Journal of Southeast University (English Edition) Vol. 28, No. 3, pp. 372 – 374 Sept. 2012 ISSN 1003—7985

Morita context of weak Hopf group coalgebras

Guo Shuangjian Dong Lihong Wang Shuanhong

(Department of Mathematics, Southeast University, Nanjing 211189, China)

Abstract: The concept of weak Hopf group coalgebras is a natural generalization of the notions of both weak Hopf algebras (quantum groupoids) and Hopf group coalgebras. Let π be a group. The Morita context is considered in the sense of weak Hopf π-coalgebras. Let H be a finite type weak Hopf π-coalgebra, and A a weak right π-H-comodule algebra. It is constructed that a Morita context connects $A\#H^*$ which is a weak smash product and the ring of coinvariants A^{coH}. This result is the generalization of that of Wang's in the paper "Morita contexts, π-Galois extensions for Hopf π-coalgebras" in 2006. Furthermore, the result is important for constructing weak π-Galois extensions.

Key words: weak Hopf π-coalgebra; Morita context; weak π-H-comodule algebra; weak smash product

doi: 10.3969/j.issn.1003 – 7985.2012.03.021

Weak Hopf algebras given in Ref. [1] are the generalizations of ordinary Hopf algebras in the following sense: the defining axioms are the same, but the multiplicativity of the counit and the comultiplicativity of the unit are replaced by weaker axioms. The easiest example of weak Hopf algebras is the groupoid algebra. A survey of weak Hopf algebras and their applications can be found in Ref. [2]. It has turned out that many results of classical Hopf algebra theory can be generalized to weak Hopf algebras, for example, the Maschke-type theorem and the Morita context over weak Hopf algebras were given by Zhang[3].

Turaev[4] introduced the notion of a modular crossed π-category. Examples of a π-category can be constructed from the so-called Hopf π-coalgebras in which one of the key points is the notion of a crossed Hopf π-coalgebra (called a Turaev π-coalgebra), π-coalgebras and Hopf π-coalgebras generalized usual coalgebras and Hopf algebras, in the sense that we recover these notions in the situation where π is the trivial group. Virelizier[5] started an algebraic study of this topic, which was further developed in Ref. [6]. As a generalization of a weak Hopf algebra, Van Daele and Wang[7] introduced the notion of a weak Hopf group-coalgebra. Motivated by the above ideas, in this paper, we construct a Morita context connecting a weak smash product $A\#H^*$ and A^{coH} for a finite type weak Hopf π-coalgebra H, where A is a weak right π-H-comodule algebra.

Throughout this paper, we let π be a group with unit 1 and k a field. All algebras are supposed to be over k, associative and unitary. All maps are k-linear and $\otimes$ means $\otimes_k$ unless otherwise specified, etc.

1 Preliminaries

In this section, we recall some basic definitions and results about weak Hopf group-coalgebras introduced by Van Daele and Wang[7].

Definition 1 A π-coalgebra is a family of k-spaces $C=\{C_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps $\Delta_{\alpha,\beta}$: $C_{\alpha\beta}\rightarrow C_\alpha\otimes C_\beta$ and a k-linear map ε: $C_1\rightarrow k$, such that the following conditions are satisfied:

$$(\Delta_{\alpha,\beta}\otimes id_{C_\gamma})\Delta_{\alpha\beta,\gamma}=(id_{C_\alpha}\otimes\Delta_{\beta,\gamma})\Delta_{\alpha\beta\gamma}\quad\forall\,\alpha,\beta,\gamma\in\pi$$
$$(id_{C_\alpha}\otimes\varepsilon)\Delta_{\alpha,1}=id_{C_\alpha}=(\varepsilon\otimes id_{C_\alpha})\Delta_{1,\alpha}\quad\forall\,\alpha\in\pi$$

We use Sweedler's notation for a comultiplication in the following way: for any $\alpha,\beta\in\pi$ and $c\in C_{\alpha\beta}$, we write $\Delta_{\alpha,\beta}(c)=c_{(1,\alpha)}\otimes c_{(2,\beta)}$.

Definition 2 A weak semi-Hopf π-coalgebra $H=\{H_\alpha,\mu_\alpha,1_\alpha,\Delta,\varepsilon\}_{\alpha\in\pi}$ is a family of algebras $\{H_\alpha,\mu_\alpha,1_\alpha\}_{\alpha\in\pi}$ and at the same time a π-coalgebra $\{H_\alpha,\Delta=\{\Delta_{\alpha,\beta}\},\varepsilon\}_{\alpha,\beta\in\pi}$ is a homomorphism of algebras such that the following conditions are satisfied:

$$(\Delta_{\alpha,\beta}\otimes id_{H_\gamma})\Delta_{\alpha\beta,\gamma}(1_{\alpha\beta\gamma})=(\Delta_{\alpha,\beta}(1_{\alpha\beta})\otimes 1_\gamma)(1_\alpha\otimes\Delta_{\beta,\gamma}(1_{\beta\gamma}))$$
$$(\Delta_{\alpha,\beta}\otimes id_{H_\gamma})\Delta_{\alpha\beta,\gamma}(1_{\alpha\beta\gamma})=(1_\alpha\otimes\Delta_{\beta,\gamma}(1_{\beta\gamma}))(\Delta_{\alpha,\beta}(1_{\alpha\beta})\otimes 1_\gamma)$$

for all $\alpha,\beta,\gamma\in\pi$, and the counit ε: $H_1\rightarrow k$ is a k-linear map satisfying the identity

$$\varepsilon(gxh)=\varepsilon(gx_{(2,1)})\varepsilon(x_{(1,1)}h)=\varepsilon(gx_{(1,1)})\varepsilon(x_{(2,1)}h)$$

for all $g,h,x\in H_1$.

Furthermore, a weak Hopf π-coalgebra $H=\{H_\alpha,\mu_\alpha,1_\alpha,\Delta,\varepsilon\}_{\alpha\in\pi}$ is a weak semi-Hopf π-coalgebra endowed with a family of k-linear maps

$$S=\{S_\alpha: H_\alpha\rightarrow H_{\alpha^{-1}}\}_{\alpha\in\pi}$$

Received 2011-08-22.

Biographies: Guo Shuangjian (1981—), male, graduate; Wang Shuanhong (corresponding author), male, doctor, professor, shuanhwang2002@yahoo.com.

Foundation item: The Scientific Research Innovation Project for College Graduates in Jiangsu Province (No. CXLX_0094).

Citation: Guo Shuangjian, Dong Lihong, Wang Shuanhong. Morita context of weak Hopf group coalgebras[J]. Journal of Southeast University (English Edition), 2012, 28(3): 372 – 374. [doi: 10.3969/j.issn.1003 – 7985.2012.03.021]

(called an antipode) such that the following conditions are satisfied:

$$\mu_\alpha(S_{\alpha^{-1}}\otimes id_{H_\alpha})\Delta_{\alpha^{-1},\alpha}(h)=1_{(1,\alpha)}\varepsilon(h1_{(2,1)})$$
$$\mu_\alpha(id_{H_\alpha}\otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}(h)=1_{(2,\alpha)}\varepsilon(1_{(1,1)}h)$$
$$S_\alpha(g_{(1,\alpha)})g_{(2,\alpha^{-1})}S_\alpha(g_{(3,\alpha)})=S_\alpha(g)$$

for all $h\in H_1$, $g\in H_\alpha$.

Now, let H be a weak Hopf π-coalgebra. Define a family of linear maps

$$\varepsilon^{t}=\{\varepsilon^{t}_\alpha: H_1\to H_\alpha\}_{\alpha\in\pi}\qquad \varepsilon^{t}_\alpha(h)=\varepsilon(1_{(1,1)}h)1_{(2,\alpha)}$$

and

$$\varepsilon^{s}=\{\varepsilon^{s}_\alpha: H_1\to H_\alpha\}_{\alpha\in\pi}\qquad \varepsilon^{s}_\alpha(h)=\varepsilon(h1_{(2,1)})1_{(1,\alpha)}$$

where ε^{t}, ε^{s} are called the π-target and π-source counital maps, and the notations

$$H^{t}:=\varepsilon^{t}(h)=\{H^{t}_\alpha=\varepsilon(H_1)\}_{\alpha\in\pi}$$
$$H^{s}:=\varepsilon^{s}(h)=\{H^{s}_\alpha=\varepsilon(H_1)\}_{\alpha\in\pi}$$

for their images.

Definition 3 Let H be a weak Hopf π-coalgebra and A an algebra. A is called a weak right π-H-comodule algebra if the following conditions hold: for all $a, b\in A$ and $\alpha,\beta\in\pi$,

1) $a1_{[0]}\otimes 1_{[1,\alpha]}=a_{[0]}\otimes\varepsilon^{t}_\alpha(a_{[1,1]})$ for all $a\in A$ and $\alpha\in\pi$;

2) $(\rho^A_\alpha\otimes id_{H_\beta})\rho^A_\beta=(id_A\otimes\Delta_{\alpha,\beta})\rho^A_{\alpha,\beta}$, for all $\alpha,\beta\in\pi$;

3) $\rho^A_\alpha(ab)=\rho^A_\alpha(a)\rho^A_\alpha(b)$, for all $a, b\in A$ and $\alpha\in\pi$;

where the coaction is denoted by $\rho^A_\alpha(a)=a_{[0]}\otimes a_{[1,\alpha]}$ for all $a\in A$ and $\alpha\in\pi$.

Let A be a weak right π-H-comodule algebra. We define

$$A^{\mathrm{coH}}=\{a\in A\mid \rho^A_\alpha(a)=a1_{[0]}\otimes 1_{[1,\alpha]}\}$$

which is called a coinvariant set of A. It is easy to see that A^{coH} is an H-subalgebra.

2 Morita Context

In what follows, let H be a finite type weak Hopf π-coalgebra with an antipode S and A a weak right π-H-comodule algebra. Then A is a weak left $H^*=\oplus_{\alpha\in\pi}H^*_\alpha$ module algebra with the structure map given by $f\to a=\oplus_{\alpha\in\pi}\langle f, a_{[1,\alpha]}\rangle a_{[0]}$ for all $a\in A$, $f\in H^*_\alpha$. So we can define a weak smash product on the k-vector space $A\otimes_{H^{*t}_\alpha}H^*_\alpha$, where H^*_α is a left H^{*t}_α-module via its multiplication and A is a right H^{*t}_α-module via $a\cdot f=a(f\cdot 1_A)$. Its multiplication is given by the following formula:

$$(a\#f)(b\#g)=ab_{[0]}\#(f\leftarrow b_{[1,\alpha]})g$$

for all $a, b\in A$, $f\in H^*_\alpha$, $g\in H^*_\beta$, where the module structure $\leftarrow$ is given by $(f\leftarrow h)(k)=f(hk)$ for all $h, k\in H_\alpha$, $f\in H^*_\alpha$.

As described above, we can easily obtain the following lemmas.

Lemma 1 With the notation as above, $A\#H^*_\alpha$ is an associative algebra with the unit $1_A\#1_{H^*_\alpha}$.

Lemma 2 With the notation as above, A is $A\#H^*$-A^{coH}-bimodule and A^{coH}-$A\#H^*$-bimodule with the module structures given by

$$(a\#f)\cdot b=\underset{\alpha\in\pi}{\oplus}ab_{[0]}\langle f, b_{[1,\alpha]}\rangle$$
$$b\cdot(a\#f)=\underset{\alpha\in\pi}{\oplus}b_{[0]}a_{[0]}\langle f, S^{-1}_{\alpha^{-1}}(b_{[1,\alpha^{-1}]}a_{[1,\alpha^{-1}]})\rangle$$

and usual A^{coH}-module structure on A.

Now we can obtain the main result of this paper.

Proposition 1 Let H be a finite type weak Hopf π-coalgebra with an antipode S and A a weak right π-H-comodule algebra, if there exists a nonzero left π-integral $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$. Then we have a Morita context $(A\#H^*, A^{\mathrm{coH}}, \tau, \mu)$, where the connecting maps are given by

$$\tau: A\otimes_{A^{\mathrm{coH}}}A\to A\#H^*_\alpha,\ \ \tau(a\otimes b)=\underset{\alpha\in\pi}{\oplus}ab_{[0]}\#\lambda_\alpha\leftarrow b_{[1,\alpha]}$$
$$\mu: A\otimes_{A\#H^*_\alpha}A\to A^{\mathrm{coH}},\ \ \mu(a\otimes b)=\underset{\alpha\in\pi}{\oplus}a_{[0]}b_{[0]}\#\langle\lambda_\alpha, a_{[1,\alpha]}b_{[1,\alpha]}\rangle$$

Proof 1) τ is an $A\#H^*_\alpha$-bimodule map and it is also A^{coH}-linear. Now let us check that τ is a left $A\#H^*_\alpha$-module map as follows: for all $f\in H^*_\alpha$,

$$\begin{aligned}&(c\#f)\tau(a\otimes b)=\\&\quad\underset{\alpha\beta\in\pi}{\oplus}ca_{[0]}b_{[0]}\#((f\leftarrow a_{[1,\alpha]})\lambda_\beta)\leftarrow b_{[1,\alpha\beta]}=\\&\quad\underset{\alpha\beta\in\pi}{\oplus}ca_{[0]}b_{[0]}\#\langle f, a_{[1,\alpha]}\rangle(\lambda_{\alpha\beta}\leftarrow b_{[1,\alpha\beta]})=\\&\quad\tau((c\#f)\cdot a\otimes b)\end{aligned}$$

Next, we check that τ is a right $A\#H^*_\alpha$-module map as follows:

$$\begin{aligned}&\tau(a\otimes b\cdot(c\#f))=\\&\quad\underset{\alpha\beta\in\pi}{\oplus}ab_{[0]}c_{[0]}\#\lambda_{\alpha\beta}\leftarrow b_{[1,\alpha](1,\alpha\beta)}c_{[1,\alpha](1,\alpha\beta)}\\&\langle f\leftarrow S^{-1}_{\beta^{-1}}(b_{[1,\alpha](2,\beta^{-1})}c_{[1,\alpha](2,\beta^{-1})}), 1_\beta\rangle=\\&\quad\underset{\alpha\beta\in\pi}{\oplus}ab_{[0]}c_{[0]}\#[\lambda_\alpha(f\leftarrow S^{-1}_{\beta^{-1}}(b_{[1,\alpha](2,\beta^{-1})}c_{[1,\alpha](2,\beta^{-1})}))]\leftarrow\\&b_{[1,\alpha](1,\alpha\beta)}c_{[1,\alpha](1,\alpha\beta)}=\\&\quad\underset{\alpha\in\pi}{\oplus}ab_{[0]}c_{[0]}\#(\lambda_\alpha\leftarrow b_{[1,\alpha]}c_{[1,\alpha]})f=\tau(a\otimes b)(c\#f)\end{aligned}$$

Finally, for all $c\in A^{\mathrm{coH}}$, we have

$$\begin{aligned}\tau(ac, b)&=\underset{\alpha\in\pi}{\oplus}acb_{[0]}\#\lambda_\alpha\leftarrow b_{[1,\alpha]}=\\&\underset{\alpha\in\pi}{\oplus}a(cb)_{[0]}\#\lambda_\alpha\leftarrow(cb)_{[1,\alpha]}=\tau(a, cb)\end{aligned}$$

2) μ is an A^{coH}-bimodule map and it is $A\#H^*_\alpha$-linear. One checks easily that μ is an A^{coH}-bimodule map, here one fact that needs to be considered is that μ is $A\#H^*_\alpha$-linear, for $a, b, c\in A$ and $f\in H^*_\alpha$,

$$\begin{aligned}&\mu(a\cdot(c\#f)\otimes b)=\\&\quad\underset{\alpha\beta\in\pi}{\oplus}a_{[0]}b_{[0]}c_{[0]}\langle\lambda_{\beta\alpha}, a_{[1,\beta](1,\beta\alpha)}c_{[1,\beta](1,\beta\alpha)}b_{[1,\beta](1,\beta\alpha)}\rangle\end{aligned}$$

$$\langle f \leftharpoonup S_{\alpha^{-1}}^{-1}(a_{[1,\beta](2,\alpha^{-1})} c_{[1,\beta](2,\alpha^{-1})}), 1_\alpha \rangle =$$
$$\bigoplus_{\alpha\beta \in \pi} a_{[0]} b_{[0]} c_{[0]} \langle \lambda_\beta, a_{[1,\beta]} c_{[1,\beta]} b_{[1,\beta\alpha](1,\beta)} \rangle \langle f, b_{[1,\beta\alpha](2,\alpha)} \rangle =$$
$$\bigoplus_{\alpha\beta \in \pi} a_{[0]} c_{[0]} b_{[0][0]} \langle \lambda_\beta, a_{[1,\beta]} c_{[1,\beta]} b_{[0](1,\beta)} \rangle \langle f, b_{[1,\alpha]} \rangle =$$
$$\mu(a \otimes (c \# f) \cdot b)$$

3) It is not difficult to verify that τ and μ satisfy associativity.

Corollary 1 Let H be a finite type Hopf π-coalgebra with an antipode S and A a right π-H-comodule algebra, if there exists a nonzero left π-integral $\lambda = (\lambda_\alpha)_{\alpha \in \pi}$. Then we have a Morita context $(A \# H^*, A^{\mathrm{coH}}, \tau, \mu)$, where the connecting maps are given by

$$\tau: A \otimes_{A^{\mathrm{coH}}} A \to A \# H_\alpha^*, \ \tau(a \otimes b) = \bigoplus_{\alpha \in \pi} ab_{[0]} \# \lambda_\alpha \leftharpoonup b_{[1,\alpha]}$$
$$\mu: A \otimes_{A \# H_\alpha^*} A \to A^{\mathrm{coH}}, \ \mu(a \otimes b) = \bigoplus_{\alpha \in \pi} a_{[0]} b_{[0]} \# \langle \lambda_\alpha, a_{[1,\alpha]} b_{[1,\alpha]} \rangle$$

References

[1] Böhm G, Nill F, Szlachányi K. Weak Hopf algebras Ⅰ: integral theory and C*-structure [J]. *Journal of Algebra,* 1999, **221**(2): 385 - 438.

[2] Nikshych D, Vainerman L. Finite quantum groupoids and their applications [J]. *Publications of the Research Institute for Mathematical Sciences,* 2002, **43**(1): 211 - 262.

[3] Zhang L Y. Maschke-type theorem and Morita context over weak Hopf algebras[J]. *Science in China: Series A,* 2006, **49**(5): 587 - 598.

[4] Turaev V. *Homotopy quantum field theory*[M]. European Mathematical Society, 2010.

[5] Virelizier A. Hopf group-coalgebras[J]. *Journal of Pure and Applied Algebra,* 2002, **171**(1): 75 - 122.

[6] Wang S H. Morita contexts, π-Galois extensions for Hopf π-coalgebras[J]. *Communications in Algebra,* 2006, **34**(2): 521 - 546.

[7] Van Daele A, Wang S H. New braided crossed categories and Drinfel'd quantum double for weak Hopf group coalgebras[J]. *Communications in Algebra,* 2008, **36**(6): 2341 - 2386.

弱 Hopf 群余代数的 Morita 关系

郭双建　董丽红　王栓宏

(东南大学数学系,南京 211189)

摘要: 弱 Hopf 群余代数是弱 Hopf 代数和 Hopf 群余代数的自然推广. 设 π 是一个群,在弱 Hopf π-余代数前提下考虑 Morita 关系,设 H 是有限型弱 Hopf 群余代数,A 是弱右 π-H-余模代数,构造了弱 smash 积 $A\#H^*$ 和余不动点 A^{coH} 的 Morita 关系. 这一结果推广了 Wang 发表于 2006 年的 Morita contexts, π-Galois extensions for Hopf π-coalgebras 一文中的结论. 此结果对于构造弱 π-Galois 扩张是非常重要的.

关键词: 弱 Hopf π-余代数;Morita 关系;弱 π-H-余模代数;弱 smash 积

中图分类号: O153.3