期系统在小扰动下平衡点附近的可约化性问题,也就是研究 $\dot{\boldsymbol{x}} = (\boldsymbol{A} + \varepsilon\boldsymbol{Q}(t))\boldsymbol{x} + \varepsilon\boldsymbol{g}(t) + \boldsymbol{h}(\boldsymbol{x}, t)$, 其中 $\boldsymbol{A}$ 可以是具有重特征值的常数矩阵;$\boldsymbol{h} = O(\boldsymbol{x}^2)(\boldsymbol{x}\to 0)$;$\boldsymbol{h}(\boldsymbol{x}, t)$, $\boldsymbol{Q}(t)$和 $\boldsymbol{g}(t)$关于 t 是解析拟周期的,且有相同的频率. 在某些非共振条件及非退化条件下,对充分小的大多数 ε,通过仿线性拟周期变换,系统可约化为具有平衡点的非线性拟周期系统.

关键词:拟周期;可约化性;非共振条件;非退化条件;KAM 迭代

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