Journal of Southeast University (English Edition) Vol. 29, No. 3, pp. 342 - 345 Sept. 2013 ISSN 1003—7985

Method of variation of parameters for solving a constrained Birkhoffian system

Zhang Yi

(College of Civil Engineering, Suzhou University of Science and Technology, Suzhou 215011, China)

Abstract: For an in-depth study on the integration problem of the constrained mechanical systems, the method of integration for the Birkhoffian system with constraints is discussed, and the method of variation of parameters for solving the dynamical equations of the constrained Birkhoffian system is provided. First, the differential equations of motion for the constrained Birkhoffian system as well as for the corresponding free Birkhoffian system are established. Secondly, a system of auxiliary equations is constructed, and the general solution of the equations is found. Finally, by varying the parameters and utilizing the properties of the generalized canonical transformation of the Birkhoffian system, the solution of the problem can be obtained. The proposed method reveals the inherent relationship between the solution of a free Birkhoffian system and that of a constrained Birkhoffian system. The research results are of universal significance, which can be further used in a variety of constrained mechanical systems, such as non-conservative systems and nonholonomic systems etc.

Key words: Birkhoffian mechanics; method of integration; method of variation of parameter; constrained Birkhoffian system

doi: 10.3969/j.issn.1003 - 7985.2013.03.020

The theory of integration for the constrained mechanical systems is an important aspect of the research for analytical dynamics. A set of beautiful methods of integration for conservative systems encountered great difficulties in reaching out to nonconservative, or nonholonomic dynamics. Therefore, it is an important research direction for analytical dynamics in providing the new versatile method of integration to complex dynamical systems. The Birkhoffian system is a quite extensive class of the dynamical system, and it is a generalization of the Hamiltonian system. The theory of integration for the Birkhoffian system is an important part of Birkhoffian dynamics[1]. The US physicist Santilli[2] studied the Birkhoff equations, the theory of transformation of the Birkhoff equations and the generalization of Galilei relativity in his monograph, and extended the Hamilton-Jacobi method to the Birkhoffian system. Galiullan et al.[3] studied the inverse problem of Birkhoffian dynamics, the integral invariants of the Birkhoffian system, and the conformal invariance etc. Mei[1,4-7] established the Poisson theory of the Birkhoffian system, the field method for integrating the Birkhoff equations, the symmetries and the conserved quantities, the inverse problems of dynamics and the integral invariants, and extended them to the generalized Birkhoffian system. Zhang[8] provided the method of variation of parameters for integrating the generalized Birkhoffian system. In recent years, some important results on the research of the theory of integration for Birkhoffian systems have been obtained[9-18]. In this paper, we will further apply the method of variation of parameters for solving the integration issues of the constrained Birkhoffian system. The method is of universal importance, and it reveals the inherent relationship between the solution of a free Birkhoffian and that of a constrained Birkhoffian system.

Received 2013-01-11.

Biography: Zhang Yi (1964—), male, doctor, professor, weidiezh@gmail.com.

Foundation item: The National Natural Science Foundation of China (No. 10972151, 11272227).

Citation: Zhang Yi. Method of variation of parameters for solving a constrained Birkhoffian system[J]. Journal of Southeast University (English Edition), 2013, 29(3): 342 - 345. [doi: 10.3969/j.issn.1003 - 7985.2013.03.020]

1 Differential Equations of Motion for a Constrained Birkhoffian System

The Birkhoff equation in the general form of a Birkhoffian system is[1-2]

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=0 \qquad \mu=1,2,\ldots,2n \tag{1}$$

where $B=B(\boldsymbol{a},\ t)$ is called Birkhoffian; $R_\mu=R_\mu(\boldsymbol{a},\ t)$ is the Birkhoff function.

Suppose that the variables $a^\mu(\mu=1,2,\ldots,2n)$ of system (1) are not independent of each other, but they are restricted by some constraints, and, as such, the system is called a constrained Birkhoffian system. If the restrictions can be expressed as the following constraint equations

$$f_\beta(\boldsymbol{a},t)=0 \qquad \beta=1,2,\ldots,g \tag{2}$$

then the restrictions added to the virtual displacements by constraints (2) are

$$\frac{\partial f_\beta}{\partial a^\mu}\delta a^\mu=0 \qquad \beta=1,2,\ldots,g \tag{3}$$

The differential equations of motion with multipliers of the constrained Birkoffian system can be expressed as[1]

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=\lambda_\beta\frac{\partial f_\beta}{\partial a^\mu}\qquad \mu=1,2,\dots,2n \tag{4}$$

Considering that the system is non-singular and from Eqs. (2) and (4), we can seek λ_β as the function of ($\boldsymbol{a}$, t) before integrating the differential equations of motion. Therefore, Eq. (4) can further be written as

$$\dot{a}^\mu=\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}+P_\nu\right)\qquad \mu=1,2,\dots,2n \tag{5}$$

where

$$P_\mu=P_\mu(\boldsymbol{a},t)=\lambda_\beta\frac{\partial f_\beta}{\partial a^\mu},\quad \Omega_{\mu\nu}=\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}$$
$$\Omega^{\mu\nu}\Omega_{\nu\tau}=\delta_{\mu\tau},\quad \det(\Omega_{\mu\nu})\neq 0 \tag{6}$$

Eq. (5) is called the differential equations of motion for the free Birkhoffian system which corresponds to the constrained Birkhoffian systems (2) and (4). As long as the initial conditions of motion satisfy the constraint equation (2), the solution of the corresponding free system (5) gives the motion of the constrained Birkhoffian system.

2 Method of Variation of Parameters for a Constrained Birkhoffian System

To solve the constrained Birkhoffian system with the method of variation of parameters, we build a system of auxiliary equations as

$$\dot{a}^\mu=\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}\right)\qquad \mu=1,2,\dots,2n \tag{7}$$

Let the general solution of Eq. (7) be

$$a^\mu=a^\mu(\alpha^1,\alpha^2,\dots,\alpha^{2n},t)\qquad \mu=1,2,\dots,2n \tag{8}$$

where α^μ is an integral constant which is the value of a^μ when $t=0$. Without loss of generality, we take α^μ as a new variable, make a variable substitution according to Eq. (8), and select

$$B^*=B^*(\alpha,t)=\left(B-\frac{\partial a^\nu}{\partial t}R_\nu\right)(\alpha,t)$$

$$R_\mu^*=R_\mu^*(\alpha,t)=\left(\frac{\partial a^\nu}{\partial \alpha^\mu}R_\nu\right)(\alpha,t)$$

$$\Omega_{\mu\nu}^*=\frac{\partial R_\nu^*}{\partial \alpha^\mu}-\frac{\partial R_\mu^*}{\partial \alpha^\nu},\quad \Omega^{\mu\nu *}\Omega_{\nu\tau}^*=\delta_{\mu\tau} \tag{9}$$

We can easily obtain [2]

$$\dot{\alpha}^\mu=\Omega^{\mu\nu *}\left(\frac{\partial B^*}{\partial \alpha^\nu}+\frac{\partial R_\nu^*}{\partial t}\right)\qquad \mu=1,2,\dots,2n \tag{10}$$

Therefore, the transformation (8) is a generalized canonical transformation, and we have[2]

$$\frac{\partial \alpha^\mu}{\partial a^\rho}\Omega^{\rho\sigma}\frac{\partial \alpha^\nu}{\partial a^\sigma}\equiv\Omega^{\mu\nu *} \tag{11}$$

Suppose that the inverse transformation of the transformation (8) is

$$\alpha^\mu=\alpha^\mu(a^1,a^2,\dots,a^{2n},t)\qquad \mu=1,2,\dots,2n \tag{12}$$

Since Eq. (12) is the first integral of Eq. (7), we have

$$\frac{\partial \alpha^\mu}{\partial t}+\frac{\partial \alpha^\mu}{\partial a^\nu}\Omega^{\nu\rho}\left(\frac{\partial B}{\partial a^\rho}+\frac{\partial R_\rho}{\partial t}\right)=0\qquad \mu=1,2,\dots,2n \tag{13}$$

Then, we find a general solution of Eq. (5).

Assume that the solution of Eq. (5) still has the form of Eq. (8), and α^μ is no longer a constant but a function of time t. Differentiating the formula (12) with respect to time t, we obtain

$$\dot{\alpha}^\mu=\frac{\partial \alpha^\mu}{\partial t}+\frac{\partial \alpha^\mu}{\partial a^\nu}\dot{a}^\nu=-\frac{\partial \alpha^\mu}{\partial a^\nu}\Omega^{\nu\rho}\left(\frac{\partial B}{\partial a^\rho}+\frac{\partial R_\rho}{\partial t}\right)+$$
$$\frac{\partial \alpha^\mu}{\partial a^\nu}\Omega^{\nu\rho}\left(\frac{\partial B}{\partial a^\rho}+\frac{\partial R_\rho}{\partial t}+P_\rho\right)=\frac{\partial \alpha^\mu}{\partial a^\nu}\Omega^{\nu\rho}P_\rho \tag{14}$$

From Eq. (11), Eq. (14) can be written as

$$\dot{\alpha}^\mu=\Omega^{\mu\sigma *}\frac{\partial a^\rho}{\partial \alpha^\sigma}P_\rho\qquad \mu=1,2,\dots,2n \tag{15}$$

Hence, we have

$$\alpha^\mu=\alpha_0^\mu+\int\Omega^{\mu\sigma *}\frac{\partial a^\rho}{\partial \alpha^\sigma}P_\rho\,\mathrm{d}t\qquad \mu=1,2,\dots,2n \tag{16}$$

Substituting (16) into (8), we can obtain the solution of the differential equation (5) of motion for the corresponding free Birkhoffian system (5). Substituting the initial conditions α_0^μ, which are the values of a^μ when $t=0$, into the constraint equation (2), we have

$$f_\beta(\alpha_0^1,\alpha_0^2,\dots,\alpha_0^{2n},t)=0\qquad \beta=1,2,\dots,g \tag{17}$$

The solution of the constrained Birkhoffian systems (2) and (4) under consideration is found by combining (16) and (17), which contains $2n-g$ independent constants. Therefore, we obtain the following proposition.

Proposition 1 For the constrained Birkhoffian systems (2) and (4), if the auxiliary equation (7) has a general solution in the form of Eq. (8), then the general solution of Eq. (5) can be written as Eq. (8), in which α^μ can be determined by Eq. (16) and the initial conditions α_0^μ satisfy Eq. (17).

3 Example

The Birkhoffian B and the Birkhoff functions R_μ of a four-dimensional Birkhoffian system are respectively

$$B = \frac{1}{2}(a^3)^2 + (a^4)^2 - g a^1 \sin\varphi$$
$$R_1 = a^3,\ R_2 = a^4,\ R_3 = R_4 = 0 \tag{18}$$

where $g,\ \varphi$ are constants. The constraint equations are

$$f_1 = a^1 - a^2 = 0,\ f_2 = a^3 - 2a^4 = 0 \tag{19}$$

We try to solve this problem by the method of this paper.

In order to solve this problem, we divide it into two steps. First, let us establish an auxiliary system and solve it. The auxiliary equation (7) gives that

$$\dot a^1 = a^3,\ \dot a^2 = 2a^4,\ \dot a^3 = g\sin\varphi,\ \dot a^4 = 0 \tag{20}$$

The solution of Eq. (20) is

$$a^1 = \alpha^1 + \alpha^3 t + \frac{1}{2} g t^2 \sin\varphi,\ a^2 = \alpha^2 + 2\alpha^4 t$$
$$a^3 = \alpha^3 + g t \sin\varphi,\ a^4 = \alpha^4 \tag{21}$$

where $\alpha^\mu (\mu = 1, \ldots, 4)$ are constants of integration.

Choose

$$B^* = -\frac{1}{2}(\alpha^3)^2 - (\alpha^4)^2 - g\alpha^1 \sin\varphi - 2gt\alpha^3 \sin\varphi - g^2 t^2 \sin^2\varphi$$
$$R_1^* = \alpha^3 + gt\sin\varphi,\ R_2^* = \alpha^4,\ R_3^* = t\alpha^3 + gt^2\sin\varphi,\ R_4^* = 2t\alpha^4$$
$$[\Omega_{\mu\nu}^*] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ [\Omega^{\mu\nu *}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{22}$$

We can easily verify the validity of Eq. (10).

Secondly, let us give a variation of parameters and calculate the motion of the system. Eq. (4) gives that

$$\dot a^1 = a^3 + \lambda_2,\ \dot a^2 = 2a^4 - 2\lambda_2,\ \dot a^3 = g\sin\varphi - \lambda_1,\ \dot a^4 = \lambda_1 \tag{23}$$

From Eqs. (23) and (19), we can obtain

$$\lambda_1 = \frac{1}{3} g\sin\varphi,\ \lambda_2 = -\frac{1}{3}(a^3 - 2a^4) \tag{24}$$

Therefore, we have

$$P_1 = \frac{1}{3} g\sin\varphi,\ P_2 = -\frac{1}{3} g\sin\varphi$$
$$P_3 = -\frac{1}{3}(a^3 - 2a^4),\ P_4 = \frac{2}{3}(a^3 - 2a^4) \tag{25}$$

Eq. (15) gives that

$$\dot\alpha^1 = -\frac{1}{3}(\alpha^3 - 2\alpha^4),\ \dot\alpha^2 = \frac{2}{3}(\alpha^3 - 2\alpha^4)$$
$$\dot\alpha^3 = -\frac{1}{3} g\sin\varphi,\ \dot\alpha^4 = \frac{1}{3} g\sin\varphi \tag{26}$$

Integrating Eq. (26), we have

$$\alpha^1 = \alpha_0^1 - \frac{1}{3}(\alpha_0^3 - 2\alpha_0^4)\ t + \frac{1}{6} g t^2 \sin\varphi$$
$$\alpha^2 = \alpha_0^2 + \frac{2}{3}(\alpha_0^3 - 2\alpha_0^4) t - \frac{1}{3} g t^2 \sin\varphi$$
$$\alpha^3 = \alpha_0^3 - \frac{1}{3} g t \sin\varphi,\ \alpha^4 = \alpha_0^4 + \frac{1}{3} g t \sin\varphi \tag{27}$$

Substituting (27) into (21), we obtain

$$a^1 = \alpha_0^1 + \frac{2}{3}(\alpha_0^3 + \alpha_0^4) t + \frac{1}{3} g t^2 \sin\varphi$$
$$a^2 = \alpha_0^2 + \frac{2}{3}(\alpha_0^3 + \alpha_0^4) t + \frac{1}{3} g t^2 \sin\varphi$$
$$a^3 = \alpha_0^3 + \frac{2}{3} g t \sin\varphi,\ a^4 = \alpha_0^4 + \frac{1}{3} g t \sin\varphi \tag{28}$$

Eq. (28) is the solutions of the corresponding free Birkhoffian system. Substituting the inertial conditions into the constraint equation (19), we have

$$\alpha_0^1 - \alpha_0^2 = 0,\ \alpha_0^3 - 2\alpha_0^4 = 0 \tag{29}$$

Eqs. (28) and (29) give the general solution of the problem under consideration, and it contains two arbitrary constants. For this problem, we can verify that its solution is given by Eqs. (28) and (29) through direct calculation.

4 Conclusion

The constrained Birkhoffian systems are a broad class of dynamical systems. The method of variation of parameters for solving the generalized Birkhoffian system is extended to the constrained Birkhoffian system in this paper. By means of this method, we can integrate a constrained Birkhoffian system in two steps. In the first step, we can construct a system of auxiliary equations whose solution is known. In the second step, we can vary the parameters, and the problem is reduced to solving Eq. (15). The research results of this paper are of universal significance, which can be applied to the systems with holonomic and nonholonomic constraints.

References

[1] Mei F X, Shi R C, Zhang Y F, et al. *Dynamics of Birkhoffian systems* [M]. Beijing: Beijing Institute of Technology Press, 1996. (in Chinese)

[2] Santilli R M. *Foundations of theoretical mechanics* Ⅱ [M]. New York: Springer-Verlag, 1983.

[3] Galiullin A S, Gafarov G G, Malaishka R P, et al. *Analytical dynamics of Helmholtz, Birkhoff and Nambu systems* [M]. Moscow: UFN, 1997. (in Russian)

[4] Mei F X. Noether theory of Birkhoffian system [J]. *Science in China: Series A*, 1993, **36**(12): 1456 − 1467.

[5] Mei F X. Poisson's theory of Birkhoffian system [J]. *Chinese Science Bulletin*, 1996, **41**(8): 641 − 645.

[6] Mei F X, Wu H B. First integral and integral invariant of Birkhoffian system [J]. *Chinese Science Bulletin*, 2000,

45(5): 412 - 414.

[7] Mei F X. On the Birkhoffian mechanics [J]. *International Journal of Non-Linear Mechanics*, 2001, **36**(5): 817 - 834.

[8] Zhang Y. The method of variation on parameters for integration of a generalized Birkhoffian system [J]. *Acta Mechanica Sinica*, 2011, **27**(6): 1059 - 1064.

[9] Guo Y X, Shang M, Luo S K. Poincaré-Cartan integral variants of Birkhoff system [J]. *Applied Mathematics and Mechanics*, 2003, **24**(1): 76 - 82.

[10] Guo Y X, Liu C, Liu S X. Generalized Birkhoffian formulation of nonholonomic systems [J]. *Communications in Mathematics*, 2010, **18**(1): 21 - 35.

[11] Luo S K, Guo Y X. Routh order reduction method of relativistic Birkhoffian systems [J]. *Communication in Theoretical Physics*, 2007, **47**(2): 209 - 212.

[12] Mei F X, Wu H B. Form invariance and new conserved quantity of generalized Birkhoffian system [J]. *Chinese Physics B*, 2010, **19**(5): 050301.

[13] Li Y M. Lie symmetries, perturbation to symmetries and adiabatic invariants of a generalized Birkhoff system [J]. *Chinese Physics Letters*, 2010, **27**(1): 010202.

[14] Zhang Y. Poisson theory and integration method of Birkhoffian systems in the event space [J]. *Chinese Physics B*, 2010, **19**(8): 080301.

[15] Zhang M J, Fang J H, Lu K. Perturbation to Mei symmetry and generalized Mei adiabatic invariants for Birkhoffian systems [J]. *International Journal of Theoretical Physics*, 2010, **49**(2): 427 - 437.

[16] Zhang Y, Zhou Y. Symmetries and conserved quantities for fractional action-like Pfaff variational problems [J]. *Nonlinear Dynamics*, 2013, **73**(1/2): 783 - 793.

[17] Zhang Y. A new method for integration of a Birkhoffian system [J]. *Journal of Southeast University*: *English Edition*, 2011, **27**(2): 188 - 191.

[18] Wu H B, Mei F X. Type of integral and reduction for a generalized Birkhoffian system [J]. *Chinese Physics B*, 2011, **20**(10): 104501.

求解约束 Birkhoff 系统的参数变异法

张　毅

(苏州科技学院土木工程学院, 苏州 215011)

摘要:为了深入研究约束力学系统的积分问题,讨论了具有约束的 Birkhoff 系统的积分方法问题,提出用参数变异法求解约束 Birkhoff 系统的动力学方程. 首先,建立约束 Birkhoff 系统及相应的自由 Birkhoff 系统的运动微分方程;其次,构建辅助方程系统,并找到其一般解;最后,变异参数并利用 Birkhoff 系统的广义正则变换的性质,获得问题的解. 该方法揭示了自由 Birkhoff 系统和约束 Birkhoff 系统的解之间的内在联系. 所提研究方法和结果具有普遍意义,可进一步应用于各种约束力学系统,例如,非保守系统和非完整约束系统等.

关键词: Birkhoff 力学;积分方法;参数变异法;约束 Birkhoff 系统

中图分类号:O316