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Adjacent vertex-distinguishing total colorings of $\overline{K}_s \vee K_t$

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Abstract: Let G be a simple graph and f be a proper total k-coloring of G. The color set of each vertex v of G is the set of colors appearing on v and the edges incident to v. The coloring f is said to be an adjacent vertex-distinguishing total coloring if the color sets of any two adjacent vertices are distinct. The minimum k for which such a coloring of G exists is called the adjacent vertex-distinguishing total chromatic number of G. The join graph of two vertex-disjoint graphs is the graph union of these two graphs together with all the edges that connect the vertices of one graph with the vertices of the other. The adjacent vertex-distinguishing total chromatic numbers of the join graphs of an empty graph of order s and a complete graph of order t are determined.

Key words: adjacent vertex-distinguishing total coloring; adjacent vertex-distinguishing total chromatic number; join graph

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Let $G=(V, E)$ be a simple graph and f: $(V\cup E)\rightarrow\{1,2,\ldots,k\}$ be a proper total k-coloring of G. Denote by $C(v)=\{f(v)\}\cup\{f(uv): uv\in E(G)\}$ and $\overline{C}(v)=\{1,2,\ldots,k\}-C(v)$ for each vertex v of G. The coloring f is said to be an adjacent vertex-distinguishing total coloring (AVDTC for short) if $C(u)\neq C(v)$ whenever $uv\in E(G)$. The minimum k for which such a coloring of G exists is called the adjacent vertex-distinguishing total chromatic number, and is denoted by $\chi_{\mathrm{at}}(G)$. Furthermore, if $C(u)\neq C(v)$ for $u, v\in V(G)$ and $u\neq v$, then f is said to be a vertex-distinguishing total coloring. The minimum k for which such a coloring of G exists is called the vertex-distinguishing total chromatic number, and is denoted by $\chi_{\mathrm{vt}}(G)$. The following result can be found in Ref. [1].

Proposition 1[1] If G is a graph of order n, then $\chi_{\mathrm{vt}}(G)\leqslant n+2$.

AVDTC is related to vertex distinguishing proper edge colorings of graphs, which is first examined by Burris and Schelp[2] and further discussed by Bazgan et al[3-4]. This type of coloring is further extended to require only adjacent vertices to be distinguished[5] and it is in turn extended to proper total colorings[6].

Let $\Delta(G)$ and $\delta(G)$ be the maximum degree and minimum degree of a graph G, respectively. By definition, it is obvious that $\chi_{\mathrm{at}}(G)\geqslant\Delta(G)+1$. It is Zhang et al. [6] who first introduced this kind of coloring. They determined $\chi_{\mathrm{at}}(G)$ for many basic families of graphs, such as trees, complete graphs and complete bipartite graphs. The following easy observation can be found in Ref. [6].

Proposition 2[6] If G is a graph with two adjacent vertices of the maximum degree, then

$$\chi_{\mathrm{at}}(G)\geqslant\Delta(G)+2$$

By deleting an edge or two from a complete graph K_{2n+1}, Zhang et al. [6] and Chen[7] obtained the adjacent vertex-distinguishing total chromatic number of such graphs.

The join graph $G\vee H$ of two vertex-disjoint graphs G and H has $V(G\vee H)=V(G)\cup V(H)$ and $E(G\vee H)=E(G)\cup E(H)\cup\{uv:\ u\in V(G),\ v\in V(H)\}$. In this work, we discuss the adjacent vertex-distinguishing total chromatic number of $\overline{K}_s\vee K_t$, where $\overline{K}_s$ is the empty graph of order s and K_t is the complete graph of order t. We can also interpret $\overline{K}_s\vee K_t$ as a complete graph $K_{(s+t)}$ by deleting all the edges of a complete subgraph K_s.

Please refer to Ref. [8] for undefined terminologies and notations in this paper.

The following useful lemma can be found in Ref. [9].

Lemma 1[9] Let n be an integer and $n\geqslant 2$, then

$$\chi_{\mathrm{at}}(K_n)=\begin{cases}n+1 & \text{if } n \text{ is even}\\ n+2 & \text{if } n \text{ is odd}\end{cases}\tag{1}$$

Theorem 1 If $s\geqslant t\geqslant 2$, then

$$\chi_{\mathrm{at}}(\overline{K}_s\vee K_t)=s+t+1$$

Proof It is obvious that $\Delta(\overline{K}_s\vee K_t)=s+t-1$. Thus $\chi_{\mathrm{at}}(\overline{K}_s\vee K_t)\geqslant\Delta(\overline{K}_s\vee K_t)+2=s+t+1$ by Proposition 2. So it only needs to give an $(s+t+1)$-AVDTC of $\overline{K}_s\vee K_t$.

If t is even, then K_t has a $(t+1)$-AVDTC by Lemma 1. Since $s\geqslant t$, we may use other s colors to properly color the edges between $\overline{K}_s$ and K_t(see Ref. [10]). For each vertex v of $\overline{K}_s$, choose an available color for v among the

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set of $s+t+1$ colors. It can be verified that the vertices of K_t have different color sets.

If t is odd, then K_t has a proper total t-coloring using colors $s+1, s+2, \dots, s+t$ (see Ref. [9]). Suppose that $V(K_t)=\{u_0, \dots, u_{t-1}\}$ and $V(\overline{K}_s)=\{v_0, \dots, v_{s-1}\}$. For integers a and b, a mod b is denoted by $[a]_b$. We use other $s+1$ colors $0, 1, \dots, s$ to properly color the edges between $\overline{K}_s$ and K_t as follows: for $0 \leqslant i \leqslant t-1$ and $0 \leqslant j \leqslant s-1$, color $u_i v_j$ by $[i+j]_{s+1}$. For each vertex v of $\overline{K}_s$, choose an available color for v from the set $\{0, 1, \dots, s+t\}$. It can be verified that the vertices of K_t have different color sets.

Theorem 2 Let $2 \leqslant s < t$, if $s+t$ is even, then

$$\chi_{\mathrm{at}}(\overline{K}_s \vee K_t) = s+t+1$$

Proof It is obvious that $\Delta(\overline{K}_s \vee K_t) = s+t-1$. Thus, $\chi_{\mathrm{at}}(\overline{K}_s \vee K_t) \geqslant \Delta(\overline{K}_s \vee K_t) + 2 = s+t+1$ by Proposition 2. So it only needs to give an $(s+t+1)$-AVDTC of $\overline{K}_s \vee K_t$.

Let $u_1 u_2 \dots u_{s+t+1}$ be a regular $(s+t+1)$-gon centered on O. All the other possible pairs (u_i, u_j) are joined by straight line segments. Suppose that $V(\overline{K}_s)=\{u_1, \dots, u_s\}$, $V(K_t)=\{u_{s+1}, \dots, u_{s+t}\}$. For $1 \leqslant i \leqslant s+t+1$, color u_i and all the edges perpendicular to Ou_i by i. Delete u_{s+t+1} (including all the edges incident to it) and all the edges within $u_1, \dots, u_s$. So we obtain a proper total $(s+t+1)$-coloring of $(\overline{K}_s \vee K_t)$.

If s is odd, then for $i = \frac{s+1}{2}, \frac{s+3}{2}, \dots, \frac{s+t}{2}$, $\overline{C}(u_{2i}) = \{i\}$, and $\overline{C}(u_{2j-1}) = \left\{\frac{s+t}{2}+j\right\}$ for $j = \frac{s+3}{2}, \frac{s+5}{2}, \dots, \frac{s+t}{2}$. It is obvious that all the vertices of K_t have different color sets.

If s is even, then $\overline{C}(u_{s+i}) = \left\{\frac{2s+t+i+1}{2}\right\}$, $i=1, 3, \dots, t-1$, and $\overline{C}(u_{s+j}) = \left\{\frac{s+j}{2}\right\}$, $j=2, 4, \dots, t$. It can be verified that all the vertices of K_t in this case also have different color sets.

Theorem 3 Let $2 \leqslant s < t$, if $(s+t)$ is odd and $t > (s+1)^2 - 2$, then

$$\chi_{\mathrm{at}}(\overline{K}_s \vee K_t) = s+t+2$$

Proof It is obvious that $\Delta(\overline{K}_s \vee K_t) = s+t-1$. Thus, $\chi_{\mathrm{at}}(\overline{K}_s \vee K_t) \geqslant \Delta(\overline{K}_s \vee K_t) + 2 = s+t+1$ by Proposition 2. Suppose that $\overline{K}_s \vee K_t$ has an $(s+t+1)$-AVDTC f.

Denote by $\mathcal{C}$ the set of $s+t+1$ colors. Note that each color in $\mathcal{C}$ is used by at most $\frac{s+t-1}{2}$ edges and at least $\left\lfloor\frac{t-1}{2}\right\rfloor$ edges. Otherwise, suppose that $c \in \mathcal{C}$ and c is used by at most $\left\lfloor\frac{t-1}{2}\right\rfloor - 1$ edges M. Since $t - 2\left(\left\lfloor\frac{t-1}{2}\right\rfloor - 1\right) = t - 2\left\lfloor\frac{t-1}{2}\right\rfloor + 2 \geqslant 3$, at least 3 vertices of K_t are not saturated by M. The incident edges of these vertices are not colored by c. So there exist two vertices of K_t, the color sets of which do not contain c; i.e., they have the same color set, and this is a contradiction.

If t is even, for $i = 1, 2, \dots, \frac{s+3}{2}$, suppose that there exist q_i colors, and each of which is used by exactly $\frac{t}{2} - 2 + i$ edges, then

$$q_1 + q_2 + \dots + q_{(s+1)/2} + q_{(s+3)/2} = s+t+1 \tag{2}$$

$$\left(\frac{t}{2}-1\right)q_1 + \frac{t}{2}q_2 + \dots + \frac{s+t-3}{2}q_{(s+1)/2} + \frac{s+t-1}{2}q_{(s+3)/2} = \frac{t^2}{2} - \frac{t}{2} + st \tag{3}$$

Taking Eq. (2) $\times \left(\frac{s+t-3}{2}\right)$ $-$ Eq. (3), we obtain

$$q_{(s+3)/2} = \frac{t+2s-s^2+3}{2} + \frac{s-1}{2}q_1 + \frac{s-3}{2}q_2 + \dots + \frac{2}{2}q_{(s-1)/2} \tag{4}$$

Since $\omega(\overline{K}_s \vee K_t) = t+1$, $\chi(\overline{K}_s \vee K_t) \geqslant t+1$, $\chi(G)$ and $\omega(G)$ are the (vertex) chromatic number and the clique number of a graph G, respectively. So there exist at most s colors that are not used by vertices of $\overline{K}_s \vee K_t$. For those colors which are used by exactly $\frac{s+t-1}{2}$ edges, there are at least $q_{(s+3)/2} - s$ colors of which being used by vertices of $\overline{K}_s \vee K_t$. Let these $q_{(s+3)/2} - s$ colors be $c_1, c_2, \dots, c_{q_{(s+3)/2}-s}$, and they appear on every color set of the vertices of K_t. Suppose that $V(K_t)=\{u_1, \dots, u_t\}$, then

$$\bigcup_{i=1}^{t} \overline{C}(u_i) \subseteq \mathcal{C} - \{c_1, c_2, \dots, c_{q_{(s+3)/2}-s}\} \tag{5}$$

That is

$$t \leqslant (s+t+1) - (q_{(s+3)/2} - s) \tag{6}$$

$$q_{(s+3)/2} \leqslant 2s+1 \tag{7}$$

From Eq. (4), we obtain

$$q_{(s+3)/2} \geqslant \frac{t+2s-s^2+3}{2} > \frac{(s+1)^2-2+2s-s^2+3}{2} = 2s+1 \tag{8}$$

which is a contradiction.

If t is odd, for $i=1, 2, \dots, \frac{s}{2}+1$, suppose that there exist q_i colors, and each of which is used by exactly $\frac{t-3}{2} + i$ edges, then

$$q_1+q_2+\dots+q_{\frac{s}{2}}+q_{\frac{s}{2}+1}=s+t+1 \tag{9}$$

$$\frac{t-1}{2}q_1+\frac{t+1}{2}q_2+\dots+\frac{s+t-3}{2}q_{s/2}+\frac{s+t-1}{2}q_{(s+2)/2}=\frac{t^2}{2}-\frac{t}{2}+st \tag{10}$$

Taking Eq. (9) $\times\left(\frac{s+t-3}{2}\right)$ − Eq. (10), we obtain

$$q_{(s+2)/2}=\frac{t+2s-s^2+3}{2}+\frac{s-2}{2}q_1+\frac{s-4}{2}q_2+\dots+\frac{2}{2}q_{(s-2)/2} \tag{11}$$

Similar to the case when t is even, there are at most s colors that are not used by vertices of $\overline{K}_s \vee K_t$. For those colors which are used by exactly $\frac{s+t-1}{2}$ edges, there are at least $q_{(s+2)/2}-s$ colors of which being used by vertices of $\overline{K}_s \vee K_t$. Let these $q_{(s+2)/2}-s$ colors be $c_1, c_2, \dots, c_{q_{(s+2)/2}-s}$, and they appear on every color set of vertices of K_t. Then

$$\bigcup_{i=1}^{t} C(u_i) \subseteq \mathcal{C}-\{c_1, c_2, \dots, c_{q_{(s+2)/2}-s}\} \tag{12}$$

That is

$$t \leqslant (s+t+1)-(q_{(s+2)/2}-s) \tag{13}$$

$$q_{(s+2)/2} \leqslant 2s+1 \tag{14}$$

Since $t>(s+1)^2-2$, from (11) we obtain

$$q_{(s+2)/2} \geqslant \frac{t+2s-s^2+3}{2} > \frac{(s+1)^2-2+2s-s^2+3}{2}=2s+1 \tag{15}$$

which is a contradiction.

Therefore, $\chi_{\mathrm{at}}(\overline{K}_s \vee K_t) \geqslant s+t+2$. It follows from Proposition 1 that

$$\chi_{\mathrm{at}}(\overline{K}_s \vee K_t) \leqslant \chi_{\mathrm{vt}}(\overline{K}_s \vee K_t) \leqslant s+t+2$$

Thus, $\chi_{\mathrm{at}}(\overline{K}_s \vee K_t)=s+t+2$. The proof is completed.

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$\overline{K}_s \vee K_t$ 的邻点可区分全着色

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摘要：设 f 是一个简单图 G 的正常全 k-着色. 对 G 的每一个顶点 v，由出现在 v 点的颜色以及和 v 点关联边的颜色构成的集合称为 v 点的颜色集. 如果图 G 的任意 2 个相邻顶点的颜色集不相同，那么 f 是图 G 的一个邻点可区分全着色. 而使得图 G 存在这样一种全着色所需要的最小整数 k 就称为 G 的邻点可区分全色数. 2 个顶点不相交的图的连接图指的是这 2 个图的并图再加上所有连接其中一个图的顶点到另外一个图的顶点的边. 确定了 s 阶空图和 t 阶完全图的连接图的邻点可区分全色数.

关键词：邻点可区分全着色；邻点可区分全色数；连接图

中图分类号：O157.5