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Existence of multi-bump solutions for coupled Schrödinger systems

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Abstract: The Schrödinger equation $-\Delta u+\lambda^2 u=|u|^{2q-2}u$ has a unique positive radial solution U_λ, which decays exponentially at infinity. Hence it is reasonable that the Schrödinger system $-\Delta u_1+u_1=|u_1|^{2q-2}u_1-\varepsilon b(x)|u_2|^q|u_1|^{q-2}u_1$, $-\Delta u_2+u_2=|u_2|^{2q-2}u_2-\varepsilon b(x)|u_1|^q|u_2|^{q-2}u_2$ has multiple-bump solutions which behave like U_λ in the neighborhood of some points. For $u=(u_1,u_2)\in H^1(\mathbf{R}^3)\times H^1(\mathbf{R}^3)$, a nonlinear functional $I_\varepsilon(u)=I_1(u_1)+I_2(u_2)-\frac{\varepsilon}{q}\int_{\mathbf{R}^3}b(x)|u_1|^q|u_2|^q\mathrm{d}x$ is defined, where $I_1(u_1)=\frac{1}{2}\|u_1\|^2-\frac{1}{2q}\int_{\mathbf{R}^3}|u_1|^{2q}\mathrm{d}x$ and $I_2(u_2)=\frac{1}{2}\|u_2\|_\omega^2-\frac{1}{2q}\int_{\mathbf{R}^3}|u_2|^{2q}\mathrm{d}x$. It is proved that the solutions of the system are the critical points of I_ε. Let Z be the smooth solution manifold of the unperturbed problem and T_zZ is the tangent space. The critical point of I_ε is rewritten as the form of $z+w$, where $w\in(T_zZ)^\perp$. Using some properties of I_ε, it is proved that there exists a critical point of I_ε close to the form $\left(\sum_{i=1}^n U(x-\xi_i),\sum_{i=1}^n V(x-\xi_i)\right)$ which is a multi-bump solution.

Key words: coupled Schrödinger system; multi-bump solution; variational reduction method

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Nonlinear Schrödinger equations (NLS) have been broadly studied in many aspects, such as existence of solitary waves, concentration and multi-bump phenomena for semiclassical states. Menyuk[1] showed that some phenomenon can be described by the following two coupled nonlinear Schrödinger equations:

$$\left.\begin{aligned}&i\phi_t+\phi_{xx}+(|\phi|^2+b|\psi|^2)\phi=0\\&i\psi_t+\psi_{xx}+(|\psi|^2+b|\phi|^2)\psi=0\end{aligned}\right\}$$

where b is a real positive constant depending on the anisotropy of the fiber. Looking for the standing wave solutions of the form:

$$\phi(x,t)=\mathrm{e}^{\mathrm{i}\omega_1^2 t}u(x),\quad \psi(x,t)=\mathrm{e}^{\mathrm{i}\omega_2^2 t}v(x)$$

and performing a rescaling of variables, it follows that (u,v) satisfies the following system:

$$\left.\begin{aligned}&-u''+u=|u|^2u+|v|^2u &&\text{in } \mathbf{R}\\&-v''+\omega^2v=|v|^2v+|u|^2v &&\text{in } \mathbf{R}\end{aligned}\right\}\tag{1}$$

where $\omega^2=\omega_2^2/\omega_1^2$. In recent years, many researchers have studied the nontrivial solutions of system (1)[2-11]. Maia et al.[8] studied the existence of solutions for a general system with perturbation terms.

In this paper, we will study the following problem:

$$\left.\begin{aligned}&-\Delta u+u=|u|^{2q-2}u-\varepsilon b(x)|v|^q|u|^{q-2}u &&\text{in } \mathbf{R}^3\\&-\Delta v+\omega^2v=|v|^{2q-2}v-\varepsilon b(x)|u|^q|v|^{q-2}v &&\text{in } \mathbf{R}^3\end{aligned}\right\}\tag{2}$$

where $\omega>0$ and $2\leqslant q<3$. Our goal is to prove that, for ε sufficiently small, system (2) possesses a nontrivial solution with two multi-bump components.

To state our main result, we first consider the following NLS equation:

$$-\Delta u+\lambda^2u=|u|^{2q-2}u \qquad \text{in } \mathbf{R}^3\tag{3}$$

where $\lambda>0$. It is well known that Eq. (3) has a unique positive radial solution U_λ, which decays exponentially at ∞[12]. Let $U=U_1$, $V=U_\omega$, then (U,V) is a solution of system (2) with $\varepsilon=0$. For $n\in\mathbf{N}$ and for sufficiently separated $\xi_1,\xi_2,\dots,\xi_n\in\mathbf{R}^3$, a solution of system (2), which is close to $\left(\sum_{i=1}^n U(x-\xi_i),\sum_{i=1}^n V(x-\xi_i)\right)$, is called an (n,n)-bump solution. Our aim is to find this kind of solution for system (2).

Throughout this paper, the following conditions for ω and $b(x)$ are satisfied:

$$\left.\begin{aligned}&\omega>\frac{q}{2q-1}\\&b(x)\text{ is continuous and } b(x)>0 \text{ for } x\in\mathbf{R}^3\\&\text{There exists } c>0 \text{ and } \sigma>0 \text{ such that } b(x)\geqslant c\mathrm{e}^{-\sigma|x|}\end{aligned}\right\}\tag{4}$$

Our main result is the following theorem.

Theorem 1 Assume that (4) is satisfied. If n satisfies

$$n<1+\frac{(2q-1)\tilde{\omega}-q}{(2q-1)\sigma}$$

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where $\widetilde{\omega} = \min\{1, \omega\}$. Then there exists $\varepsilon(n) > 0$ such that, for $0 < \varepsilon < \varepsilon(n)$, the system (2) has an (n, n)-bump solution.

1 Abstract Setting

To prove our existence result we shall find critical points of a class of perturbation functionals. Below we outline the so-called finite dimensional reduction. The procedure has been widely used in the literature (see Refs. [13 − 18]).

Consider a Hilbert space H and let $I_\varepsilon \in C^2(H, \mathbf{R})$ be a functional. We assume that there exists a smooth manifold $Z \subset H$ such that

$$\| I'_\varepsilon(z) \| \to 0 \tag{5}$$

as $\varepsilon \to 0$ uniformly for $z \in Z$.

Let us consider the tangent space $T_z Z$ of Z at z and let $W = (T_z Z)^\perp$. We shall look for the critical points of I_ε of the form $z + w$ with $w \in W$. We first consider the auxiliary equation:

$$PI'_\varepsilon(z + w) = 0$$

where P denotes the projection onto W. In order to solve this equation, we assume that there exists $\gamma > 0$ such that, for ε small enough,

$$\| [PI''_\varepsilon(z)]^{-1} \|_{L(W, W)} \leqslant \gamma \tag{6}$$

uniformly for $z \in Z$.

Fix $z \in Z$, and define $B_{\varepsilon,\gamma} = \{ w \in W: \ \| w \| \leqslant 2\gamma \| I'_\varepsilon(z) \| \}$. We assume that

$$\| I''_\varepsilon(z + w) - I''_\varepsilon(z) \| \to 0 \tag{7}$$

as $\varepsilon \to 0$ uniformly for $z \in Z$ and $w \in B_{\varepsilon,\gamma}$. Define the map $S_\varepsilon: \ B_{\varepsilon,\gamma} \to W$,

$$S_\varepsilon(w) = w - [PI''_\varepsilon(z)]^{-1}(PI'_\varepsilon(z + w)) \tag{8}$$

According to the definition (8), the fixed point of S_ε is a solution of $PI'_\varepsilon(z + w) = 0$. It is possible to show that for ε small enough, the map S_ε maps the ball $B_{\varepsilon,\gamma}$ into itself and is a contraction. As a consequence, there exists a unique $w_{\varepsilon,z} \in B_{\varepsilon,\gamma}$ such that $S_\varepsilon(w_\varepsilon) = w_\varepsilon$. Moreover, one has that

$$\| w_{\varepsilon,z} \| \leqslant 2\gamma \| I'_\varepsilon(z) \|$$

Finally, one considers the reduced functional

$$\widetilde{I}_\varepsilon(z) = I_\varepsilon(z + w_{\varepsilon,z}) \qquad z \in Z$$

and proves that if $z_\varepsilon \in Z$ is a critical point of $\widetilde{I}_\varepsilon$, then $u_\varepsilon = z_\varepsilon + w_{\varepsilon,z_\varepsilon}$ is a critical point of I_ε.

2 Variational Reduction

In this section, we solve the auxiliary equation by verifying formulae (5), (6) and (7). We will use the notation $E = H^1(\mathbf{R}^3)$, endowed with two equivalent scalar products and norms

$$(u \mid v) = \int_{\mathbf{R}^3} (\nabla u \cdot \nabla v + uv)\,\mathrm{d}x, \quad \| u \|^2 = (u \mid u)$$

$$(u \mid v)_\omega = \int_{\mathbf{R}^3} (\nabla u \cdot \nabla v + \omega^2 uv)\,\mathrm{d}x, \quad \| u \|^2_\omega = (u \mid u)_\omega$$

We will work on the space $H = E \times E$, equipped with the scalar product and norm:

$$(u \mid v) = (u_1 \mid v_1) + (u_2 \mid v_2)_\omega, \quad \| u \|^2 = \| u_1 \|^2 + \| u_2 \|^2_\omega$$

for $u = (u_1, u_2), v = (v_1, v_2) \in H$. For $u_1, \ u_2 \in E$, let

$$I_1(u_1) = \frac{1}{2} \| u_1 \|^2 - \frac{1}{2q} \int_{\mathbf{R}^3} | u_1 |^{2q} \mathrm{d}x$$

$$I_2(u_2) = \frac{1}{2} \| u_2 \|^2_w - \frac{1}{2q} \int_{\mathbf{R}^3} | u_2 |^{2q} \mathrm{d}x$$

For $u = (u_1, u_2) \in H$, we define the functional of (2),

$$I_\varepsilon(u) = I_1(u_1) + I_2(u_2) - \frac{\varepsilon}{q} \int_{\mathbf{R}^3} b(x) | u_1 |^q | u_2 |^q \mathrm{d}x$$

Then the solutions of (2) are the critical points of I_ε.

For any positive integer n satisfying the assumption of Theorem 1 and $0 < \delta < \dfrac{(2q-1)(\widetilde{\omega} - \sigma(n-1)) - q}{(2q-1)(\widetilde{\omega} - \sigma(n-1))}$, define

$$T_\varepsilon = \left\{ \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in (\mathbf{R}^3)^n: \ | \xi_i - \xi_j | > (1-\delta) \ln \frac{1}{\varepsilon} \text{ for } i \neq j \right\}$$

where $\xi_i = (\xi_{i1}, \xi_{i2}, \xi_{i3}) \in \mathbf{R}^3$, $Z = \left\{ z_\xi = (z_{1\xi}, z_{2\xi}) = \left(\sum_{i=1}^n U(x - \xi_i), \sum_{i=1}^n V(x - \xi_i) \right), \xi \in T \right\}$. U and V are the solutions of (3) with $\lambda = 1$ and $\lambda^2 = \omega^2$, respectively, and

$$T_{z_\xi} Z = \operatorname{span} \left\{ D^i_\alpha z_\xi = \left(\frac{\partial U(x - \xi_i)}{\partial x_\alpha}, \frac{\partial V(x - \xi_i)}{\partial x_\alpha} \right), \alpha = 1, 2, 3; \ i = 1, 2, \ldots, n \right\}$$

$$W_\xi = (T_{z_\xi} Z)^\perp = \{ u \in H: (u \mid v) = 0 \quad \forall v \in T_{z_\xi} Z \}$$

We also denote $U(x - \xi_i)$ and $V(x - \xi_i)$ by U_{ξ_i} and V_{ξ_i}, respectively.

In the following we shall solve the auxiliary equation by verifying formulae (5), (6) and (7).

Lemma 1 The perturbation functional I_ε satisfies (5). More precisely, there exists $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$ we have $\| I'_\varepsilon(z_\xi) \| \leqslant C\varepsilon^{(2q-1)/(2q)(1-\delta)\widetilde{\omega}}$ for any $\xi \in T_\varepsilon$.

Proof

$$I'_\varepsilon(z_\xi)[(\phi_1, \phi_2)] = \left(I'_1(z_{1\xi})[\phi_1] + \varepsilon \int_{\mathbf{R}^3} b(x) z^{q-1}_{1\xi} \phi_1 z^q_{2\xi} \mathrm{d}x \right) +$$

$$\left(I_2'(z_{2\xi})[\phi_2] + \varepsilon\int_{\mathbf{R}^3} b(x) z_{1\xi}^q z_{2\xi}^{q-1}\phi_2 \mathrm{d}x\right)$$

For the first two terms, we have

$$\left| I_1'(z_{1\xi})[\phi_1] + \varepsilon\int_{\mathbf{R}^3} b(x) z_{1\xi}^{q-1}\phi_1 z_{2\xi}^q \mathrm{d}x \right| \leqslant C\Big(\int_{\mathbf{R}^3}\sum_{i\neq j} U_{\xi_i}^{2q-1} U_{\xi_j}\mathrm{d}x\Big)^{(2q-1)/(2q)} \|\varphi_1\| + C\varepsilon\|\varphi_1\| \leqslant C\varepsilon^{(2q-1)/(2q)(1-\delta)}\|\varphi_1\|$$

Similarly,

$$\left| I_2'(z_{2\xi})[\phi_2] + \varepsilon\int_{\mathbf{R}^3} b(x) z_{1\xi}^q z_{2\xi}^{q-1}\phi_2 \mathrm{d}x \right| \leqslant C\varepsilon^{(2q-1)/(2q)\omega(1-\delta)}\|\varphi_2\|$$

Lemma 2 Assumption (6) in Section 1 holds. More precisely, there exists $0<\varepsilon_2<\varepsilon_1$ such that for $0<\varepsilon<\varepsilon_2$, we have $\|[PI''_\varepsilon(z_\xi)]^{-1}\|\leqslant\gamma$ on W_ξ for any $\xi\in T_\varepsilon$.

Proof Since $\|PI''_\varepsilon(z_\xi) - PI''_0(z_\xi)\|\leqslant C\varepsilon$, it suffices to prove that $\|[PI''_0(z_\xi)]^{-1}\|\leqslant\gamma$. Let

$$z_i = z_{\xi_i} - \sum_{\alpha=1}^{3}\sum_{j\neq i}(z_{\xi_i}\mid D_\alpha^j z_\xi)\,\|D_\alpha^j z_\xi\|^{-2} D_\alpha^j z_\xi$$

then $z_i\in(T_{z_\xi}Z)^\perp$. Denoting $V_1=\operatorname{span}\{z_i(x)\mid i=1,2,\dots,n\}$, we have $W_\xi = V_1\oplus V_2$ and $V_2 = V_1^\perp\cap W_\xi$. One can verify that ① $I''_0(z_\xi)|_{V_1}$ is negative definite; ② $I''_0(z_\xi)|_{V_2}$ is positive definite. Therefore, $PI''_0(z_\xi)|_{W_\xi}$ is invertible.

Lemma 3 Assumption (7) in Section 1 holds.

Proof Direct computation yields

$$\begin{aligned}(I''_\varepsilon(z_\xi+w) - I''_\varepsilon(z_\xi))[(\phi_1,\phi_2)][(\psi_1,\psi_2)] \leqslant\\ C\Big(\int_{\mathbf{R}^3}|w_1|^{2q} + |w_1|\big(\sum_{i=1}^n U_{\xi_i}\big)^{2q-1} +\\ |w_1|^{2q-1}\sum_{i=1}^n U_{\xi_i}\mathrm{d}x\Big)^{(2q-2)/(2q)}\|\varphi_1\|\,\|\psi_1\| +\\ C\Big(\int_{\mathbf{R}^3}|w_2|^{2q} + |w_2|\big(\sum_{i=1}^n V_{\xi_i}\big)^{2q-1} +\\ |w_2|^{2q-1}\sum_{i=1}^n V_{\xi_i}\mathrm{d}x\Big)^{(2q-2)/(2q)}\|\varphi_2\|\,\|\psi_2\| +\\ C\varepsilon\|\varphi\|\,\|\psi\|\end{aligned}$$

Using the Sobolev embedding theorem, for sufficiently small ε, we have

$$\begin{aligned}(I''_\varepsilon(z_\xi+w) - I''_\varepsilon(z_\xi))[(\phi_1,\phi_2)][(\psi_1,\psi_2)]\leqslant\\ C(|w|_{L^{2q}}^{2q-2} + |w|_{L^{2q}}^{(2q-2)/(2q)} + |w|_{L^{2q}}^{(2q-1)(2q-2)/(2q)})\|\phi\|\,\|\psi\| +\\ C\varepsilon\|\phi\|\,\|\psi\|\leqslant C(\|w\|^{(2q-2)/(2q)}+\varepsilon)\|\phi\|\,\|\psi\|\end{aligned}$$

From Lemma 1 and $\|w\|\leqslant 2\gamma\|I'_\varepsilon(z)\|$, we obtain assumption (7).

Lemma 4 There exists $0<\varepsilon_3<\varepsilon_2$ such that for $0<\varepsilon<\varepsilon_3$ the auxiliary equation $PI'_\varepsilon(z_\xi+w)=0$ has a unique $w_{\varepsilon,\xi}\in W_\xi$ satisfying $\|w_{\varepsilon,\xi}\|\to 0$ as $\varepsilon\to 0$ uniformly for $\xi\in T_\varepsilon$.

Proof According to Lemma 2, the map S_ε defined by (8) makes sense. One can easily verify that S_ε is a contraction and maps $B_{\varepsilon,\gamma}$ into itself. Hence the auxiliary equation has a unique solution $w_{\varepsilon,\xi}\leqslant 2\gamma\|I'_\varepsilon(z_\xi)\|$ for ε sufficiently small.

3 The Regularity of $w_{\varepsilon,\xi}$ and Estimation of Its Derivative

In this section we will study the regularity of $w_{\varepsilon,\xi}$ and estimate its derivative, and then we will use the critical point of the reduced functional to construct the solution of (2).

Lemma 5 For $0<\varepsilon<\varepsilon_3$, $w_{\varepsilon,\xi}$ is C^1 with respect to $\xi\in T_\varepsilon$.

We omit the proof.

In the following we estimate $\partial_\xi w_{\varepsilon,\xi}$.

Lemma 6 There holds $\partial_\xi w_{\varepsilon,\xi}\to 0$ as $\varepsilon\to 0$ for $\xi\in T_\varepsilon$.

Proof Since

$$0 = \frac{\partial H(\xi, w_{\varepsilon,\xi}, a, \varepsilon)}{\partial\xi} + \frac{\partial H(\xi, w_{\varepsilon,\xi}, a,\varepsilon)}{\partial(w_\varepsilon(\xi),\alpha)}\frac{\partial(w_{\varepsilon,\xi}, a)}{\partial\xi}$$

we have

$$\left\|\frac{\partial w_{\varepsilon,\xi}}{\partial\xi}\right\|\leqslant C\left(\left\|I'_\varepsilon(z_\xi + w_{\varepsilon,\xi})\left[\frac{\partial z_\xi}{\partial\xi}\right]\right\| + |a| + \|w_{\varepsilon,\xi}\|\right) \tag{9}$$

We estimate each term in the right hand side of (9). From Lemma 4, we have

$$\|w_{\varepsilon,\xi}\|\leqslant C\|I'_\varepsilon(\xi)\| \tag{10}$$

Secondly,

$$|a_i^\alpha|\leqslant C(\|w_{\varepsilon,\xi}\| + \|I'_\varepsilon(\xi)\|)\leqslant C\|I'_\varepsilon(\xi)\| \tag{11}$$

Finally, we can deduce

$$\left\|I''_\varepsilon(z_\xi+w_{\varepsilon,\xi})\left[\frac{\partial z_\xi}{\partial\xi}\right]\right\|\leqslant C(O_\varepsilon(1) + \|w_\varepsilon(\xi)\|^{(2q-2)/(2q)} + \varepsilon) \tag{12}$$

Using Lemma 1 and combining (9) to (12), we obtain $\left\|\dfrac{\partial w_\varepsilon(\xi)}{\partial\xi}\right\|\to 0$ as $\varepsilon\to 0$ uniformly for $\xi\in T_\varepsilon$.

Defining $\tilde{I}_\varepsilon(\xi) = I_\varepsilon(z_\xi + w_\varepsilon(\xi))$, we have the following lemma.

Lemma 7 If $\xi\in T_\varepsilon$ is a critical point of $\tilde{I}_\varepsilon$, then $z_\xi + w_\varepsilon(\xi)$ is a critical point of I_ε.

Proof We consider manifold $Z_\varepsilon = \{z_\xi + w_\varepsilon(\xi): \xi\in T_\varepsilon\}$. Since $\xi\in T_\varepsilon$ is a critical point of $\tilde{I}_\varepsilon$, then one has that $(I_\varepsilon(z_\xi + w_\varepsilon(\xi))|_{Z_\varepsilon})' = 0$, namely,

$$I'_\varepsilon(z_\xi + w_\varepsilon(\xi))[D_\alpha^i z_\xi + D_\alpha^i w_\varepsilon(\xi)] = 0 \tag{13}$$

Since $I'_\varepsilon(z_\xi + w_\varepsilon(\xi))\in T_{z_\xi}Z$, one has that

$$\| I'_\varepsilon(z_\xi + w_\varepsilon(\xi)) \| = \sup_{v\in T_\xi Z} | I'_\varepsilon(z_\xi + w_\varepsilon(\xi))[v] | \tag{14}$$

Therefore, using Lemma 6 and combining (13) and (14), one has that $\| I'_\varepsilon(z_\xi + w_\varepsilon(\xi)) \| \leqslant O_\varepsilon(1) \| I'_\varepsilon(z_\xi + w_\varepsilon(\xi)) \|$ for ε small enough, which implies $I'_\varepsilon(z_\xi + w_\varepsilon(\xi)) = 0$.

4 Proof of Theorem 1

In this section, we will prove that the reduced functional has a critical point in T_ε for ε small enough in order to complete the proof of Theorem 1.

We consider the reduced functional

$$\tilde{I}_\varepsilon(\xi) = I_\varepsilon(z_\xi + w_{\varepsilon,\xi}) = I_\varepsilon(z_\xi) + I'_\varepsilon(z_\xi)[w_\varepsilon, \xi] + O(\| w_{\varepsilon,\xi} \|^2)$$

From Lemma 1 and Lemma 4 it follows that

$$\tilde{I}_\varepsilon(\xi) = I_1(z_{1\xi}) + I_2(z_{2\xi}) + \frac{\varepsilon}{q}\int_{\mathbf{R}^3} z_{1\xi}^q z_{2\xi}^q \mathrm{d}x + O\Big(\varepsilon^2\Big(\int_{\mathbf{R}^3} b(x) z_{1\xi}^q z_{2\xi}^q \mathrm{d}x\Big)^{(2q-2)/q}\Big) + O\Big(\sum_{i<j}\int_{\mathbf{R}^3} U_{\xi_i}^{2q-1} U_{\xi_j}\mathrm{d}x\Big)^{(2q-1)/q} + O\Big(\sum_{i<j}\int_{\mathbf{R}^3} V_{\xi_i}^{2q-1} V_{\xi_j}\mathrm{d}x\Big)^{(2q-1)/q}$$

where

$$I_1(z_{1\xi}) = \frac{1}{2}(z_{1\xi} | z_{1\xi}) - \frac{1}{2q}\int_{\mathbf{R}^3}(z_{1\xi})^{2q}\mathrm{d}x = nc_0 + \frac{n}{2q}\| U \|_{L^{2q}}^{2q} - \frac{1}{2q}\| z_{1\xi} \|_{L^{2q}}^{2q} + \sum_{i<j}\int_{\mathbf{R}^3} U_{\xi_i}^{2q-1} U_{\xi_j}\mathrm{d}x$$

with $c_0 = I_1(U)$ and

$$I_2(z_{2\xi}) = n\tilde{c}_0 + \frac{n}{2q}\| V \|_{L^{2q}}^{2q} - \frac{1}{2q}\| z_{2\xi} \|_{L^{2q}}^{2q} + \sum_{i<j}\int_{\mathbf{R}^3} V_{\xi_i}^{2q-1} V_{\xi_j}\mathrm{d}x$$

with $\tilde{c}_0 = I_2(V)$.

In the following, we argue by the method introduced in Refs.[19] and [20]. We define $M_\varepsilon = \sup\{\tilde{I}_\varepsilon(\xi): \xi \in T_\varepsilon\}$. One can deduce the following lemma.

Lemma 8 Assume that $n \geqslant 2$. Then there exists $0 < \varepsilon_4 < \varepsilon_3$ such that for $0 < \varepsilon < \varepsilon_4$, $M_\varepsilon > \sup\Big\{\tilde{I}_\varepsilon(\xi): \xi \in T_\varepsilon$ and $| \xi_i - \xi_j | \in \Big[(1-\delta)\ln\frac{1}{\varepsilon}, (1-\delta)\ln\frac{1}{\varepsilon}+1\Big]$ for some $i \neq j\Big\}$.

For any $0 < \varepsilon < \varepsilon_4$, let $\xi^k(\varepsilon) = (\xi_1^k(\varepsilon), \xi_2^k(\varepsilon), \ldots, \xi_n^k(\varepsilon)) \in T_\varepsilon$ be a maximizing sequence of $\tilde{I}_\varepsilon$ for $k = 1, 2, \ldots$. Then Lemma 8 implies that $\inf_k \min_{i\neq j} | \xi_i^k(\varepsilon) - \xi_j^k(\varepsilon) | \geqslant \Big((1-\delta)\ln\frac{1}{\varepsilon}+1\Big)$. Therefore, passing to subsequence if necessary, we may assume that either $\lim_{k\to\infty}\xi_i^k(\varepsilon) = \xi_i^0(\varepsilon) \in \mathbf{R}^3$ with $| \xi_i^0(\varepsilon) - \xi_j^0(\varepsilon) | \geqslant \Big((1-\delta)\ln\frac{1}{\varepsilon}+1\Big)$ for $i\neq j$ or $\lim_{k\to\infty} | \xi_i^k(\varepsilon) | = \infty$. Define $\pi(\varepsilon) = \{1 \leqslant i \leqslant n: | \xi_i^k(\varepsilon) | \to \infty \text{ as } k\to\infty\}$ for $0 < \varepsilon < \varepsilon_4$.

Lemma 9 Assume that $n \geqslant 2$. Then there exists $0 < \varepsilon(n) < \varepsilon_4$ such that for $0 < \varepsilon < \varepsilon(n)$ $\pi(\varepsilon) = \varnothing$.

Proof We argue by contradiction and assume that $\pi(\varepsilon) \neq \varnothing$ along a sequence $\varepsilon_m \to 0$. Since $\pi(\varepsilon_m) \supset \pi(\varepsilon_{m+1})$, we may assume that $\pi(\varepsilon_m) = \{1, 2, \ldots, j_n\}$ for all $m \in \mathbf{N}$ and for $1 \leqslant j_n \leqslant n$. We consider two cases: 1) $j_n < n$; 2) $j_n = n$. For convenience of notations, we shall denote $\varepsilon = \varepsilon_m$ and $\xi_i^k = \xi_i^k(\varepsilon_m)$.

1) $j_n < n$

Denoting

$$\xi_*^k = (\xi_{j_n+1}^k, \xi_{j_n+2}^k, \ldots, \xi_n^k)$$
$$\xi_*^k \to \xi_*^0 = (\xi_{j_n+1}^0, \xi_{j_n+2}^0, \ldots, \xi_n^0)$$

we have

$$\tilde{I}_\varepsilon(\xi_*^k) = (n-j_n)c_0 - \frac{1}{2q}\| z_{1\xi_*^k} \|_{L^{2q}}^{2q} + \frac{n-j_n}{2q}\| U \|_{L^{2q}}^{2q} + \sum_{j_n+1\leqslant i<j}\int_{\mathbf{R}^3} U_{\xi_i^k}^{2q-1} U_{\xi_j^k}\mathrm{d}x + (n-j_n)\tilde{c}_0 - \frac{1}{2q}\| z_{2,\xi_*^k} \|_{L^{2q}}^{2q} + \frac{n-j_n}{2q}\| V \|_{L^{2q}}^{2q} + \sum_{j_n+1\leqslant i<j}\int_{\mathbf{R}^3} V_{\xi_i^k}^{2q-1} V_{\xi_j^k}\mathrm{d}x + \frac{\varepsilon}{q}\int_{\mathbf{R}^3} b(x) z_{1\xi_*^k}^q z_{2\xi_*^k}^q \mathrm{d}x + O(\varepsilon^{(2q-1)/q\tilde{\omega}(1-\delta)})$$

Hence

$$\tilde{I}_\varepsilon(\xi^k) - \tilde{I}_\varepsilon(\xi_*^k) = j_n c_0 - \frac{1}{2q}\| z_{1\xi^k} \|_{L^{2q}}^{2q} + \frac{1}{2q}\| z_{1\xi_*^k} \|_{L^{2q}}^{2q} + \frac{j_n}{2q}\| U \|_{L^{2q}}^{2q} + j_n\tilde{c}_0 - \frac{1}{2q}\| z_{2\xi^k} \|_{L^{2q}}^{2q} + \frac{1}{2q}\| z_{2\xi_*^k} \|_{L^{2q}}^{2q} + \frac{j_n}{2q}\| V \|_{L^{2q}}^{2q} + \sum_{i<j}\int_{\mathbf{R}^3} U_{\xi_i^k}^{2q-1} U_{\xi_j^k}\mathrm{d}x - \sum_{j_n+1\leqslant i<j}\int_{\mathbf{R}^3} U_{\xi_i^k}^{2q-1} U_{\xi_j^k}\mathrm{d}x + \sum_{i<j}\int_{\mathbf{R}^3} V_{\xi_i^k}^{2q-1} V_{\xi_j^k}\mathrm{d}x - \sum_{j_n+1\leqslant i<j}\int_{\mathbf{R}^3} V_{\xi_i^k}^{2q-1} V_{\xi_j^k}\mathrm{d}x + \frac{\varepsilon}{q}\int_{\mathbf{R}^3} b(x)(z_{1\xi^k}^q z_{2\xi^k}^q - z_{1\xi_*^k}^q z_{2\xi_*^k}^q)\mathrm{d}x + O(\varepsilon^{(2q-1)/q\tilde{\omega}(1-\delta)}) \tag{15}$$

We have

$$-\frac{1}{2q}\| z_{1\xi^k} \|_{L^{2q}}^{2q} + \frac{1}{2q}\| z_{1\xi_*^k} \|_{L^{2q}}^{2q} + \frac{j_n}{2q}\| U \|_{L^{2q}}^{2q} \leqslant -\frac{2q-1}{q}\sum_{i=1}^{j_n}\sum_{j=i+1}^{n}\int_{\mathbf{R}^3} U_{\xi_i^k}^{2q-1} U_{\xi_j^k}\mathrm{d}x \tag{16}$$

$$-\frac{1}{2q}\| z_{2\xi^k} \|_{L^{2q}}^{2q} + \frac{1}{2q}\| z_{2\xi_*^k} \|_{L^{2q}}^{2q} + \frac{j_n}{2q}\| V \|_{L^{2q}}^{2q} \leqslant -\frac{2q-1}{q}\sum_{i=1}^{j_n}\sum_{j=i+1}^{n}\int_{\mathbf{R}^3} V_{\xi_i^k}^{2q-1} V_{\xi_j^k}\mathrm{d}x \tag{17}$$

By (15), (16) and (17), we have

$$\tilde{I}_\varepsilon(\xi^k) - \tilde{I}_\varepsilon(\xi_*^k) \leqslant j_n c_0 + j_n\tilde{c}_0 + \frac{\varepsilon}{q}\int_{\mathbf{R}^3} b(x)(z_{1\xi^k}^q z_{2\xi^k}^q - z_{1\xi_*^k}^q z_{2\xi_*^k}^q)\mathrm{d}x + O(\varepsilon^{(2q-1)/q\tilde{\omega}(1-\delta)})$$

Letting $k\to\infty$ and noticing that there holds $|\xi_i^k|\to\infty$ for $i=1,2,\dots,j_n$, we have

$$M_\varepsilon\leqslant j_nc_0+j_n\tilde{c}_0+\tilde{I}_\varepsilon(\xi_*^0)+O(\varepsilon^{(2q-1)/q\tilde{\omega}(1-\delta)}) \quad (18)$$

On the other hand, let $\xi_s^\varepsilon=\left((4s-2n-2)\dfrac{2q-1}{2q}(1-\delta)\ln\dfrac{1}{\varepsilon},0,0\right)\in\mathbf{R}^3$, $s=1,2,\dots,n$, then the open balls $B_\varepsilon\left(\xi_s^\varepsilon,\dfrac{2q-1}{q}(1-\delta)\ln\dfrac{1}{\varepsilon}\right)$ are mutually disjoint. Thus there are j_n integers from $\{1,2,\dots,n\}$ denoted by $s_1<s_2<\dots<s_{j_n}$ such that for $i=1,\dots,j_n$, $j=j_n+1,\dots,n$, we have

$$|\xi_{s_i}^\varepsilon-\xi_j^0|\geqslant\frac{2q-1}{q}(1-\delta)\ln\frac{1}{\varepsilon} \quad (19)$$

for $i=1,2,\dots,j_n$; $j=j_n+1,\dots,n$.

Letting $\xi_i^\varepsilon=\xi_{s_i}^\varepsilon$, we have

$$|\xi_i^\varepsilon-\xi_j^\varepsilon|>\frac{2q-1}{q}(1-\delta)\ln\frac{1}{\varepsilon}\qquad 1\leqslant i<j\leqslant j_n \quad (20)$$

$$|\xi_i^\varepsilon|\leqslant(n-1)\frac{2q-1}{q}(1-\delta)\ln\frac{1}{\varepsilon}\qquad i=1,2,\dots,j_n \quad (21)$$

Letting $\xi^\varepsilon=(\xi_1^\varepsilon,\dots,\xi_{j_n}^\varepsilon,\xi_{j_n+1}^0,\dots,\xi_n^0)$ and using (16) and (17), we have $\xi^\varepsilon\in T_\varepsilon$. Denoting ξ_j^ε by ξ_j^0 for $j_n+1\leqslant j\leqslant n$, we have

$$\begin{aligned}\tilde{I}_\varepsilon(\xi^\varepsilon)-\tilde{I}_\varepsilon(\xi_*^0)&=j_nc_0-\frac{1}{2q}\|z_{1\xi^\varepsilon}\|_{L^{2q}}^{2q}+\\&\frac{1}{2q}\|z_{1\xi_*^0}\|_{L^{2q}}^{2q}+\frac{j_n}{2q}\|U\|_{L^{2q}}^{2q}+j_n\tilde{c}_0-\\&\frac{1}{2q}\|z_{2\xi^\varepsilon}\|_{L^{2q}}^{2q}+\frac{1}{2q}\|z_{2\xi_*^0}\|_{L^{2q}}^{2q}+\frac{j_n}{2q}\|V\|_{L^{2q}}^{2q}+\\&\sum_{i<j}\int_{\mathbf{R}^3}U_{\xi_i^\varepsilon}^{2q-1}U_{\xi_j^\varepsilon}\mathrm{d}x-\sum_{j_n+1\leqslant i<j}\int_{\mathbf{R}^3}U_{\xi_i^\varepsilon}^{2q-1}U_{\xi_j^\varepsilon}\mathrm{d}x+\\&\sum_{i<j}\int_{\mathbf{R}^3}V_{\xi_i^\varepsilon}^{2q-1}V_{\xi_j^\varepsilon}\mathrm{d}x-\sum_{j_n+1\leqslant i<j}\int_{\mathbf{R}^3}V_{\xi_i^\varepsilon}^{2q-1}V_{\xi_j^\varepsilon}\mathrm{d}x+\\&\frac{\varepsilon}{q}\int_{\mathbf{R}^3}b(x)(z_{1\xi^\varepsilon}^qz_{2\xi^\varepsilon}^q-z_{1\xi_*^0}^qz_{2\xi_*^0}^q)\mathrm{d}x+O(\varepsilon^{(2q-1)/q\tilde{\omega}(1-\delta)})\end{aligned} \quad (22)$$

We have

$$-\frac{1}{2q}\|z_{1\xi^\varepsilon}\|_{L^{2q}}^{2q}+\frac{1}{2q}\|z_{1\xi_*^0}\|_{L^{2q}}^{2q}+\frac{j_n}{2q}\|U\|_{L^{2q}}^{2q}\geqslant-C\sum_{i=1}^{j_n}\sum_{j=i+1}^{n}\int_{\mathbf{R}^3}U_{\xi_i^\varepsilon}^{2q-1}U_{\xi_j^\varepsilon}\mathrm{d}x \quad (23)$$

$$-\frac{1}{2q}\|z_{2\xi^\varepsilon}\|_{L^{2q}}^{2q}+\frac{1}{2q}\|z_{2\xi_*^0}\|_{L^{2q}}^{2q}+\frac{j_n}{2q}\|V\|_{L^{2q}}^{2q}\geqslant-C\sum_{i=1}^{j_n}\sum_{j=i+1}^{n}\int_{\mathbf{R}^3}V_{\xi_i^\varepsilon}^{2q-1}V_{\xi_j^\varepsilon}\mathrm{d}x \quad (24)$$

Using (19) and (20), we have

$$-\int_{\mathbf{R}^3}U_{\xi_i^\varepsilon}^{2q-1}U_{\xi_j^\varepsilon}\mathrm{d}x\geqslant-C\varepsilon^{(2q-1)/q(1-\delta)} \quad (25)$$

$$-\int_{\mathbf{R}^3}V_{\xi_i^\varepsilon}^{2q-1}V_{\xi_j^\varepsilon}\mathrm{d}x\geqslant-C\varepsilon^{(2q-1)/q(1-\delta)\omega} \quad (26)$$

Using (22) to (26), we have

$$\begin{aligned}\tilde{I}_\varepsilon(\xi^\varepsilon)-\tilde{I}_\varepsilon(\xi_*^0)\geqslant{}&j_nc_0+j_n\tilde{c}_0-C\varepsilon^{(2q-1)/q\tilde{\omega}(1-\delta)}+\\&\frac{\varepsilon}{q}\int_{\mathbf{R}^3}b(x)(z_{1\xi^\varepsilon}^qz_{2\xi^\varepsilon}^q-z_{1\xi_*^0}^qz_{2\xi_*^0}^q)\mathrm{d}x\end{aligned} \quad (27)$$

Using assumption of $b(x)$ and (21), we have

$$\frac{\varepsilon}{q}\int_{\mathbf{R}^3}b(x)(z_{1\xi^\varepsilon}^qz_{2\xi^\varepsilon}^q-z_{1\xi_*^0}^qz_{2\xi_*^0}^q)\mathrm{d}x\geqslant C\varepsilon^{1+(2q-1)/q(1-\delta)\sigma(n-1)} \quad (28)$$

Combining (27) and (28), we have

$$\tilde{I}_\varepsilon(\xi^\varepsilon)-\tilde{I}_\varepsilon(\xi_*^0)\geqslant j_nc_0+j_n\tilde{c}_0-C\varepsilon^{(2q-1)/q\tilde{\omega}(1-\delta)}+C\varepsilon^{1+(2q-1)/q(1-\delta)\sigma(n-1)}$$

According to the choice of δ, when ε is small enough, we have

$$M_\varepsilon\geqslant\tilde{I}_\varepsilon(\xi^\varepsilon)\geqslant j_nc_0+j_n\tilde{c}_0+\tilde{I}_\varepsilon(\xi_*^0)+C\varepsilon^{1+(2q-1)/q(1-\delta)\sigma(n-1)}$$

which contradicts (18).

2) $j_n=n$

Taking $\xi^\varepsilon=(\xi_1^\varepsilon,\xi_2^\varepsilon,\dots,\xi_n^\varepsilon)$, where $\xi_i^\varepsilon=\left((4i-2n-2)(2q-1)/(2q)(1-\delta)\ln\dfrac{1}{\varepsilon},0,0\right)\in\mathbf{R}^3$ $i=1,2,\dots,n$, we have

have

$$M_\varepsilon\leftarrow\tilde{I}_\varepsilon(\xi^k)\leqslant nc_0+n\tilde{c}_0+C\varepsilon^{(2q-1)/q\tilde{\varepsilon}(1-\delta)}$$

$$M_\varepsilon\geqslant\tilde{I}_\varepsilon(\xi^\varepsilon)\geqslant nc_0+n\tilde{c}_0+C\varepsilon^{1+(2q-1)/q(1-\delta)\sigma(n-1)}$$

It follows from the two cases that $\pi(\varepsilon)=\varnothing$.

Proof of Theorem 1 For $n\geqslant2$, according to Lemma 9, if $0<\varepsilon<\varepsilon(n)$, $\tilde{I}_\varepsilon$ can achieve its maximum at some point $\xi=(\xi_1,\xi_2,\dots,\xi_n)\in T_\varepsilon$. Hence, according to Lemma 7 we know that $z_\xi+w_\varepsilon(\xi)$ is a critical point of I_ε.

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薛定谔方程组多峰解的存在性

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摘要：Schrödinger 方程 $-\Delta u + \lambda^2 u = |u|^{2q-2}u$ 有唯一的正径向对称解 U_λ，当 $r \to \infty$ 时 U_λ 指数衰减到零. 因此可以预料薛定谔方程组 $-\Delta u_1 + u_1 = |u_1|^{2q-2}u_1 - \varepsilon b(x)|u_2|^q|u_1|^{q-2}u_1$，$-\Delta u_2 + u_2 = |u_2|^{2q-2}u_2 - \varepsilon b(x)|u_1|^q|u_2|^{q-2}u_2$ 存在在某些点附近形同 U_λ 的多峰解. 对于 $u = (u_1, u_2) \in H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$ 定义非线性泛函 $I_\varepsilon(u) = I_1(u_1) + I_2(u_2) - \dfrac{\varepsilon}{q}\int_{\mathbf{R}^3} b(x)|u_1|^q|u_2|^q \mathrm{d}x$，其中 $I_1(u_1) = \dfrac{1}{2}\|u_1\|^2 - \dfrac{1}{2q}\int_{\mathbf{R}^3}|u_1|^{2q}\mathrm{d}x$，$I_2(u_2) = \dfrac{1}{2}\|u_2\|_\omega^2 - \dfrac{1}{2q}\int_{\mathbf{R}^3}|u_2|^{2q}\mathrm{d}x$. 证明了此泛函的临界点就是薛定谔方程组的解. 设 Z 为非扰动问题的解流形，T_zZ 为此流形的切空间. 寻求 I_ε 的形如 $z + w$ 的临界点，其中 $w \in (T_zZ)^\perp$. 应用 I_ε 的性质，证明了 I_ε 存在近似于 $\left(\sum_{i=1}^n U(x - \xi_i), \sum_{i=1}^n V(x - \xi_i)\right)$ 的多峰解.

关键词：耦合薛定谔方程组；多峰解；变分约化方法

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