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Crossed modules of Lie color algebras

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Abstract: The linear operations of the equivalent classes of crossed modules of Lie color algebras are studied. The set of the equivalent classes of crossed modules is proved to be a vector space, which is isomorphic with the homogeneous components of degree zero of the third cohomology group of Lie color algebras. As an application of this theory, the crossed modules of Witt type Lie color algebras is described, and the result is proved that there is only one equivalent class of the crossed modules of Witt type Lie color algebras when the abelian group Γ is equal to Γ^+. Finally, for a Witt type Lie color algebra, the classification of its crossed modules is obtained by the isomorphism between the third cohomology group and the crossed modules.

Key words: crossed modules of Lie color algebras; Witt type Lie color algebra; third cohomology; isomorphism
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Lie color algebras[1] are the natural generalizations of Lie algebras and Lie superalgebras, which are special cases of Hopf algebras in braided monoidal categories[2]. The enveloping algebras of Lie color algebras provide a large class of examples of graded Hopf algebras which are neither commutative nor cocommutative. In recent years, Lie color algebras have become an interesting field of mathematics and physics. The cohomology group of Lie color algebras were introduced by Scheunert and Zhang[3]. The crossed modules of Lie color algebras were studied by Pei and Zhou[4] for further description of the cohomology group, which are generalizations of the crossed modules of Lie algebras and special cases of the crossed products of Hopf algebras.

Our present work is motivated by the results and methods in the Lie algebras case[5]. This paper introduces the linear operations of the equivalent classes of the crossed modules of Lie color algebras. It mainly relates the crossed modules with the third cohomology group of Lie color algebras. And its main theorem claims that the crossed modules of Lie color algebras classify the corresponding third cohomology group, which extends our previous results[5].

The reader is referred to Scheunert[1] as general references about Lie color algebras, to Scheunert and Zhang[3] about cohomology groups, to Kassel and Loday[6] about the crossed modules and to Zhou[7] about the Witt type Lie color algebras. Throughout this paper, k is assumed to be a field of characteristic zero; $k^* = k \setminus \{0\}$ is the group of units of k; spaces, algebras and modules are all over k. Assume that Γ is an abelian group and ε is the skew-symmetric bicharacter on Γ. The crossed module of a Lie color algebra L is denoted by (R, L, ∂) where $\partial: R \to L$ is a homomorphism of Lie color algebras. The equivalent class of (R, L, ∂) is denoted by $[(R, L, \partial)]$. Let P be another Lie color algebra and M be a P-module; then the set of equivalent classes is denoted by $S(P, L; M)$, and the third cohomology group of the chain complex $C^2(P, L; M)$ is denoted by $H^3(P, L; M)$ where $R = M \oplus N$, $N = \operatorname{Ker} \nu$, $\nu: L \to P$ is a surjective homomorphism of Lie color algebras.

1 Crossed Modules of Lie Color Algebras

Lemma 1[4] Let $[(R, L, \partial)]$ be an equivalent class in $S(P, L; M)$, and then there exists a crossed module $(\bar{R}, L, \bar{\partial})$ in $[(R, L, \partial)]$ such that $\bar{\partial}(m, n) = n$.

Let $[(R_1, L, \partial_1)]$ and $[(R_2, L, \partial_2)]$ be two equivalent classes in $S(P, L; M)$. Two representative elements are taken from the two classes respectively, denoted by (R_1, L, ∂_1) and (R_2, L, ∂_2), such that $\partial_1(m, n) = \partial_2(m, n) = n$. We can construct a new crossed module (R', L, ∂') as follows:

$$[(m, n), (m', n')]' = [(m, n), (m', n')]_1 + [(m, n), (m', n')]_2 - (0, [n, n])$$
$$L \times R \to R, \quad x(m, n) = x(m, n) + x(m, n) - ((\nu x)m, [x, n])$$
$$\partial': R \to L, \quad (m, n) \mapsto n$$

where $R = R'$ is a vector space. Define the operation of addition in the set $S(P, L; M)$ by

$$[(R_1, L, \partial_1)] + [(R_2, L, \partial_2)] = [(R', L, \partial')]$$

It is not difficult to verify that the definition is well-defined.

Similarly, for any $a \in k$ and the equivalent class $[(R, L, \partial)]$, one can define the operation of scalar multiplica-

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tion in $S(P,L;M)$ by

$$a[(R,L,\partial)]=[(R_a,\ L,\ \partial_a)]$$

where $R_a=R$ is a vector space, and the crossed module structure is defined as

$$[(m,n),\ (m',n')]_a=a[(m,n),\ (m',n')]-(a-1)(0,\ [n,n'])$$
$$L\times R_a\to R_a,\ x_a(m,n)=ax(m,n)-(a-1)((vx)m,\ [x,n])$$
$$\partial_a:R_a\to L,\ (m,n)\mapsto n$$

Proposition 1 Let $\nu: L\to P$ be a surjective homomorphism of Lie color algebras and M a P-module. Then $S(P,L;M)$ is a vector space under the linear operations above.

Proof Let $a=0$ in the operation of scalar multiplication. One can directly check that $[(R_0,\ L,\ \partial_0)]$ is the zero element in $S(P,L;M)$ and thus show that $S(P,L;M)$ is not an empty set. For any equivalent class $[(R,L,\partial)]$ in $S(P,L;M)$, $[(R_{-1},\ L,\ \partial_{-1})]$ is the negative element of $[(R,L,\partial)]$. It is easy to verify the other rules of vector spaces.

Theorem 1 Let $\nu: L\to P$ be a surjective homomorphism of Lie color algebras and M a P-module. Then there is an isomorphism between vector spaces of the homogeneous components of degree zero of the third cohomology $H^3(P,\ L;\ M)$ and the equivalent classes of crossed modules $S(P,L;M)$.

Proof Straightforward from Proposition 1 and Theorem 1[5].

2 The Crossed Modules of Witt Type Lie Color Algebras

Let $f:\Gamma\to k^*$ be a function, $S(f)=\{\alpha\in\Gamma\mid f(\alpha)\neq 0\}$ be the support of f and $L=\bigoplus_{\alpha\in\Gamma}ke_\alpha$ be a Γ-graded vector space together with a basis $\{e_\alpha\mid\alpha\in\Gamma\}$ and a multiplication $[e_\alpha,e_\beta]=(f(\beta)-\varepsilon(\alpha,\beta)f(\alpha))e_{\alpha+\beta}$. Suppose that ε is non-degenerate and Γ is 2-torsion free. L is called a Witt type Lie color algebra if

$$f(\alpha)=0 \text{ or } f(0)$$
$$(f(\beta)-\varepsilon(\alpha,\beta)f(\alpha))(f(\alpha+\beta)-f(\alpha)-f(\beta)+f(0))=0$$

Now we describe the crossed modules of Witt type Lie color algebras in the case of $S(f)=\Gamma$. Without loss of generality, let $f(0)=1$. Then the multiplication in L becomes $[e_\alpha,e_\beta]=(1-\varepsilon(\alpha,\beta))e_{\alpha+\beta}$. Let $P=\dfrac{L}{ke_0}$, M be a one-dimensional P-module, and then we will study $S(P,L;M)$ in the following part. Suppose that (R,L,∂) is such a crossed module, and then $\dim R=2$ by the four term exact sequence. Let $M=\{r\}$, $R=\{r,r'\}$, and then $[r,r']=0$, $\partial\neq 0$ by the definition of crossed modules. Since $\operatorname{Im}\partial=\ker\nu$ is a one-dimensional ideal of L, let $\operatorname{Im}\partial=ke_\alpha$ for some $\alpha\in\Gamma$. Therefore, for all $\beta\in\Gamma$, $[e_\alpha,e_\beta]=(1-\varepsilon(\alpha,\beta))\cdot e_{\alpha+\beta}\in ke_\alpha$, it follows that $\varepsilon(\alpha,\beta)=1$ for all $\beta\in\Gamma$. Thus $\alpha=0$ and $\operatorname{Im}\partial=ke_0$ since ε is non-degenerate. So we can take the crossed modules of L as follows:

$$e_\alpha\cdot r=k_\alpha r+l_\alpha r',\ e_\alpha\cdot r'=k'_\alpha r+l'_\alpha r'$$
$$\partial(r)=0,\ \partial(r')=ae_0\qquad a\neq 0$$

where $k_\alpha,l_\alpha,k'_\alpha,l'_\alpha,a\in k$ for all $\alpha\in\Gamma$. Since (R,L,∂) is a crossed module, then

$$l_\alpha ae_0=\partial(e_\alpha\cdot r)=[e_\alpha,\partial(r)]=0$$
$$l'_\alpha ae_0=\partial(e_\alpha\cdot r')=[e_\alpha,\partial(r)]=0$$

It follows that $l_\alpha=l'_\alpha=0$. Hence

$$e_\alpha\cdot r=k_\alpha r,\ e_\alpha\cdot r'=k'_\alpha r\qquad \alpha\in\Gamma \tag{1}$$

Theorem 2 Let $L=\bigoplus_{\alpha\in\Gamma}ke_\alpha$ be a Witt type Lie color algebra, $P=\dfrac{L}{ke_0}$ and M be a one-dimensional P-module. Suppose that $\Gamma_+=\{\alpha\in\Gamma\mid\varepsilon(\alpha,\alpha)=1\}=\Gamma$. Then $S(P,L;M)$ is trivial.

Proof By Eq. (1), we have $ak_0r=ae_0\cdot r=\partial(r')\cdot r=[r',r]=0$. Thus $k_0=0$ since $a\neq 0$. Also we have

$$\left.\begin{aligned}[e_\alpha,e_\beta]\cdot r&=(1-\varepsilon(\alpha,\beta))e_{\alpha+\beta}\cdot r=(1-\varepsilon(\alpha,\beta))k_{\alpha+\beta}r\\ [e_\alpha,e_\beta]\cdot r&=e_\alpha e_\beta r-\varepsilon(\alpha,\beta)e_\beta e_\alpha\cdot r=(1-\varepsilon(\alpha,\beta))k_\alpha k_\beta r\end{aligned}\right\} \tag{2}$$

Thus $(1-\varepsilon(\alpha,\beta))k_{\alpha+\beta}=(1-\varepsilon(\alpha,\beta))k_\alpha k_\beta$. By the assumption that $\Gamma_+=\Gamma$, we have $\varepsilon(\alpha,-\alpha)=-1$ for all $\alpha\in\Gamma$.

Suppose that $\beta=-\alpha$ in Eq. (2), we can derive that $2k_\alpha k_{-\alpha}=0$. Furthermore, $k_\alpha=0$ or $k_{-\alpha}=0$. By taking $k_\alpha=0$ and $\beta=-2\alpha$, we have $3(k_{-\alpha}-k_\alpha k_{-2\alpha})=3k_{-\alpha}=0$, thus $k_{-\alpha}=0$. Since α is an arbitrary element in Γ, $k_\alpha=0$ for all $\alpha\in\Gamma$. Similarly, we can show that $k'_\alpha=0$ for all $\alpha\in\Gamma$. So the action of L on R is trivial.

Suppose that we have two such crossed modules (R,L,∂) and (R',L,∂'), where

$$R=\{r,r'\mid[r,r']=0;L\cdot R=0;\partial(r)=0,\partial(r')=ae_0(a\neq 0)\}$$
$$R'=\{r,r''\mid[r,r'']=0;L\cdot R'=0;\partial(r)=0,\partial(r'')=a'e_0(a'\neq 0)\}$$

It is easy to see that they are equivalent by $f:R\to R',r\mapsto r,r'\mapsto\dfrac{a}{a'}r''$, which is an isomorphism of Lie color algebras. This completes the proof.

Corollary 1 Let $L=\bigoplus_{\alpha\in\Gamma}ke_\alpha$ be a Witt type Lie color algebra, $P=\dfrac{L}{ke_0}$, and M be a one-dimensional P-module. Suppose that $\Gamma_+=\{\alpha\in\Gamma\mid\varepsilon(\alpha,\alpha)=1\}=\Gamma$. Then the homogeneous components of degree zero of $H^3(P,L;M)$ is zero.

Proof Straightforward from Theorem 1 and Theorem 2.

Theorem 3 Let $L=\bigoplus_{\alpha\in\Gamma} ke_\alpha$ be a Witt type Lie color algebra, $P=\frac{L}{ke_0}$, and M be a one-dimensional P-module and $\Gamma_+\neq\Gamma$. Suppose that there are two such crossed modules (R,L,∂) and (R',L,∂') as follows:

$$R=\{r,r'\mid[r,r']=0;e_\alpha\cdot r=k_\alpha r,e_\alpha\cdot r'=k'_\alpha r;\partial(r)=0,\partial(r')=ae_0(a\neq0)\}$$
$$R'=\{r,r''\mid[r,r'']=0;e_\alpha\cdot r=t_\alpha r,e_\alpha\cdot r''=t'_\alpha r;\partial(r)=0,\partial(r'')=a'e_0(a'\neq0)\}$$

Then (R,L,∂) and (R',L,∂') are equivalent if and only if there exists s in k such that

$$k_a=t_a,\ \frac{k'_a}{a}-\frac{t'_a}{a'}=\frac{s}{a}t_a$$

Proof To verify the sufficient condition, let $f:R\to R',r\mapsto r,r'\mapsto\frac{a}{a'}r''$. It is not difficult to check that f satisfies the equivalent conditions of the crossed modules. The verification of the necessary condition is straightforward. This completes the proof.

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李着色代数的交叉模

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摘要:研究了李着色代数的交叉模等价类集合中的线性运算. 证明了交叉模的等价类集合是一个线性空间,而且它与李着色代数的三阶上同调的零次齐次部分空间同构. 作为这个理论的一个应用,刻画了 Witt 型李着色代数的交叉模,当交换群 Γ 等于 Γ^+ 时,证明了 Witt 型李着色代数的交叉模等价类只有一个. 最后,根据三阶上同调与交叉模之间的同构关系,对 Witt 型李着色代数的交叉模进行了分类.

关键词:李着色代数的交叉模;Witt 型李着色代数; 三阶上同调; 同构
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