Journal of Southeast University (English Edition) Vol. 29, No. 3, pp. 352 - 354 Sept. 2013 ISSN 1003—7985

Ground states for asymptotically periodic quasilinear Schrödinger equations with critical growth

Zhang Hui Zhang Fubao

(Department of Mathematics, Southeast University, Nanjing 211189, China)

Abstract: For a class of asymptotically periodic quasilinear Schrödinger equations with critical growth, the existence of ground states is proved. First, applying a change of variables, the quasilinear Schrödinger equations are reduced to semilinear Schrödinger equations, in which the corresponding functional is well defined in $H^1(\mathbf{R}^N)$. Moreover, there is a one-to-one correspondence between ground states of the semilinear Schrödinger equations and the quasilinear Schrödinger equations. Then the mountain-pass theorem is used to find nontrivial solutions for the semilinear Schrödinger equations. Finally, under certain monotonicity conditions, using the Nehari manifold method and the concentration compactness principle, the nontrivial solutions are found to be exactly the same as the ground states of the semilinear Schrödinger equations.

Key words: quasilinear Schrödinger equation; variational method; ground state; critical growth

doi: 10.3969/j. issn. 1003 - 7985. 2013. 03. 022

1 Introduction and Statement of Main Result

As the models of physical phenomena, the quasilinear Schrödinger equation

$$-\Delta u+V(x)u-u\Delta(u^2)=\tilde{l}(x,u) \tag{1}$$

has been extensively studied in recent years. For the detailed physical applications, one can see Ref. [1].

Many studies about Eq. (1) have focused on the existence of nontrivial solutions[2-8]. Especially, the study concerning the ground states of Eq. (1) has attracted many researchers' attention due to its great physical interests. In Ref. [2], the authors obtained the ground states of Eq. (1) with $\tilde{l}(x,u)=\lambda|u|^{q-2}u$. Later, Liu et al.[3] studied positive and sign-changing ground states of Eq. (1) with $\tilde{l}(x,u)=|u|^{q-2}u$ and general quasilinear equations. Recently, Liu et al.[4] completed the results in Ref. [3] of Eq. (1) with $\tilde{l}(x,u)=|u|^{2^*-2}u+|u|^{q-2}u$.

Inspired by Refs. [4 - 5], we are interested in the existence of ground states for asymptotically periodic quasilinear Schrödinger equation (1). We consider

$$\begin{gathered}-\Delta u+V(x)u-u\Delta(u^2)=K(x)|u|^{22^*-2}u+g(x,u)\\ u\in H^1(\mathbf{R}^N)\end{gathered} \tag{2}$$

Let F be a class of functions $h\in C(\mathbf{R}^N)\cap L^\infty(\mathbf{R}^N)$, such that for every $\epsilon>0$ the set $\{x\in\mathbf{R}^N: |h(x)|\geqslant\epsilon\}$ has a finite Lebesgue measure. Suppose that $V, K\in C(\mathbf{R}^N)$ satisfies the following conditions:

H$_1$) There exists a constant $a_0>0$ and a function $V_p\in C(\mathbf{R}^N)$, 1-periodic in x_i, $1\leqslant i\leqslant N$, such that $V-V_p\in F$ and $V_p(x)\geqslant V(x)\geqslant a_0$, $x\in\mathbf{R}^N$.

H$_2$) There exists a function $K_p\in C(\mathbf{R}^N)$, 1-periodic in x_i, $1\leqslant i\leqslant N$, and a point $x_0\in\mathbf{R}^N$ such that $K-K_p\in F$ and

① $K(x)\geqslant K_p(x)>0$, $x\in\mathbf{R}^N$;

② $K(x)=|K|_\infty+O(|x-x_0|^{N-2})$, as $x\to x_0$.

Let $G(x,u)=\int_0^u g(x,s)\,\mathrm{d}s$ and assume that $g\in C(\mathbf{R}^N\times\mathbf{R},\mathbf{R})$ satisfies

H$_3$) $g(x,u)=o(u)$ uniformly in x as $u\to0$;

H$_4$) $|g(x,u)|\leqslant a(1+|u|^{q-1})$, for some $a>0$ and $4\leqslant q<22^*$;

H$_5$) $u\longmapsto\dfrac{g(x,u)}{|u|^3}$ is nondecreasing on $(-\infty,0)$ and $(0,\infty)$.

H$_6$) There exists a neighborhood of x_0 given by H$_2$), $\Omega\subseteq\mathbf{R}^N$, such that

① $\dfrac{G(x,u)}{|u|^{22^*-1}}\to\infty$ uniformly in Ω as $|u|\to\infty$ if $3\leqslant N<10$;

② $\dfrac{G(x,u)}{u^4}\to\infty$ uniformly in Ω as $|u|\to\infty$ if $N\geqslant10$.

H$_7$) There exists a constant $q_1\in(2,22^*)$, functions $h\in F$ and $g_p\in C(\mathbf{R}^N\times\mathbf{R},\mathbf{R})$ such that

① g_p is 1-periodic in x_i, $1\leqslant i\leqslant N$;

② $|g(x,u)-g_p(x,u)|\leqslant|h(x)|(|u|+|u|^{q_1-1})$, $x\in\mathbf{R}^N$;

③ $G(x,u)\geqslant G_p(x,u):=\int_0^u g_p(x,s)\,\mathrm{d}s$;

④ $u\longmapsto\dfrac{g_p(x,u)}{|u|^3}$ is nondecreasing on $(-\infty,0)$ and

Received 2012-09-02.

Biographies: Zhang Hui (1987—), female, graduate; Zhang Fubao (corresponding author), male, doctor, professor, zhangfubao@seu.edu.cn.

Foundation item: The Scientific Innovation Research of College Graduates in Jiangsu Province (No. CXLX_0069).

Citation: Zhang Hui, Zhang Fubao. Ground states for asymptotically periodic quasilinear Schrödinger equations with critical growth[J]. Journal of Southeast University (English Edition), 2013, 29(3): 352 - 354. [doi: 10.3969/j. issn. 1003 - 7985. 2013. 03. 022]

$(0, \infty)$.

Theorem 1 If H_1) to H_7) hold, then the problem (2) has a ground state.

Remark 1 H_3) and H_5) imply that

$$0 \leqslant G(x, u) \leqslant \frac{1}{4} g(x, u) u \qquad \forall u \in \mathbf{R},\ x \in \mathbf{R}^N \quad (3)$$

2 Variational Setting

In the sequel, $\int_{\mathbf{R}^N} h(x)\,\mathrm{d}x$ is represented by $\int h$. $S_1 = \{u \in H^1(\mathbf{R}^N):\ \|u\|^2 = 1\}$. Observe that the functional of the problem (2)

$$J(u) = \frac{1}{2}\int (1 + 2u^2) |\nabla u|^2 + \frac{1}{2}\int V(x) u^2 - \frac{1}{22^*}\int K(x) |u|^{22^*} - \int G(x, u)$$

is not well defined in $H^1(\mathbf{R}^N)$. Choose the change f defined by

$$f'(t) = \frac{1}{(1 + 2f^2(t))^{1/2}} \text{ on } [0, \infty)$$

$$f(t) = -f(-t) \text{ on } (-\infty, 0]$$

and set $v = f^{-1}(u)$, then we obtain

$$I(v) = \frac{1}{2}\int |\nabla v|^2 + \frac{1}{2}\int V(x) f^2(v) - \frac{1}{22^*}\int K(x) |f(v)|^{22^*} - \int G(x, f(v))$$

which is well defined in $H^1(\mathbf{R}^N)$ by the properties of f (see Ref. [5]). The critical points of I are weak solutions of

$$-\Delta v + V(x) f'(v) f(v) = K(x) |f(v)|^{22^*-2} f(v) f'(v) + g(x, f(v)) f'(v) \qquad v \in H^1(\mathbf{R}^N) \quad (4)$$

Similar to Ref. [5], we first prove that there is a nontrivial solution for Eq. (4). We know that the results obtained under (V), (K), (g_1), (g_2) and (g_5) in Ref. [5] still hold since the conditions H_1) to H_4) and H_6) are the same as (V), (K), (g_1), (g_2) and (g_5), respectively. However, H_5) and H_7) are different from (g_3) and (g_4) in Ref. [5]; in the following, we verify whether the results under (g_3) and (g_4) still hold.

Lemma 1 Let H_1) to H_5) hold. Then, the $(\mathrm{Ce})_b$ ($b > 0$) sequence v_n of I satisfying

$$I(v_n) \to b, \quad \|I'(v_n)\|(1 + \|v_n\|) \to 0 \quad (5)$$

is bounded.

Proof As in the proof of Lemma 4 in Ref. [5], we only need to show that $\int |f(v_n)|^{22^*}$ is bounded.

By (5), we have

$$b + o_n(1) = I(v_n) - \frac{1}{2}\langle I'(v_n), v_n\rangle = \frac{1}{2}\int [V(x) f^2(v_n) - V(x) f'(v_n) f(v_n) v_n] + \frac{1}{2}\int \left[K(x) |f(v_n)|^{22^*-2} f(v_n) f'(v_n) v_n - \frac{1}{2^*} K(x) |f(v_n)|^{22^*}\right] + \int \left[\frac{1}{2} g(x, f(v_n)) f'(v_n) v_n - G(x, f(v_n))\right] := I_1 + I_2 + I_3 \quad (6)$$

By Lemma 1 (8) in Ref. [5], we obtain

$$I_1 \geqslant 0,\ I_2 \geqslant \frac{1}{2N} \inf K \int |f(v_n)|^{22^*} \quad (7)$$

For I_3, using Lemma 1 (8) in Ref. [5] and inequality (3), we have

$$\frac{1}{2} g(x, f(v_n)) f'(v_n) v_n \geqslant \frac{1}{4} g(x, f(v_n)) f(v_n)$$

Then from (3) it follows that $I_3 \geqslant 0$. Combining with (6) and (7), $\int |f(v_n)|^{22^*}$ is bounded. This ends the proof.

In Ref. [5], the authors supposed that $|g(x,u) - g_p(x,u)| \leqslant h(x) |u|^{q_3 - 1}$, $q_3 \in [2, 22^*)$, and we assume that $|g(x,u) - g_p(x,u)| \leqslant h(x)(|u| + |u|^{q_1 - 1})$, $q_1 \in (2, 22^*)$. So Lemma 9 in Ref. [5] holds under H_1), H_2) and H_7). Following the outline in Ref. [5], we have the following lemma.

Lemma 2 Let H_1) to H_7) hold. Then the problem (4) possesses a nontrivial solution u such that $I(u) = c$ with $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0$, where $\Gamma = \{\gamma \in C([0,1], H^1(\mathbf{R}^N)) : \gamma(0) = 0, I(\gamma(1)) \leqslant 0\}$.

In order to find ground states, we also need to introduce the Nehari manifold. The Nehari manifold corresponding to Eq. (4) is

$$M = \{u \in H^1(\mathbf{R}^N) \setminus \{0\} : \langle I'(u), u\rangle = 0\}$$

First, we give the following lemma in which the simple proof is left to the reader.

Lemma 3 Let H_1) to H_5) hold. Then $I(tu) \to -\infty$ as $t \to \infty$, $u \in H^1(\mathbf{R}^N) \setminus \{0\}$.

Inspired by Ref. [6], we have

Lemma 4 Let H_1) to H_6) hold. Then for all $v \in H^1(\mathbf{R}^N) \setminus \{0\}$, there exists a unique $t_v > 0$ such that $t_v v \in M$ and $I(t_v v) = \max_{t>0} I(tv)$.

Proof For any $v \in H^1(\mathbf{R}^N) \setminus \{0\}$, as in the proof of Lemma 2 (I_1) in Ref. [5], we can easily obtain $I(tv) > 0$ when t is small enough. Lemma 3 implies that $I(tv) \to -\infty$ as $t \to \infty$. Therefore, there exists $t_v > 0$ such that $I(t_v v) = \max_{t>0} I(tv)$, $\left.\frac{\mathrm{d}I(tv)}{\mathrm{d}t}\right|_{t=t_v} = 0$. Then $t_v v \in M$. Set $\zeta(t) = I(tv)$, $t > 0$. Then t_v is the maximum point of ζ.

Note that

$$\zeta'(t)=\langle I'(tv),v\rangle=t\Big(\,|\nabla v|_2^2+\frac{1}{t}\int V(x)f'(tv)f(tv)v-\frac{1}{t}\int K(x)\,|f(tv)|^{22^*-2}f(tv)f'(tv)v-\frac{1}{t}\int g(x,f(tv))f'(tv)v\Big):=t(\,|\nabla v|_2^2+\Phi_1(t)+\Phi_2(t)+\Phi_3(t))$$

By Lemma 1 (8) in Ref. [5] and the fact that $f''(tv)=-2f(tv)f'^4(tv)$, we obtain

$$t^2\Phi'_1(t)=\int[f'^2(tv)tv^2-2f^2(tv)f'^4(tv)tv^2-f(tv)f'(tv)v]V(x)\leqslant\int[f(tv)f'(tv)v-2f^2(tv)f'^4(tv)tv^2-f(tv)f'(tv)v]V(x)<0$$

So Φ_1 is decreasing.

Using Lemma 1 (8) and (10) in Ref. [5], and the fact that $f''(tv)=-2f(tv)f'^4(tv)$, as before, we obtain that $t^2\Phi'_2(t)\leqslant 0$. Then Φ_2 is nonincreasing.

Note that $\Phi_3(\mathrm{t})=-\int\frac{g(x,f(tv))}{|f(tv)|^3}\frac{|f(tv)|^3f'(tv)v}{t}$. Let $B_3(t)=\frac{|f(tv)|^3f'(tv)v}{t}$. As before, we obtain that $t^2B'_3(t)\geqslant 0$. Then Φ_3 is nonincreasing by H_5). So the maximum point of ζ is unique. This ends the proof.

Define $\hat{m}$: $H^1(\mathbf{R}^N)\setminus\{0\}\to M$ by $\hat{m}(u)=t_u u$ and $m:=\hat{m}|_{S_1}$. Then m is a bijection from S_1 to M. Let $c^*=\inf_M I$, and Lemma 4 implies that

$$c^*=\inf_{u\in S_1}I(m(u))=\inf_{u\in H^1(\mathbf{R}^N)\setminus\{0\}}I(\hat{m}(u))=\inf_{u\in H^1(\mathbf{R}^N)\setminus\{0\}}\max_{t\geqslant 0}I(tu) \tag{8}$$

Lemma 5 Let H_1) to H_6) hold. Then $c^*\geqslant c$.

Proof For $u\in H^1(\mathbf{R}^N)\setminus\{0\}$, Lemma 3 implies that $\lim_{t\to\infty}I(tu)=-\infty$. Then there exists t_0 large enough such that $I(t_0u)<0$. Choose $\gamma_0(t):=tt_0u$, $0\leqslant t\leqslant 1$. Therefore, $c\leqslant\max_{0\leqslant t\leqslant 1}I(\gamma_0(t))$. So $c\leqslant\inf_{u\in H^1(\mathbf{R}^N)\setminus\{0\}}\max_{t\in[0,1]}I(tu)$. From (8) it follows that $c^*\geqslant c$.

3 Proof of Theorem 1

Proof By Lemma 2, we assume that there is a nontrivial solution w with $I(w)=c$. Then $w\in M$. So $I(w)\geqslant c^*$. Note that $I(w)=c$ and $c^*\geqslant c$, and we obtain $I(w)\leqslant c^*$. So $I(w)=c^*$. Then we can easily infer that w is a ground state for Eq. (4). We complete the proof.

References

[1] Kurihara S. Large-amplitude quasi-solitons in superfluid films [J]. *Journal of the Physical Society of Japan*, 1981, **50**(10): 3262 − 3267.

[2] Liu J, Wang Z. Soliton solutions for quasilinear Schrödinger equations Ⅰ [J]. *Proceedings of the American Mathematical Society*, 2003, **131**(2): 441 − 448.

[3] Liu J, Wang Y, Wang Z. Solutions for quasilinear Schrödinger equations via the Nehari method [J]. *Communications in Partial Differential Equations*, 2004, **29** (5/6): 879 − 901.

[4] Liu X, Liu J, Wang Z. Ground states for quasilinear Schrödinger equation with critical growth [J]. *Calculus of Variations and Partial Differential Equations*, 2013, **46**(3/4): 641 − 669.

[5] Silva E A B, Vieira G F. Quasilinear asymptotically periodic elliptic equations with critical growth [J]. *Calculus of Variations and Partial Differential Equations*, 2012, **39**(1/2): 1 − 33.

[6] Szulkin A, Weth T. The method of Nehari manifold [C]//*Handbook of Nonconvex Analysis and Applications*. Boston, USA: International Press, 2010: 597 − 632.

[7] Do J M, Miyagaki O H, Soares S H M. Soliton solutions for quasilinear Schrödinger equations with critical growth [J]. *Journal of Differential Equations*, 2010, **248**(4): 722 − 744.

[8] Silva E A B, Vieira G F. Quasilinear asymptotically periodic Schrödinger equations with subcritical growth [J]. *Nonlinear Analysis: Theory, Methods and Applications*, 2010, **72**(6): 2935 − 2949.

具有临界增长的渐进周期拟线性 Schrödinger 方程的基态解

张　慧　　张福保

(东南大学数学系,南京 211189)

摘要:针对一类具有临界增长的渐进周期的拟线性 Schrödinger 方程,证明了基态解的存在性. 首先利用一个变量代换,将拟线性 Schrödinger 方程转化为半线性 Schrödinger 方程. 半线性 Schrödinger 方程的泛函在 $H^1(\mathbf{R}^N)$ 中定义良好,并且半线性 Schrödinger 方程和拟线性 Schrödinger 方程的基态解是一一对应的. 然后利用山路引理证明了半线性 Schrödinger 方程的非平凡解的存在性. 最后,在适当的单调性条件下,运用 Nehari 流形的方法和集中紧性引理证明了得到的非平凡解恰好是半线性 Schrödinger 方程的基态解.

关键词:拟线性 Schrödinger 方程;变分方法;基态解;临界增长

中图分类号:O175.2