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Structure theorem for Hopf group-coalgebra

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Abstract: Let π be a group with a unit 1; H is a Hopf π-coalgebra and A is a right π-H-comodule algebra. First, the notion of a two-sided relative (A, H)-Hopf π-comodule is introduced; then it is obtained that $\mathrm{Hom}_A^H(M, N) \otimes H$ and $\mathrm{HOM}_A(M, N)$ are isomorphic as right Hopf π-H-comodules, where $\mathrm{Hom}_A^H(M, N)$ denotes the space of right A-module right H-comodule morphisms and $\mathrm{HOM}_A(M, N)$ denotes the rational space of a space $\mathrm{Hom}_A(M, N)$ of right A-module morphisms. Secondly, the structure theorem of endomorphism algebras of two-sided relative (A, H)-Hopf π-comodules is established; that is, $\mathrm{End}_A^H(M)\#H$ and $\mathrm{END}_A(M, N)$ are isomorphic as right Hopf π-H-comodules and algebras.

Key words: Hopf group-coalgebra; Hopf group-comodule algebra; two-sided relative Hopf group-comodule

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Let H be a Hopf algebra and A a right H-comodule algebra. Let ${}_H\mathscr{M}_A^H$ denote the category of two-sided (A, H)-Hopf modules[1-2]. For any $M \in {}_H\mathscr{M}_A^H$, Ulbrich[2] showed the following result:

$$\mathrm{End}_A^H(M)\#H \cong \mathrm{END}_A(M)$$

as right H-comodule algebras, where $\mathrm{END}_A(M)$ denotes the rational space of a space $\mathrm{End}_A(M)$ of right A-module morphisms and $\mathrm{End}_A^H(M)$ denotes the space of right A-module right H-comodule morphisms.

As a generalization of a Hopf algebra, Turaev[3] introduced and studied the notions of Hopf π-coalgebras. Further study is referred to Virelizier[4] and Wang[5-7]. It is now very natural to ask whether we can extend the main result in Ref. [2] to the setting of Hopf π-coalgebras. This is the motivation of this paper.

1 Preliminaries

In this section we recall some basic definitions and results about Hopf π-coalgebras introduced by Turaev[3]. Throughout this paper, let k be a field. The reader is referred to Sweedler[8] about Hopf algebras.

1.1 Semi-Hopf π-coalgebra

Recall from Turaev[3] that a π-coalgebra is a family of k-spaces $C = \{C_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps $\Delta = \{\Delta_{\alpha,\beta}: C_{\alpha\beta} \rightarrow C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$ and a k-linear map $\varepsilon: C_1 \rightarrow k$, such that Δ is coassociative in the sense that,

$$(\Delta_{\alpha,\beta}\otimes id_{C_\gamma})\Delta_{\alpha\beta,\gamma} = (id_{C_\alpha}\otimes\Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma} \qquad \forall\,\alpha,\beta,\gamma\in\pi$$
$$(id_{C_\alpha}\otimes\varepsilon)\Delta_{\alpha,1} = id_{C_\alpha} = (\varepsilon\otimes id_{C_\alpha})\Delta_{1,\alpha} \qquad \forall\,\alpha\in\pi$$

A semi-Hopf π-coalgebra is a π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon)$ such that each H_α is an algebra with multiplication m_α and unit element $1_\alpha \in H_\alpha$; and for all $\alpha, \beta \in \pi$, $\Delta_{\alpha,\beta}$ and ε: $H_1 \rightarrow k$ are algebra maps.

1.2 Right π-H-comodule algebra

Let C be a π-coalgebra and M a k-vector space. Recall from Wang[5] that a right π-C-comodulelike object is a couple $M = (M, \rho^M = \{\rho_\alpha^M\}_{\alpha\in\pi})$, where for any $\alpha \in \pi$, $\rho_\alpha^M: M \rightarrow M \otimes C_\alpha$ is a k-linear morphism, which is denoted by $\rho_\alpha^M(m) = m_{[0,0]} \otimes m_{[1,\alpha]}$, such that 1) $(\rho_\alpha^M \otimes id_{C_\beta})\rho_\beta^M = (id_M \otimes \Delta_{\alpha,\beta})\rho_{\alpha\beta}^M$, for any $\alpha, \beta \in \pi$; 2) $(id_M \otimes \varepsilon)\rho_1^M = id_M$.

Let H be a semi-Hopf π-coalgebra and A an algebra. A is called a right π-H-comodule algebra if the following conditions hold: 1) A is a right π-H-comodulelike object; 2) $\rho_\alpha^A(ab) = \rho_\alpha^A(a)\rho_\alpha^A(b)$; (3) $\rho_\alpha^A(1_A) = 1_A \otimes 1_\alpha$.

1.3 Relative Hopf π-comodule

Let H be a semi-Hopf π-coalgebra and A a right π-H-comodule algebra. If the k-space M is a right π-H-comodulelike object, and M is a left A-module such that, for any $a \in A$, $m \in M$,

$$\rho_\alpha^M(a \cdot m) = a_{[0,0]} \cdot m_{[0,0]} \otimes a_{[1,\alpha]} m_{[1,\alpha]}$$

then M is called a left-right relative (A, H)-Hopf π-comodule.

Similarly, we can define a right relative (A, H)-Hopf π-comodule. Let A be a right π-H-comodule algebra. Then A is called a left-right (or right) relative (A, H)-Hopf π-comodule with a left (or right) A-module structure given by its multiplication. The category of a left-right (or right) relative (A, H)-Hopf π-comodule is de-

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noted by ${}_{A}\mathscr{M}^{H}$ (or $\mathscr{M}_{A}^{H}$).

2 Structure Theorem of Endomorphism Algebras of Two-Sided Relative (A, H)-Hopf π-Comodule

In this section, we always assume that H is a semi-Hopf π-coalgebra and each component H_α is projective, and A is a right π-H-comodule algebra.

Definition 1 The k-vector space M is called a two-sided relative (A, H)-Hopf π-comodule if 1) M is a left π-H-and right A bimodule; 2) M is a right relative (A, H)-Hopf π-comodule; and 3) M is a left-right Hopf π-H-comodule.

In what follows, ${}_{H}\mathscr{M}_{A}^{H}$ is called the category of two-sided relative (A, H)-Hopf π-comodules. The morphisms in this category are left π-H-module, right A-module and right π-H-comodule maps.

Example 1 Let $M \in \mathscr{M}_{A}^{H}$. Then the tensor product $\{H_\alpha \otimes M\}_{\alpha\in\pi}$ is a two-sided relative (A, H)-Hopf π-comodule whose actions and coactions are given by

$$g \rightarrow (h\otimes m) = gh\otimes m,\ (h\otimes m) \leftarrow a = h\otimes m \cdot a$$
$$\forall\, g, h\in H_\alpha;\ a\in A;\ m\in M$$
$$\rho(h\otimes m) = h_{(1,\alpha)}\otimes m_{[0,0]}\otimes h_{(2,\beta)} m_{(1,\beta)}$$
$$\forall\, h\in H_{\alpha\beta},\ m\in M$$

In particular, $H\otimes A = \{H_\alpha\otimes A\}_{\alpha\in\pi}$ is a two-sided relative (A, H)-Hopf π-comodule.

Let $M, N\in \mathscr{M}_{A}^{H}$. Define $\rho_\beta(f)\in \mathrm{Hom}_A(M, N\otimes H_\beta)$ by

$$\rho_\beta(f)(m) = f(m_{[0,0]})_{[0,0]}\otimes f(m_{[0,0]})_{[1,\beta]} S_{\beta-1}(m_{[1,\beta-1]}) \tag{1}$$

for any $f\in \mathrm{Hom}_A(M, N)$, $m\in M$, where $N\otimes H_\beta$ is a right A-module via $(n\otimes h)\cdot a = n\cdot a\otimes h$ for any $n\in N$, $h\in H_\beta$, $a\in A$. Then, it is easy to see that $\rho_\beta(f)$ is a right A-module map.

A right A-linear f: $M\rightarrow N$ is called rational if there exists an element $f_{[0,0]}\otimes f_{[1,\beta]}\in \mathrm{Hom}_A(M, N)\otimes H_\beta$ such that

$$f_{[0,0]}(m)\otimes f_{[1,\beta]} = f(m_{[0,0]})_{[0,0]}\otimes f(m_{[0,0]})_{[1,\beta]} S_{\beta-1}(m_{[1,\beta-1]}) \tag{2}$$

for any $m\in M$. Define

$$\mathrm{Hom}_A(M, N) = \{f\in \mathrm{Hom}_A(M, N) \mid f \text{ is rational}\}$$

Since each H_β is projective, $\mathrm{Hom}_A(M, N)$ may be viewed as a submodule of $\mathrm{Hom}_A(M, N\otimes H_\beta)$. And by Eqs. (1) and (2), for any $f\in \mathrm{Hom}_A(M, N)$, we know that

$$\rho_\beta(f) = f_{[0,0]}\otimes f_{[1,\beta]}$$

Remark Let $M, N\in \mathscr{M}_{A}^{H}$. Then we can obtain that Eq. (2) is equivalent to

$$\rho_\beta(f) = f_{[0,0]}(m_{[0,0]})\otimes f_{[1,\beta]} m_{[1,\beta]}$$

for any $m\in M$ and $f\in \mathrm{Hom}_A(M, N)$.

As described above, we can easily obtain the following lemmas.

Lemma 1 Let $M, N\in \mathscr{M}_{A}^{H}$. Then the following conclusions hold.

1) $\rho(f) = \{\rho_\beta(f)\in \mathrm{Hom}_A(M, N) \mid \otimes H_\beta\}_{\beta\in\pi}$ for any $f\in \mathrm{Hom}_A(M, N)$, and $\mathrm{Hom}_A(M, N)$ is a right π-H-comodulelike object;

2) $\mathrm{END}_A(M, N)$ is a right π-H-comodule algebra;

3) $\mathrm{Hom}_A(M, N)^{\mathrm{co}H} = \mathrm{Hom}_A^H(M, N)$. In particular, $\mathrm{Hom}(M, N)^{\mathrm{co}H} = \mathrm{Hom}^H(M, N)$.

Lemma 2 Let $M, N\in {}_{H}\mathscr{M}_{A}^{H}$. Then $\mathrm{HOM}_A(M, N)$ is a right Hopf π-H-comodule, where the module action is defined by $(f\leftarrow h)(m) = \mathrm{f}(h\cdot m)$.

Lemma 3 Let $M\in {}_{H}\mathscr{M}_{A}^{H}$. Define

$$(h\rightarrow f)(m) = h_{(1,\alpha)}\cdot f(S_{\alpha^{-1}}(h_{(2,\alpha^{-1})})\cdot m) \tag{3}$$

for any $h\in H_1$, $f\in \mathrm{End}_A^H(M)$. Then

1) $\mathrm{End}_A^H(M)$ is a left H_1-module algebra via action "$\rightarrow$" as in Eq. (3).

2) There exists an algebra map φ: $\mathrm{End}_A^H(M)\#H\rightarrow \mathrm{END}_A(M)$, $f\#h\mapsto f\leftarrow h$, where $f\in \mathrm{End}_A^H(M)$ and $h\in H_\alpha$.

Now we can obtain the main result of this paper.

Theorem 1 Let $N\in \mathscr{M}_{A}^{H}$. Then the following assertions hold.

1) There exists an isomorphism of right relative (A, H)-Hopf π-comodules,

$$N\cong \mathrm{HOM}_A(A, N)$$

where $\mathrm{HOM}_A(A, N)$ is a right A-module via $f\cdot a(b) = f(ab)$.

2) Moreover, assume that $M\in {}_{H}\mathscr{M}_{A}^{H}$. Then

$$\mathrm{Hom}_A^H(M, N)\otimes H\rightarrow \mathrm{HOM}_A(M, N) \qquad g\otimes h\mapsto g\leftarrow h$$

is an isomorphism of right Hopf π-H-comodules, where $g\in \mathrm{HOM}_A(A, N)$ and $h\in H_\alpha$. Furthermore,

$$\mathrm{End}_A^H(M)\#H\rightarrow \mathrm{END}_A(M) \qquad g\#h\mapsto g\leftarrow h$$

is an isomorphism of right Hopf π-H comodules and algebras.

Proof We have a well-defined map

$$\phi: N\rightarrow \mathrm{HOM}_A(A, N) \qquad \phi(n)(a) = n\cdot a$$

since $\rho_\beta(\phi(n))(a) = \phi(n_{[0,0]})(a)\otimes n_{[1,\beta]}$ for any $n\in N$ and $a\in A$. It is easy to show that φ: $\mathrm{HOM}_A(A, N)\rightarrow N$, $f\mapsto f(1_A)$ is the inverse of ϕ.

In a similar way to Lemma 2, we know that $\mathrm{HOM}_A(A, N)$ is a right relative (A, H)-Hopf π-comodule. It is easy to prove that φ is a right relative (A, H)-Hopf π-comodule map.

The conclusion follows from Theorem 2.7 in Ref. [4] and 3) of Lemma 1, Lemma 2 and Lemma 3, which completes our proof.

By Theorem 1, we have the following remark.

Remark 1) By 1) of Theorem 1, $A\cong \mathrm{END}_A(A)$, which is an isomorphism of right π-H-comodule algebras. In particular, $H\cong \mathrm{END}_H(H)$, which is an isomorphism of algebras.

2) Let $N \in \mathscr{M}_H^H$. By 1) of Theorem 1, we have $N \cong \mathrm{HOM}_H(H, N)$ as right Hopf π-H-comodules.

3) Let $M \in \mathscr{M}_A^H$. Then, by Example 1 and 2) of Theorem 1, there exists an isomorphism of right Hopf π-H-comodules and algebras

$$\mathrm{End}_A^H(H \otimes M) \# H \cong \mathrm{END}_A(H \otimes M)$$

i.e., for any $a \in \pi$, $\mathrm{End}_A^H(H_\alpha \otimes M) \# H \cong \mathrm{END}_A(H_\alpha \otimes M)$. In particular, $\mathrm{End}_A^H(H \otimes A) \# H \cong \mathrm{END}_A(H \otimes A)$ is an isomorphism of the algebras.

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Hopf 群余代数的结构定理

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摘要:设 π 是一个带有单位元 1 的群,H 是一个 Hopf π-余代数, A 是一个右 π-H-余模代数. 首先,引入双边相对(A,H)-Hopf π-余模的概念,进而得到了 $\mathrm{Hom}_A^H(M,N) \otimes H$ 和 $\mathrm{HOM}_A(M,N)$ 作为右 Hopf π-H-余模是同构的结论, 其中 $\mathrm{Hom}_A^H(M,N)$ 表示右 A-模和右 H-余模同态作成的空间, $\mathrm{HOM}_A(M,N)$ 表示右 A-模同态构成空间 $\mathrm{Hom}_A(M,N)$ 的有理空间. 其次, 得到了双边相对(A,H)-Hopf π-余模的自同态代数的结构定理, 即 $\mathrm{End}_A^H(M) \# H$ 和 $\mathrm{END}_A(M,N)$ 作为右 Hopf π-H-余模和代数是同构的.

关键词: Hopf 群余代数;Hopf 群余模余代数;双边相对 Hopf 群余模

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