Journal of Southeast University (English Edition) Vol. 28, No. 4, pp. 451 − 456 Dec. 2012 ISSN 1003—7985

# Cracked elastic substrate strip with functionally graded coating under thermal-mechanical loading

Miao Fusheng[1,2] Liu Junqiao[1,3] Li Xing[4]

([1] School of Civil Engineering and Water Conservancy, Ningxia University, Yinchuan 750021, China)
([2] Academic Journal Center, Ningxia University, Yinchuan 750021, China)
([3] Faculty of Applied Mathematics, Yuncheng University, Yuncheng 044000, China)
([4] School of Mathematics and Computer Science, Ningxia University, Yinchuan 750021, China)

**Abstract:** This paper investigates the functionally graded coating bonded to an elastic strip with a crack under thermal-mechanical loading. Considering some new boundary conditions, it is assumed that the temperature drop across the crack surface is the result of the thermal conductivity index which controls heat conduction through the crack region. By the Fourier transforms, the thermal-elastic mixed boundary value problems are reduced to a system of singular integral equations which can be approximately solved by applying the Chebyshev polynomials. The numerical computation methods for the temperature, the displacement field and the thermal stress intensity factors (TSIFs) are presented. The normal temperature distributions (NTD) with different parameters along the crack surface are analyzed by numerical examples. The influence of the crack position and the thermal-elastic non-homogeneous parameters on the TSIFs of modes Ⅰ and Ⅱ at the crack tip is presented. Results show that the variation of the thickness of the graded coating has a significant effect on the temperature jump across the crack surfaces when keeping the thickness of the substrate constant, and the thickness of functionally graded material (FGM) coating has a significant effect on the crack in the substrate. The results can be expected to be used for the purpose of gaining better understanding of the thermal-mechanical behavior of graded coatings.

**Key words:** thermal-mechanical loading; singular integral equations; functionally graded coating; thermal stress intensity factors (TSIFs)

**doi:** 10.3969/j. issn. 1003 − 7985. 2012. 04. 014

The functionally graded materials (FGM) have been introduced in coating designs as an alternative to the conventional coating because such materials can reduce the magnitude of residual thermal stresses and increase the bonding strength. Many investigators studied the fracture behavior of the FGM to optimize the design of FGM components[1−9]. In recent years, Zhou et al.[9] studied the partially insulated interface crack between a graded orthotropic coating and a homogeneous orthotropic substrate under heat flux. Ding et al.[10] obtained the thermal stress intensity factors for the interface crack in a functionally graded layered structure under the thermal loading in 2011.

However, little attention has been paid to an elastic substrate strip with a partially insulated crack under thermal-mechanical loading. It is vital that an elastic strip is weakened by a crack withstanding high thermal loads and the thermal loads is resisted by the bonded FGM coating. In this paper, the crack in the elastic substrate strip with the FGM coating under thermal-mechanical loading is investigated. The graded coating is assumed to be perfectly bonded to the homogeneous substrate strip. The coating/substrate system is subjected to mechanical loads and thermal loads. It is assumed that all the material properties are some exponential functions of $y$. By using the Fourier transforms, the problem is reduced to a singular integral equation. The equations are solved numerically and the stress intensity factor versus time for various material constants is calculated. The problem is solved under the assumption of generalized plane stress conditions.

## 1 Statement of the Problem

The geometry of the problem is shown in the Fig. 1. The graded coating of thickness $h_3$ is bonded to a homogenous strip with thickness $h_1 + h_2$. The substrate strip contains a partially insulated crack of length $2c$ along the $x$-axis and is subjected to both the thermal loading $Q_0$ and the mechanical loading $\omega_1(x)$ and $\omega_2(x)$. The thermal-mechanical properties are modeled as follows:

$$\left.\begin{aligned} k(y) &= k_0\exp(\delta(y-h_2)) \\ \mu(y) &= \mu_0\exp(\beta(y-h_2)) \\ \alpha(y) &= \alpha_0\exp(\gamma(y-h_2)) \end{aligned}\right\} \qquad (1)$$

where $k_0$, $\mu_0$ and $\alpha_0$ are the heat conductivity, the elasticity modulus and the thermal expansion coefficient in the

**Received** 2012-08-14.

**Biographies:** Miao Fusheng (1972—), male, graduate, associate professor, miaofsh@ nxu. edu. cn; Li Xing (corresponding author), male, doctor, professor, li_x@ nxu. edu. cn.

**Foundation items:** The National Natural Science Foundation of China (No. 10962008, 51061015), Specialized Research Fund for the Doctoral Program of Higher Education of China(No. 20116401110002).

**Citation:** Miao Fusheng, Liu Junqiao, Li Xing. Cracked elastic substrate strip with functionally graded coating under thermal-mechanical loading[J]. Journal of Southeast University (English Edition), 2012, 28 (4): 451 − 456. [doi: 10.3969/j. issn. 1003 − 7985. 2012. 04. 014]

substrate, respectively; $\delta$, $\beta$, $\gamma$ are the graded parameters controlling the variation of the heat conductivity, the elasticity modulus and the thermal expansion coefficient in the FGM coating, respectively.

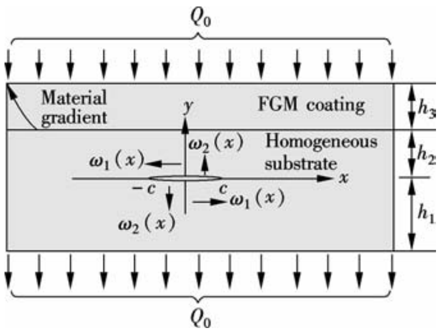


**Fig. 1** Geometry of the problem

Let $T_j (j=1,2,3)$ be the temperature; the subscripts $j=1,2,3$ refer to three regions. The heat equations can be written as

$$\left.\begin{aligned} &\nabla^2 T_1 + \delta \frac{\partial T_1}{\partial y} = 0 && h_2 \leqslant y \leqslant h_2 + h_3 \\ &\nabla^2 T_2 = 0 && 0 \leqslant y < h_2 \\ &\nabla^2 T_3 = 0 && -h_1 \leqslant y < 0 \end{aligned}\right\} \tag{2}$$

The thermal boundary conditions are given as

$$\left.\begin{aligned} &k \frac{\partial T_1}{\partial y} = Q_0 && y = h_2 + h_3,\ |x| < +\infty \\ &k_0 \frac{\partial T_3}{\partial y} = -Q_0 && y = -h_1,\ |x| < +\infty \end{aligned}\right\} \tag{3}$$

$$-k_0 \frac{\partial T_2(x,0^+)}{\partial y} = -k_0 \frac{\partial T_3(x,0^-)}{\partial y} = RQ_0 \qquad |x| \leqslant c \tag{4}$$

$$T_1(x,h_2^+) = T_2(x,h_2^-),\ \frac{\partial T_1(x,h_2^+)}{\partial y} = \frac{\partial T_2(x,h_2^-)}{\partial y} \tag{5}$$

$$T_2(x,0^+) = T_3(x,0^-),\ \frac{\partial T_2(x,0^+)}{\partial y} = \frac{\partial T_3(x,0^-)}{\partial y} \qquad |x| > c \tag{6}$$

Eq. (4) describes the partial insulation of the crack surfaces. In this case, we assume that the crack allows some heat flux which is only a certain percentage of the flux $Q_0$. $Q_0$ is the perfect conduction case. $R$ is the thermal conductivity index.

The thermal-elastic equations in the graded coating strip are given by

$$(\kappa+1)\frac{\partial^2 u}{\partial x^2} + (\kappa-1)\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 v}{\partial x \partial y} + \beta(\kappa-1)\frac{\partial u}{\partial y} + \beta(\kappa-1)\frac{\partial v}{\partial x} = 4\alpha \frac{\partial T}{\partial x} \tag{7}$$

$$(\kappa-1)\frac{\partial^2 v}{\partial x^2} + (\kappa+1)\frac{\partial^2 v}{\partial y^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \beta(\kappa+1)\frac{\partial v}{\partial y} + \beta(3-\kappa)\frac{\partial u}{\partial x} = 4\alpha\left[(\beta+\gamma)T + \frac{\partial T}{\partial y}\right] \tag{8}$$

The equations in the strip are

$$(\kappa+1)\frac{\partial^2 u_j}{\partial x^2} + (\kappa-1)\frac{\partial^2 u_j}{\partial y^2} + 2\frac{\partial^2 v_j}{\partial x \partial y} = 4\alpha_0 \frac{\partial T_j}{\partial x} \tag{9}$$

$$(\kappa-1)\frac{\partial^2 v_j}{\partial x^2} + (\kappa+1)\frac{\partial^2 v_j}{\partial y^2} + 2\frac{\partial^2 u_j}{\partial x \partial y} = 4\alpha_0 \frac{\partial T_j}{\partial y} \tag{10}$$

The related mechanical boundary conditions are given by

$$\begin{aligned} &\tau_{xy(2)}(x,0^+) = \tau_{xy(3)}(x,0^-) = \omega_1(x) \\ &\sigma_{yy(2)}(x,0^+) = \sigma_{yy(3)}(x,0^-) = \omega_2(x) \end{aligned} \qquad |x| \leqslant c \tag{11}$$

$$\tau_{xy}(x,y) = \sigma_{yy}(x,y) = 0 \qquad y = h_2 + h_3,\ |x| < +\infty \tag{12}$$

$$\tau_{xy(3)}(x,y) = \sigma_{yy(3)}(x,y) = 0 \qquad y = -h_1,\ |x| < +\infty \tag{13}$$

$$\left.\begin{aligned} &\tau_{xy(2)}(x,0^+) = \tau_{xy(3)}(x,0^-) = 0 \\ &\sigma_{yy(2)}(x,0^+) = \sigma_{yy(3)}(x,0^-) \end{aligned} \qquad y=0,\ |x| > c \right\} \tag{14}$$

$$\left.\begin{aligned} &u_2(x,0^+) = u_3(x,0^-) = 0 \\ &v_2(x,0^+) = v_3(x,0^-) \end{aligned} \qquad y=0,\ |x| > c \right\} \tag{15}$$

$$\tau_{xy}(x,h_2^+) = \tau_{xy(2)}(x,h_2^-),\ \sigma_{yy}(x,h_2^+) = \sigma_{yy(2)}(x,h_2^-) \tag{16}$$

$$u(x,h_2^+) = u_2(x,h_2^-),\ v(x,h_2^+) = v_2(x,h_2^-) \tag{17}$$

where the subscripts 2, 3 refer to different regions. We define the dimensionless quantities as

$$\begin{aligned} &(\bar{x}, \bar{y}, \overline{h_1}, \overline{h_2}, \overline{h_3}) = \frac{(x, y, h_1, h_2, h_3)}{c} \\ &\overline{(T_1, T_2, T_3)} = \frac{(T_1, T_2, T_3)}{-Q_0 c/k_0} \end{aligned} \tag{18}$$

$$(\overline{\tau_{xy}}, \overline{\sigma_{xx}}, \overline{\sigma_{yy}}, \overline{\tau_{xy(j)}}, \overline{\sigma_{xx(j)}}, \overline{\sigma_{yy(j)}}) = \frac{(\tau_{xy}, \sigma_{xx}, \sigma_{yy}, \tau_{xy(j)}, \sigma_{xx(j)}, \sigma_{yy(j)})}{-\mu_0 Q_0 \alpha_0 c/k_0} \qquad j=2,3 \tag{19}$$

$$\begin{aligned} &(\bar{u}, \bar{v}, \overline{u_j}, \overline{v_j}) = \frac{(u, v, u_j, v_j)}{-Q_0 \alpha_0 c^2/k_0} \qquad j=2,3 \\ &(\bar{\delta}, \bar{\beta}, \bar{\gamma}) = (\delta, \beta, \gamma)c \end{aligned} \tag{20}$$

For simplicity in what follows, the bar appearing with the dimensionless quantities is omitted.

## 2 Heat Conduction

The dimensionless temperature can be expressed as

$$(T_1(x,y), T_2(x,y), T_3(x,y)) = (T_{11}(y), T_{21}(y), T_{31}(y)) + (T_{12}(x,y), T_{22}(x,y), T_{32}(x,y)) \tag{21}$$

And it can be easily obtained that

$$\left.\begin{array}{ll} T_{11}(y)=\left(\dfrac{1}{\delta}+h_2\right)-\dfrac{1}{\delta}\mathrm{e}^{-\delta(y-h_2)} & h_2\leqslant y\leqslant h_2+h_3 \\ T_{21}(y)=y & 0\leqslant y<h_2 \\ T_{31}(y)=y & -h_1\leqslant y<0 \end{array}\right\} \tag{22}$$

We introduce the Fourier integral transforms as follows:

$$\left.\begin{aligned} F(\lambda,y) &= \int_{-\infty}^{+\infty} f(x,y)\,\mathrm{e}^{-\mathrm{i}x\lambda}\,\mathrm{d}x \\ f(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\lambda,y)\,\mathrm{e}^{\mathrm{i}x\lambda}\,\mathrm{d}\lambda \end{aligned}\right\} \tag{23}$$

The solutions to $T_{12}(x,y)$, $T_{22}(x,y)$, $T_{32}(x,y)$ can be expressed as

$$T_{12}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}[A_1(\lambda)\mathrm{e}^{\lambda_1 y}+A_2(\lambda)\mathrm{e}^{\lambda_2 y}]\,\mathrm{e}^{\mathrm{i}x\lambda}\,\mathrm{d}\lambda \tag{24}$$

$$T_{j2}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}[A_{2j-1}(\lambda)\mathrm{e}^{-|\lambda|y}+A_{2j}(\lambda)\mathrm{e}^{|\lambda|y}]\,\mathrm{e}^{\mathrm{i}x\lambda}\,\mathrm{d}\lambda \qquad j=2,3 \tag{25}$$

where $\lambda_{1,2}=\dfrac{1}{2}(-\delta\pm\sqrt{\delta^2+4\lambda^2})$.

We introduce the following density function:

$$\varphi(x)=\frac{\partial}{\partial x}[T(x,0^+)-T(x,0^-)]=\frac{\partial}{\partial x}[T_{22}(x,0^+)-T_{32}(x,0^-)] \tag{26}$$

and it follows that

$$\int_{-1}^{+1}\varphi(t)\,\mathrm{d}t=0,\ \varphi(x)=0 \qquad |x|\geqslant 1 \tag{27}$$

The unknown functions $A_k(\lambda)$ $(k=1,2,\ldots,6)$ can be given by the boundary conditions.

We obtain the following integral equation in which the unknown function is $\varphi(x)$.

$$\frac{1}{\pi}\int_{-1}^{1}\left(\frac{1}{t-x}+k(x,t)\varphi(t)\,\mathrm{d}t\right)=R-1 \tag{28}$$

where $k(x,t)$ is the known function.

The equation can be solved[4], and its solution can be expressed as

$$\varphi(t)=\frac{1}{\sqrt{1-t^2}}\sum_{j=1}^{\infty}a_jT_j(t) \tag{29}$$

where $T_j(t)$ is the first kind Chebyshev polynomials; $a_j$ is the constant to be determined. A system of linear algebraic equations can be obtained by using the Gauss-Chebyshev formula.

$$\sum_{j=1}^{\infty}a_jU_{j-1}(x)+\sum_{j=1}^{\infty}a_jk_j(x)=R-1 \qquad |x|<1 \tag{30}$$

where $U_{j-1}(x)$ is the second kind Chebyshev polynomials, and $k_j(x)$ is an improper integral.

The following equation can be solved by a system of algebraic equations (see Ref. [11]).

$$\sum_{j=1}^{N}a_j\frac{\sin\left(\dfrac{jl\pi}{N+1}\right)}{\sin\left(\dfrac{l\pi}{N+1}\right)}+\sum_{j=1}^{N}a_jk_j\left(\frac{l\pi}{N+1}\right)=R-1 \qquad j,l=1,2,\ldots,N \tag{31}$$

## 3 Displacement Field

From Eqs. (7) to (10), we can obtain that

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\sum_{k=1}^{4}C_k(\lambda)\mathrm{e}^{m_ky}\right)\mathrm{e}^{\mathrm{i}x\lambda}\,\mathrm{d}\lambda+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(T_{f1}(\lambda,y)\right)\mathrm{e}^{\mathrm{i}x\lambda}\,\mathrm{d}\lambda \tag{32}$$

$$v(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\sum_{k=1}^{4}C_k(\lambda)S_{1k}(\lambda)\mathrm{e}^{m_ky}\right)\mathrm{e}^{\mathrm{i}x\lambda}\,\mathrm{d}\lambda+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(T_{f2}(\lambda,y)\right)\mathrm{e}^{\mathrm{i}x\lambda}\,\mathrm{d}\lambda \tag{33}$$

where $C_k(\lambda)$ $(k=1,2,3,4)$ is to be determined; $T_{f1}(\lambda,y)$, $S_{1k}(\lambda)$ and $T_{f2}(\lambda,y)$ are some known functions; and $m_k$ $(k=1,2,3,4)$ is the root of the following equation:

$$m^4+2\beta m^3+(\beta^2-2\lambda^2)m^2-2\beta\lambda^2m+\lambda^2\left(\lambda^2+\frac{\beta^2(3-\kappa)}{1+\kappa}\right)=0 \tag{34}$$

The stresses for the coating are given as

$$\frac{\sigma_{yy}}{\mathrm{e}^{\beta(y-h_2)}}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\sum_{k=1}^{4}C_k(\lambda)S_{2k}(\lambda)\mathrm{e}^{m_ky}+T_{f3}(\lambda,y)\right]\mathrm{e}^{\mathrm{i}x\lambda}\,\mathrm{d}\lambda+F_{c5}\mathrm{e}^{\gamma y}+F_{c6}\mathrm{e}^{(\gamma-\delta)y} \tag{35}$$

$$\frac{\tau_{xy}}{\mathrm{e}^{\beta(y-h_2)}}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\sum_{k=1}^{4}C_k(\lambda)S_{3k}(\lambda)\mathrm{e}^{m_ky}+T_{f4}(\lambda,y)\right]\mathrm{e}^{\mathrm{i}x\lambda}\,\mathrm{d}\lambda \tag{36}$$

where $S_{2k}(\lambda)$, $S_{3k}(\lambda)$, $F_{c5}$, $F_{c6}$, $T_{f3}(\lambda,y)$ and $T_{f4}(\lambda,y)$ are some known functions.

Applying the Fourier transforms to Eqs. (7) to (10), we can obtain the stresses in the substrate.

$$\sigma_{(j)yy}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}[(F_{e1(j)}(\lambda,y)+F_{e2}(\lambda)A_{2j-1})\mathrm{e}^{-|\lambda|y}+(F_{e3(j)}(\lambda,y)+F_{e4}(\lambda)A_{2j})\mathrm{e}^{|\lambda|y}+F_{e5}(\lambda)y]\,\mathrm{e}^{\mathrm{i}x\lambda}\,\mathrm{d}\lambda-\frac{4}{\kappa-1}y \qquad j=2,3 \tag{37}$$

$$\tau_{(j)yy}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}[(F_{f1(j)}(\lambda,y)+F_{f2}(\lambda)A_{2j-1})\mathrm{e}^{-|\lambda|y}+(F_{f3(j)}(\lambda,y)+F_{f4}(\lambda)A_{2j})\mathrm{e}^{|\lambda|y}+F_{f5}(\lambda)y]\,\mathrm{e}^{\mathrm{i}x\lambda}\,\mathrm{d}\lambda \qquad j=2,3 \tag{38}$$

where $F_{e1(j)}$, $F_{e3(j)}$, $F_{f1(j)}$, $F_{f3(j)}$, $F_{e2}$, $F_{e4}$, $F_{e5}$, $F_{f2}$, $F_{f4}$,

$F_{f5}$ are some known functions. $C_{k(j)}(\lambda)$ $(k = 1, 2, 3, 4)$ can be computed by the boundary conditions. We now introduce the density functions that satisfy the following single-value conditions:

$$\int_{-1}^{+1} \varphi_j(t)\,\mathrm{d}t = 0,\ \varphi_j(x) = 0 \qquad |x| \geqslant 1;\ j = 1,2 \tag{39}$$

Applying the boundary conditions, we obtain the following system of integral equations:

$$\int_{-1}^{+1} \left[ \left( \frac{1}{t - x} + K_{11}(x,t) \right) \varphi_1(t) + K_{12}(x,t)\varphi_2(t) \right] = \pi\omega_3(x) \tag{40}$$

$$\int_{-1}^{+1} \left[ \left( \frac{1}{t - x} + K_{22}(x,t) \right) \varphi_2(t) + K_{21}(x,t)\varphi_1(t) \right] = \pi\omega_4(x) \tag{41}$$

where $K_{ij}(x, t)$ $(i = 1, 2;\ j = 1, 2)$ is the Fredholm kernel, and $\omega_3(x)$, $\omega_4(x)$ contain $\omega_1(x)$, $\omega_2(x)$.

The integral equations are solved by

$$\varphi_1(t) = \frac{1}{\sqrt{1 - t^2}} \sum_{j=1}^{\infty} b_j T_j(t),\ \varphi_2(t) = \frac{1}{\sqrt{1 - t^2}} \sum_{j=1}^{\infty} c_j T_j(t) \tag{42}$$

where $b_j$, $c_j$ are to be determined.

Eqs. (39) to (41) can be converted into a system of linear equations[11].

$$\sum_{j=1}^{N} b_j [U_{j-1}(x_r) + k_{11j}(x_r)] + \sum_{j=1}^{N} c_j k_{12j}(x_r) = \omega_3(x_r) \qquad r = 1, 2, \ldots, N \tag{43}$$

$$\sum_{j=1}^{N} b_j k_{21j}(x_r) + \sum_{j=1}^{N} c_j [U_{j-1}(x_r) + k_{22j}(x_r)] = \omega_4(x_r) \qquad r = 1, 2, \ldots, N \tag{44}$$

where $k_{mnj}(x_r)$ is the improper integral.

The stress intensity factors at the crack tips are defined as

$$K_{\mathrm{I}}(1) = \frac{-2}{1 + \kappa} \sum_{j=1}^{N} b_j,\ K_{\mathrm{I}}(-1) = \frac{2}{1 + \kappa} \sum_{j=1}^{N} (-1)^j b_j \tag{45}$$

$$K_{\mathrm{II}}(1) = \frac{-2}{1 + \kappa} \sum_{j=1}^{N} c_j,\ K_{\mathrm{II}}(-1) = \frac{2}{1 + \kappa} \sum_{j=1}^{N} (-1)^j c_j \tag{46}$$

## 4 Numerical Results and Discussion

### 4.1 Temperature field

In this section, the numerical results of the temperature distribution along the crack surface are analyzed.

Fig. 2 depicts the effect of the thickness $h_3$ on the normal temperature distribution (NTD) as $\delta = -1$, $\delta = 1$, $R = 0$ and $h_1 = 3$. It can be obtained that the variation of the thickness of the graded coating has a significant effect on the temperature jump across the crack surfaces when keeping the thickness of the substrate constant.

Fig. 3 depicts the effect of $h_1$ on the NTD. It is possible that the thickness of the substrate strip and the

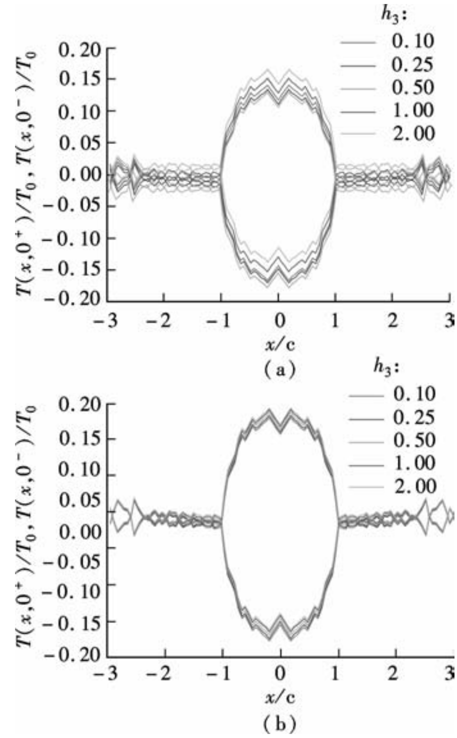


**Fig. 2** The effect of $h_3$ on the NTD. (a) $\delta = 1$, $h_1 = 3$; (b) $\delta = -1$, $h_1 = 3$

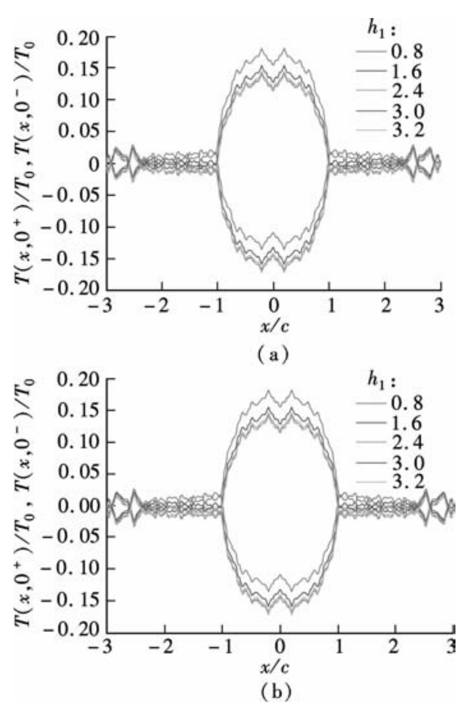


**Fig. 3** The effect of $h_1$ on the NTD. (a) $\delta = 1$, $h_3 = 0.5$; (b) $\delta = -1$, $h_3 = 0.5$

position of the crack are closely related to the NTD. The variation of $h_1$ has an obvious influence on the temperature jump across the crack surface when $h_1$ is smaller. The temperature jump across the crack upper and lower surfaces is more obvious.

## 4.2 Thermal stress intensity factors (TSIFs) at crack tip

In this section, the thermal stress factors are normalized by $K_0 = \mu_0 Q_0 \alpha_0 c / k_0$. The normalized TSIFs are obtained by the numerical technique. The influence of the crack position and the thermal-elastic parameters on the TSIFs of mode Ⅰ is presented in Fig. 4 to Fig. 8; the situation of mode Ⅱ is similar to mode Ⅰ.

Fig. 4 shows the effects of $\delta$ on the TSIFs when keeping $\beta = 1$, $\gamma = 1$ and thickness $h_2 = 1$ constant, and the mechanical loading $\omega_1(x) = 0$, $\omega_2(x) = -1$. These results indicate that the absolute values of the TSIFs decline rapidly with the increase in $h_3/h_1$ when keeping $\delta$ constant. The TSIFs decrease as $\delta$ increases. One reason for this is that the FGM is quite effective in reducing thermal stress. This means that the graded parameter controlling the variation of the heat conductivity has a vital influence on the thermal fracture of coating/substrate structure.

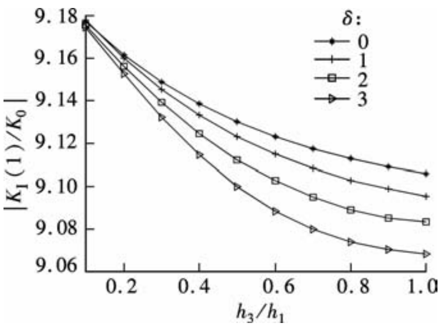

**Fig. 4** Effects of $\delta$ on the TSIFs of mode Ⅰ at the crack tip

Fig. 5 depicts the effects of the graded parameters controlling the variation of the elasticity modulus $\beta$ on the TSIFs as $\delta = 1$, $\gamma = 1$, $h_2 = 1$, $\omega_1(x) = 0$, $\omega_2(x) = -1$. The absolute values $K_{\mathrm{I}}$ decrease with the increase in $h_3/h_1$, but decreases as $\beta$ increases. These results imply that the FGM coating has obvious effects on the crack in the substrate.

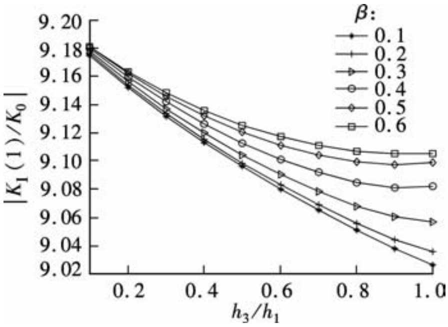

**Fig. 5** Effects of $\beta$ on the TSIFs of mode Ⅰ at the crack tip

Fig. 6 illustrates the effects of the graded parameters controlling the variation of the thermal expansion coefficient $\gamma$ on the TSIFs when $\delta = 1$, $\beta = 1$, $h_2 = 1$. The absolute values of $K_{\mathrm{I}}$ decreases with the increase in $h_3/h_1$ and $\gamma$. The absolute values of $K_{\mathrm{II}}$ decreases with the increase in $h_3/h_1$, but increases with the increase in $\gamma$. These results suggest that the thermal expansion coefficient of the FGM coating is correlated with the crack tip TSIFs.

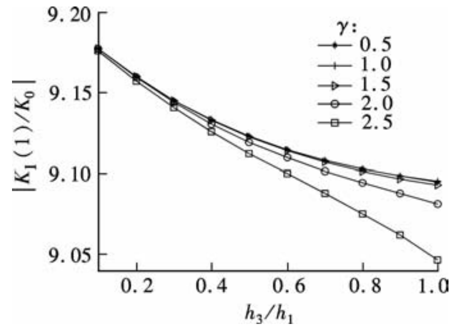

**Fig. 6** Effects of $\gamma$ on the TSIFs of mode Ⅰ at the crack tip

The effects of $h_2$ on the TSIFs when $\delta = 1$, $\beta = 1$, $\gamma = 1$ are displayed in Fig. 7. The absolute values of $K_{\mathrm{I}}$ decrease with the increase in $h_3/h_1$ and $h_2$. The absolute values of $K_{\mathrm{II}}$ decrease when $h_3/h_1$ increases, but increase as $h_2$ increases. It indicates that the position of the crack is closely related to the absolute values of mode Ⅰ and mode Ⅱ at the crack tip.

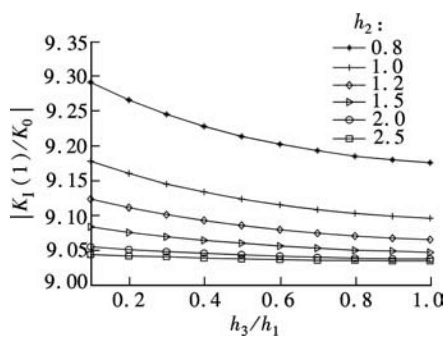

**Fig. 7** Effects of $h_2$ on the TSIFs of mode Ⅰ at the crack tip

Fig. 8 presents the effects of the thermal conductivity index $R$ on the TSIFs when $\delta = 1$, $\beta = 1$, $\gamma = 1$. The absolute values of the TSIFs decrease with the increase in $h_3/h_1$ and $R$. The absolute values of the TSIFs decrease

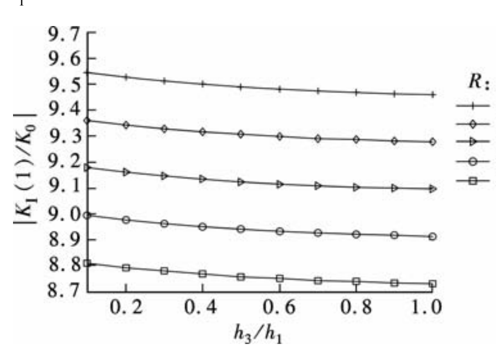

**Fig. 8** Effects of $R$ on the TSIFs of mode Ⅰ at the crack tip

very slowly with the increase in $h_3/h_1$ as the thermal conductivity index $R$ is a constant. But when the thermal conductivity index $R$ changes, the TSIFs vary very quickly. It is likely that the variation of $R$ has a significant effect on the absolute values of TSIFs.

## 5 Conclusion

The problem of the FGM coating bonded to an elastic substrate strip with a partially insulated crack under the mechanical and thermal loading is investigated. Using the Fourier transforms, the thermal-elastic mixed boundary value problems are reduced to a system of singular integral equations. The singular integral equations are solved by applying the Chebyshev polynomials. The numerical results of the NTD and the TSIFs are displayed graphically for several degenerated problems. The influence factors on the NTD and the TSIFs have been discussed in detail. It shows that the heat conductivity index $R$ and the thickness of the coating have a significant influence on the NTD on the crack surface. The influences of the parameters on the TSIFs is quite significant. These results can be used to gain better understanding of the fracture behavior of graded coating. We can optimize the coating-substrate structure to control the NTD and the TSIFs by adjusting the thickness of the FGM coating and the parameters.

## References

[1] Chen J. Determination of thermal stress intensity factors for interface crack in a graded orthotropic coating-substrate structure[J]. *International Journal of Fracture*, 2005, **133**(4): 303-328.

[2] Delale F, Erdogan F. Interface crack in a non-homogeneous elastic medium[J]. *International Journal of Engineering Sciences*, 1988, **26**(6): 559-568.

[3] Erdogan F. Fracture mechanics of functionally graded materials[J]. *Composites Engineering*, 1995, **5**(7): 753-770.

[4] Erdogan F, Wu B H. The surface crack problem for a plate with functionally graded properties[J]. *ASME Journal of Applied Mechanics*, 1997, **64**(3): 449-456.

[5] El-Borgi S, Erdogan F, Hidri L. A partially insulated embedded crack in an infinite functionally graded medium under thermal-mechanical loading[J]. *International Journal of Engineering Sciences*, 2004, **42**(3/4): 371-393.

[6] Jin Z H, Feng Y Z. Thermal fracture resistance of a functionally graded coating with periodic edge crack[J]. *Surface and Coatings Technology*, 2008, **202**(17): 4189-4197.

[7] Noda N, Jin Z H. Steady thermal stresses in an infinite nonhomogeneous elastic solid containing a crack[J]. *Journal of Thermal Stresses*, 1993, **16**(2): 181-196.

[8] Zhou Y T, Li X, Qin J Q. Transient thermal stress analysis of orthotropic functionally graded materials with a crack[J]. *Journal of Thermal Stresses*, 2007, **30**(12): 1211-1231.

[9] Zhou Y T, Li X, Yu D H. A partially insulated interface crack between a graded orthotropic coating and a homogeneous orthotropic substrate under heat flux supply[J]. *International Journal of Solids and Structures*, 2010, **47**(6): 768-778.

[10] Ding S H, Li X. Thermal stress intensity factors for an interface crack in a functionally graded layered structures[J]. *Arch Appl Mech*, 2011, **81**(7): 943-955.

[11] Erdogan F, Gupta G D. On the numerical solution of singular integral equations[J]. *Quarterly of Applied Mathematics*, 1972, **29**(1): 525-534.

# 热机荷载下含梯度涂层的弹性条中裂纹问题

苗福生[1,2]　刘俊俏[1,3]　李　星[4]

([1] 宁夏大学土木与水利工程学院，银川 750021)
([2] 宁夏大学学术期刊中心，银川 750021)
([3] 运城学院应用数学系，运城 044000)
([4] 宁夏大学数学计算机学院，银川 750021)

**摘要**：研究了热机荷载作用下含功能梯度材料涂层的裂纹弹性底层条问题，提出一些新的边界条件，假设裂纹面上的温度降低是由通过裂纹的控制热传导的因子造成，利用傅里叶积分变换，将热弹性混合边值问题转化为一组奇异积分方程，奇异积分方程组可以利用 Chebyshev 多项式逼近方法近似求解. 给出了温度、位移场和热应力强度因子的数值计算方法. 通过算例分析了不同几何参数下裂纹表面标准温度的分布，并讨论了裂纹位置和热弹性非均匀参数对Ⅰ、Ⅱ型裂纹尖端标准热应力强度因子的影响. 结果表明：弹性底层厚度不变时，梯度涂层厚度对裂纹表面的温度分布有重要的影响；梯度涂层厚度的变化对底层的裂纹有重要的影响. 研究结果有助于对梯度涂层结构热机行为的理解.

**关键词**：热机载荷；奇异积分方程；功能梯度涂层；热应力强度因子

**中图分类号**：O241.8