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Direct linear discriminant analysis based on column pivoting QR decomposition and economic SVD

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Abstract: A direct linear discriminant analysis algorithm based on economic singular value decomposition (DLDA/ESVD) is proposed to address the computationally complex problem of the conventional DLDA algorithm, which directly uses ESVD to reduce dimension and extract eigenvectors corresponding to nonzero eigenvalues. Then a DLDA algorithm based on column pivoting orthogonal triangular (QR) decomposition and ESVD (DLDA/QR-ESVD) is proposed to improve the performance of the DLDA/ESVD algorithm by processing a high-dimensional low rank matrix, which uses column pivoting QR decomposition to reduce dimension and ESVD to extract eigenvectors corresponding to nonzero eigenvalues. The experimental results on ORL, FERET and YALE face databases show that the proposed two algorithms can achieve almost the same performance and outperform the conventional DLDA algorithm in terms of computational complexity and training time. In addition, the experimental results on random data matrices show that the DLDA/QR-ESVD algorithm achieves better performance than the DLDA/ESVD algorithm by processing high-dimensional low rank matrices.

Key words: direct linear discriminant analysis; column pivoting orthogonal triangular decomposition; economic singular value decomposition; dimension reduction; feature extraction

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The direct linear discriminant analysis (DLDA) is an important method for dimension reduction and feature extraction in many applications such as face recognition[1−3], microarray data classification[4], text classification[5]. Yu and Yang[1] first proposed the DLDA algorithm based on eigenvalue decomposition (DLDA/EVD) by utilizing the information of the range space of between-class scatter matrix $\boldsymbol{S}_{\mathrm{b}}$ and within-class scatter matrix $\boldsymbol{S}_{\mathrm{w}}$ for face identification. In recent years, many approaches have been brought to improve the DLDA algorithm. Song et al.[2] proposed a PD-LDA algorithm by introducing a parameter β to improve the recognition rate; however, the improvement is not obvious and the choice of parameter β is difficult. Paliwal and Sharma[4] developed an improved DLDA algorithm to improve classification accuracy for DNA datasets; however, it is improper to deal with high-dimensional data such as face recognition.

Dimension reduction and eigenvectors extraction corresponding to nonzero eigenvalues are the main tasks of the DLDA algorithm. To achieve the two tasks, Yu and Yang's algorithm adopts the principal component analysis (PCA) method and EVD; Song and Paliwal's[2,4] algorithms use singular value decomposition (SVD). All the algorithms mentioned above are computationally complex. In this paper, two improved DLDA algorithms are proposed to reduce the computational complexity of the conventional DLDA algorithm.

In this paper, we propose the DLDA/ESVD algorithm that directly uses economic singular value decomposition (ESVD) to reduce dimension and extract eigenvectors corresponding to nonzero eigenvalues. Then we further propose the DLDA/QR-ESVD algorithm that uses high-performance column pivoting orthogonal triangular (QR) decomposition to reduce dimension and ESVD to extract eigenvectors corresponding to nonzero eigenvalues. The proposed two algorithms are efficient and outperform the conventional DLDA algorithm in terms of computational complexity. In addition, the DLDA/QR-ESVD algorithm achieves better performance than DLDA/ESVD algorithm by processing high-dimensional low rank matrices.

1 Direct Linear Discriminant Analysis

A brief overview of the DLDA algorithm is presented here. The DLDA algorithm aims to find a projection matrix that diagonalizes both within-class scatter matrix $\boldsymbol{S}_{\mathrm{w}}$ and between-class scatter matrix $\boldsymbol{S}_{\mathrm{b}}$ simultaneously. In the DLDA algorithm, within-class scatter matrix $\boldsymbol{S}_{\mathrm{w}}$ and between-class scatter matrix $\boldsymbol{S}_{\mathrm{b}}$ are defined as [6]

$$\boldsymbol{S}_{\mathrm{w}} = \frac{1}{n}\sum_{i=1}^{c}\sum_{j=1}^{n_i}(\boldsymbol{x}_i^j - \boldsymbol{\mu}_i)(\boldsymbol{x}_i^j - \boldsymbol{\mu}_i)^{\mathrm{T}} \tag{1}$$

$$\boldsymbol{S}_{\mathrm{b}} = \sum_{i=1}^{c}\frac{n_i}{n}(\boldsymbol{\mu}_i - \boldsymbol{\mu})(\boldsymbol{\mu}_i - \boldsymbol{\mu})^{\mathrm{T}} \tag{2}$$

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The precursors[3] $\boldsymbol{H}_\mathrm{w}$ and $\boldsymbol{H}_\mathrm{b}$ of the within-class scatter and between-class matrices in Eqs. (1) and (2) are

$$\boldsymbol{H}_\mathrm{w} = \frac{1}{\sqrt{n}}(\boldsymbol{X}_1 - \boldsymbol{\mu}_1\boldsymbol{e}_1, \ldots, \boldsymbol{X}_c - \boldsymbol{\mu}_c\boldsymbol{e}_c) \tag{3}$$

$$\boldsymbol{H}_\mathrm{b} = \frac{1}{\sqrt{n}}(\sqrt{n_1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}), \ldots, \sqrt{n_c}(\boldsymbol{\mu}_c - \boldsymbol{\mu})) \tag{4}$$

where $\boldsymbol{e}_i = [1, \ldots, 1] \in \mathbf{R}^{n_i}$; $\boldsymbol{X}_i$ is the data matrix of the i-th class; $\boldsymbol{\mu}_i$ is the mean of the i-th class, and $\boldsymbol{\mu}$ is the mean of the training samples; c is the number of training sample classes and n is the number of training samples. It is easy to verify that $\boldsymbol{S}_\mathrm{w} = \boldsymbol{H}_\mathrm{w}\boldsymbol{H}_\mathrm{w}^\mathrm{T}$ and $\boldsymbol{S}_\mathrm{b} = \boldsymbol{H}_\mathrm{b}\boldsymbol{H}_\mathrm{b}^\mathrm{T}$, where $\boldsymbol{S}_\mathrm{w} \in \mathbf{R}^{m\times m}$, $\boldsymbol{S}_\mathrm{b} \in \mathbf{R}^{m\times m}$, $\boldsymbol{H}_\mathrm{w} \in \mathbf{R}^{m\times n}$, $\boldsymbol{H}_\mathrm{b} \in \mathbf{R}^{m\times c}$. The procedure of the DLDA/EVD algorithm can be summarized as follows.

Step 1 Calculate an orthogonal matrix $\boldsymbol{V}$ using PCA such that $\boldsymbol{V}^\mathrm{T}\boldsymbol{S}_\mathrm{b}\boldsymbol{V} = \boldsymbol{\Lambda}$. The null space of $\boldsymbol{S}_\mathrm{b}$ carries no useful discriminant information[2], so the zero-value diagonal elements of $\boldsymbol{\Lambda}$ are discarded. Let $\boldsymbol{Y}$ be the r columns of $\boldsymbol{V}$ corresponding to the nonzero eigenvalues of $\boldsymbol{S}_\mathrm{b}$; thus, $\boldsymbol{Y}^\mathrm{T}\boldsymbol{S}_\mathrm{b}\boldsymbol{Y} = \boldsymbol{D}_\mathrm{b} > 0$. Let $\boldsymbol{Z} = \boldsymbol{Y}\boldsymbol{D}_\mathrm{b}^{-1/2}$; obviously $\boldsymbol{Z}^\mathrm{T}\boldsymbol{S}_\mathrm{b}\boldsymbol{Z} = \boldsymbol{I}_\mathrm{b}$.

Step 2 Find an orthogonal matrix $\boldsymbol{U}$ using eigenvalue decomposition such that $\boldsymbol{U}^\mathrm{T}\boldsymbol{Z}^\mathrm{T}\boldsymbol{S}_\mathrm{w}\boldsymbol{Z}\boldsymbol{U} = \boldsymbol{D}_\mathrm{w}$. It is easy to verify that the matrix $\boldsymbol{U}^\mathrm{T}\boldsymbol{Z}^\mathrm{T}$ simultaneously diagonalizes both $\boldsymbol{S}_\mathrm{b}$ and $\boldsymbol{S}_\mathrm{w}$. The projection matrix of the DLDA/EVD algorithm is $\boldsymbol{W} = \boldsymbol{D}_\mathrm{w}^{-1/2}\boldsymbol{U}^\mathrm{T}\boldsymbol{Z}^\mathrm{T} = \boldsymbol{D}_\mathrm{w}^{-1/2}\boldsymbol{U}^\mathrm{T}\boldsymbol{D}_\mathrm{b}^{-1/2}\boldsymbol{Y}^\mathrm{T}$.

2 Proposed algorithms

First, the DLDA/ESVD algorithm is presented in detail, and then we further present the DLDA/QR-ESVD algorithm, which can obtain better performance than the DLDA/ESVD algorithm by processing a high-dimensional low rank matrix.

2.1 DLDA/ESVD algorithm

In the DLDA/ESVD algorithm, the rectangular matrix $\boldsymbol{H}_\mathrm{b}$ in $\boldsymbol{S}_\mathrm{b} = \boldsymbol{H}_\mathrm{b}\boldsymbol{H}_\mathrm{b}^\mathrm{T}$ can be decomposed into orthogonal matrix $\boldsymbol{Q}_\mathrm{b} \in \mathbf{R}^{m\times c}$, diagonal matrix $\boldsymbol{D}_\mathrm{b} \in \mathbf{R}^{c\times c}$, and orthogonal matrix $\boldsymbol{V}_\mathrm{b} \in \mathbf{R}^{c\times c}$ using the ESVD[7] as

$$\boldsymbol{H}_\mathrm{b} = \boldsymbol{Q}_\mathrm{b}\boldsymbol{D}_\mathrm{b}\boldsymbol{V}_\mathrm{b} \tag{5}$$

Discard the zero-value diagonal elements of $\boldsymbol{D}_\mathrm{b}$ with corresponding orthogonal eigenvectors of $\boldsymbol{Q}_\mathrm{b}$ and $\boldsymbol{V}_\mathrm{b}$, and then we obtain $\widetilde{\boldsymbol{Q}}_\mathrm{b} \in \mathbf{R}^{m\times r}$, $\widetilde{\boldsymbol{D}}_\mathrm{b} \in \mathbf{R}^{r\times r}$ and $\widetilde{\boldsymbol{V}}_\mathrm{b} \in \mathbf{R}^{r\times c}$, where $\mathrm{rank}(\boldsymbol{H}_\mathrm{b}) = r$. It is easy to verify that

$$\boldsymbol{H}_\mathrm{b} = \widetilde{\boldsymbol{Q}}_\mathrm{b}\widetilde{\boldsymbol{D}}_\mathrm{b}\widetilde{\boldsymbol{V}}_\mathrm{b} \tag{6}$$

Substituting Eq. (6) into $\boldsymbol{S}_\mathrm{b} = \boldsymbol{H}_\mathrm{b}\boldsymbol{H}_\mathrm{b}^\mathrm{T}$, we obtain

$$\boldsymbol{S}_\mathrm{b} = \widetilde{\boldsymbol{Q}}_\mathrm{b}\widetilde{\boldsymbol{D}}_\mathrm{b}\widetilde{\boldsymbol{V}}_\mathrm{b}(\widetilde{\boldsymbol{Q}}_\mathrm{b}\widetilde{\boldsymbol{D}}_\mathrm{b}\widetilde{\boldsymbol{V}}_\mathrm{b})^\mathrm{T} = \widetilde{\boldsymbol{Q}}_\mathrm{b}\widetilde{\boldsymbol{D}}_\mathrm{b}\widetilde{\boldsymbol{V}}_\mathrm{b}\widetilde{\boldsymbol{V}}_\mathrm{b}^\mathrm{T}\widetilde{\boldsymbol{D}}_\mathrm{b}{}^\mathrm{T}\widetilde{\boldsymbol{Q}}_\mathrm{b}{}^\mathrm{T} = \widetilde{\boldsymbol{Q}}_\mathrm{b}\widetilde{\boldsymbol{D}}_\mathrm{b}{}^2\widetilde{\boldsymbol{Q}}_\mathrm{b}{}^\mathrm{T} \tag{7}$$

Thus, it is easy to verify that

$$(\widetilde{\boldsymbol{Q}}_\mathrm{b}\widetilde{\boldsymbol{D}}_\mathrm{b}{}^{-1})^\mathrm{T}\boldsymbol{S}_\mathrm{b}(\widetilde{\boldsymbol{Q}}_\mathrm{b}\widetilde{\boldsymbol{D}}_\mathrm{b}{}^{-1}) = \boldsymbol{I}_\mathrm{r} \tag{8}$$

Let $\boldsymbol{Z} = \widetilde{\boldsymbol{Q}}_\mathrm{b}(\widetilde{\boldsymbol{D}}_\mathrm{b}^{-1})^{-1}$, and the dimension of symmetric matrix $\boldsymbol{S}_\mathrm{w}$ can be reduced by $\boldsymbol{Z}$ as $\boldsymbol{Z}^\mathrm{T}\boldsymbol{S}_\mathrm{w}\boldsymbol{Z}$. The symmetric matrix $\boldsymbol{Z}^\mathrm{T}\boldsymbol{S}_\mathrm{w}\boldsymbol{Z}$ can be decomposed into orthogonal matrix $\boldsymbol{U}_\mathrm{w} \in \mathbf{R}^{r\times r}$ and diagonal matrix $\boldsymbol{D}_\mathrm{w} \in \mathbf{R}^{r\times r}$ by using the ESVD, and it can be presented as

$$\boldsymbol{Z}^\mathrm{T}\boldsymbol{S}_\mathrm{w}\boldsymbol{Z} = \boldsymbol{U}_\mathrm{w}\boldsymbol{D}_\mathrm{w}\boldsymbol{U}_\mathrm{w}^\mathrm{T} \tag{9}$$

Since

$$(\boldsymbol{U}_\mathrm{w}^\mathrm{T}\boldsymbol{Z}^\mathrm{T})\boldsymbol{S}_\mathrm{b}(\boldsymbol{U}_\mathrm{w}^\mathrm{T}\boldsymbol{Z}^\mathrm{T})^\mathrm{T} = \boldsymbol{U}_\mathrm{w}^\mathrm{T}(\widetilde{\boldsymbol{Q}}_\mathrm{b}\widetilde{\boldsymbol{D}}_\mathrm{b}^{-1})^\mathrm{T}\boldsymbol{S}_\mathrm{b}(\widetilde{\boldsymbol{Q}}_\mathrm{b}\widetilde{\boldsymbol{D}}_\mathrm{b}^{-1})\boldsymbol{U}_\mathrm{w} = \boldsymbol{U}_\mathrm{w}^\mathrm{T}\boldsymbol{I}_\mathrm{r}\boldsymbol{U}_\mathrm{w} = \boldsymbol{I}_\mathrm{r}$$

$$(\boldsymbol{U}_\mathrm{w}^\mathrm{T}\boldsymbol{Z}^\mathrm{T})\boldsymbol{S}_\mathrm{w}(\boldsymbol{U}_\mathrm{w}^\mathrm{T}\boldsymbol{Z}^\mathrm{T})^\mathrm{T} = \boldsymbol{U}_\mathrm{w}^\mathrm{T}\boldsymbol{Z}^\mathrm{T}\boldsymbol{S}_\mathrm{w}\boldsymbol{Z}\boldsymbol{U}_\mathrm{w} = \boldsymbol{U}_\mathrm{w}^\mathrm{T}\boldsymbol{U}_\mathrm{w}\boldsymbol{D}_\mathrm{w}\boldsymbol{U}_\mathrm{w}^\mathrm{T}\boldsymbol{U}_\mathrm{w} = \boldsymbol{D}_\mathrm{w}$$

matrix $\boldsymbol{U}_\mathrm{w}^\mathrm{T}\boldsymbol{Z}^\mathrm{T}$ can diagonalize both within-class scatter matrix $\boldsymbol{S}_\mathrm{w}$ and between-class scatter matrix $\boldsymbol{S}_\mathrm{b}$ simultaneously, and the projection matrix of the DLDA/ESVD algorithm is $\boldsymbol{W} = \boldsymbol{D}_\mathrm{w}^{-1/2}\boldsymbol{U}_\mathrm{w}^\mathrm{T}\boldsymbol{Z}^\mathrm{T} = \boldsymbol{D}_\mathrm{w}^{-1/2}\boldsymbol{U}_\mathrm{w}^\mathrm{T}\widetilde{\boldsymbol{D}}_\mathrm{b}^{-1}\widetilde{\boldsymbol{Q}}_\mathrm{b}{}^\mathrm{T}$. The DLDA/ESVD algorithm is summarized as below:

Step 1 $\boldsymbol{H}_\mathrm{b}$ is decomposed as $\boldsymbol{H}_\mathrm{b} = \boldsymbol{Q}_\mathrm{b}\boldsymbol{D}_\mathrm{b}\boldsymbol{V}_\mathrm{b}$ by using the ESVD. Discarding the zero-value diagonal elements of $\boldsymbol{D}_\mathrm{b}$ with corresponding orthogonal eigenvectors of $\boldsymbol{Q}_\mathrm{b}$ and $\boldsymbol{V}_\mathrm{b}$, and then we obtain $\widetilde{\boldsymbol{Q}}_\mathrm{b} \in \mathbf{R}^{m\times r}$, $\widetilde{\boldsymbol{D}}_\mathrm{b}^{-1} \in \mathbf{R}^{r\times r}$, $\widetilde{\boldsymbol{V}}_\mathrm{b} \in \mathbf{R}^{r\times c}$, where rank $(\boldsymbol{H}_\mathrm{b}) = r$, $\boldsymbol{Z} = \widetilde{\boldsymbol{Q}}_\mathrm{b}(\widetilde{\boldsymbol{D}}_\mathrm{b}^{-1})^{-1}$.

Step 2 $\boldsymbol{Z}^\mathrm{T}\boldsymbol{S}_\mathrm{w}\boldsymbol{Z}$ is decomposed as $\boldsymbol{Z}^\mathrm{T}\boldsymbol{S}_\mathrm{w}\boldsymbol{Z} = \boldsymbol{U}_\mathrm{w}\boldsymbol{D}_\mathrm{w}\boldsymbol{U}_\mathrm{w}^\mathrm{T}$ by using the ESVD, and the projection matrix of the DLDA/ESVD algorithm is $\boldsymbol{W} = \boldsymbol{D}_\mathrm{w}^{-1/2}\boldsymbol{U}_\mathrm{w}^\mathrm{T}\boldsymbol{Z}^\mathrm{T} = \boldsymbol{D}_\mathrm{w}^{-1/2}\boldsymbol{U}_\mathrm{w}^\mathrm{T}\widetilde{\boldsymbol{D}}_\mathrm{b}^{-1}\widetilde{\boldsymbol{Q}}_\mathrm{b}{}^\mathrm{T}$.

2.2 DLDA/QR-ESVD algorithm

From Step 1 of the DLDA/ESVD algorithm, if we can directly acquire $\widetilde{\boldsymbol{Q}}_\mathrm{b}$ without calculating $\boldsymbol{Q}_\mathrm{b}$ by processing a high-dimensional low rank matrix, the performance of the DLDA/ESVD algorithm can be further improved, while the column pivoting QR decomposition can solve this problem. In the DLDA/QR-ESVD algorithm, the rectangular matrix $\boldsymbol{H}_\mathrm{b}$ in $\boldsymbol{S}_\mathrm{b} = \boldsymbol{H}_\mathrm{b}\boldsymbol{H}_\mathrm{b}^\mathrm{T}$ is first decomposed into orthogonal matrix $\boldsymbol{Q}_\mathrm{b} \in \mathbf{R}^{m\times r}$, upper triangular matrix $\boldsymbol{R}_\mathrm{b} \in \mathbf{R}^{r\times r}$, and permutation matrix $\boldsymbol{E} \in \mathbf{R}^{r\times c}$ ($\boldsymbol{E}\boldsymbol{E}^\mathrm{T} = \boldsymbol{I}_{r\times r}$, $\mathrm{rank}(\boldsymbol{H}_\mathrm{b}) = r$) using column pivoting QR decomposition[7] as

$$\boldsymbol{H}_\mathrm{b} = \boldsymbol{Q}_\mathrm{b}\boldsymbol{R}_\mathrm{b}\boldsymbol{E} \tag{10}$$

Substituting Eq. (6) into $\boldsymbol{S}_\mathrm{b} = \boldsymbol{H}_\mathrm{b}\boldsymbol{H}_\mathrm{b}^\mathrm{T}$, we obtain

$$\boldsymbol{S}_\mathrm{b} = (\boldsymbol{Q}_\mathrm{b}\boldsymbol{R}_\mathrm{b}\boldsymbol{E})(\boldsymbol{Q}_\mathrm{b}\boldsymbol{R}_\mathrm{b}\boldsymbol{E})^\mathrm{T} = \boldsymbol{Q}_\mathrm{b}\boldsymbol{R}_\mathrm{b}\boldsymbol{R}_\mathrm{b}^\mathrm{T}\boldsymbol{Q}_\mathrm{b}^\mathrm{T} \tag{11}$$

Then matrix $\boldsymbol{R}_\mathrm{b}$ can be decomposed by the ESVD as

$$\boldsymbol{R}_\mathrm{b} = \boldsymbol{U}_\mathrm{b}\boldsymbol{D}_\mathrm{b}\boldsymbol{V}_\mathrm{b} \tag{12}$$

where both $\boldsymbol{U}_\mathrm{b}$ and $\boldsymbol{V}_\mathrm{b}$ are orthogonal matrix; $\boldsymbol{D}_\mathrm{b}$ is a diagonal matrix; and $\boldsymbol{U}_\mathrm{b} \in \mathbf{R}^{r\times r}$, $\boldsymbol{D}_\mathrm{b} \in \mathbf{R}^{r\times r}$, $\boldsymbol{V}_\mathrm{b} \in \mathbf{R}^{r\times r}$.

Substituting Eq. (12) into Eq. (11), we obtain

$$\boldsymbol{S}_\mathrm{b} = \boldsymbol{Q}_\mathrm{b}\boldsymbol{U}_\mathrm{b}\boldsymbol{D}_\mathrm{b}\boldsymbol{V}_\mathrm{b}\boldsymbol{E}\boldsymbol{E}^\mathrm{T}\boldsymbol{V}_\mathrm{b}^\mathrm{T}\boldsymbol{D}_\mathrm{b}^\mathrm{T}\boldsymbol{Q}_\mathrm{b}^\mathrm{T} = \boldsymbol{Q}_\mathrm{b}\boldsymbol{U}_\mathrm{b}\boldsymbol{D}_\mathrm{b}^2\boldsymbol{U}_\mathrm{b}^\mathrm{T}\boldsymbol{Q}_\mathrm{b}^\mathrm{T}$$

Thus, it is easy to verify that

$$(\boldsymbol{Q}_\mathrm{b}\boldsymbol{U}_\mathrm{b}\boldsymbol{D}_\mathrm{b}^{-1})^\mathrm{T}\boldsymbol{S}_\mathrm{b}(\boldsymbol{Q}_\mathrm{b}\boldsymbol{U}_\mathrm{b}\boldsymbol{D}_\mathrm{b}^{-1}) = \boldsymbol{I}_\mathrm{r} \tag{13}$$

Let $\boldsymbol{Z} = \boldsymbol{Q}_\mathrm{b}\boldsymbol{U}_\mathrm{b}\boldsymbol{D}_\mathrm{b}^{-1}$. The dimension of symmetric matrix $\boldsymbol{S}_\mathrm{w}$ can be reduced by $\boldsymbol{Z}$ as $\boldsymbol{Z}^\mathrm{T}\boldsymbol{S}_\mathrm{w}\boldsymbol{Z}$. The symmetric matrix

$\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{S}_{\mathrm{w}}\boldsymbol{Z}$ can be decomposed into orthogonal matrix $\boldsymbol{U}_{\mathrm{w}} \in \mathbf{R}^{r\times r}$, and diagonal matrix $\boldsymbol{D}_{\mathrm{w}} \in \mathbf{R}^{r\times r}$ using the ESVD is

$$\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{S}_{\mathrm{w}}\boldsymbol{Z} = \boldsymbol{U}_{\mathrm{w}}\boldsymbol{D}_{\mathrm{w}}\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}} \tag{14}$$

Since

$$(\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}})\boldsymbol{S}_{\mathrm{b}}(\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}(\boldsymbol{Q}_{\mathrm{b}}\boldsymbol{U}_{\mathrm{b}}\boldsymbol{D}_{\mathrm{b}}^{-1})^{\mathrm{T}}\boldsymbol{S}_{\mathrm{b}}(\boldsymbol{Q}_{\mathrm{b}}\boldsymbol{U}_{\mathrm{b}}\boldsymbol{D}_{\mathrm{b}}^{-1})\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}} = \boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}\boldsymbol{I}_{r}\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}} = \boldsymbol{I}_{r}$$

$$(\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}})\boldsymbol{S}_{\mathrm{w}}(\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{S}_{\mathrm{w}}\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}} = \boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}\boldsymbol{U}_{\mathrm{w}}\boldsymbol{D}_{\mathrm{w}}\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}} = \boldsymbol{D}_{\mathrm{w}}$$

matrix $\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}}$ can diagonalize both within-class scatter matrix $\boldsymbol{S}_{\mathrm{w}}$ and between-class scatter matrix $\boldsymbol{S}_{\mathrm{b}}$ simultaneously, and the projection matrix of DLDA/QR-ESVD algorithm is $\boldsymbol{W} = \boldsymbol{D}_{\mathrm{w}}^{-1/2}\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}} = \boldsymbol{D}_{\mathrm{w}}^{-1/2}\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}\boldsymbol{D}_{\mathrm{b}}^{-1}\boldsymbol{U}_{\mathrm{b}}^{\mathrm{T}}\boldsymbol{Q}_{\mathrm{b}}^{\mathrm{T}}$. The DLDA/QR-ESVD algorithm is summarized as follows:

Step 1 $\boldsymbol{H}_{\mathrm{b}}$ is decomposed as $\boldsymbol{H}_{\mathrm{b}} = \boldsymbol{Q}_{\mathrm{b}}\boldsymbol{R}_{\mathrm{b}}\boldsymbol{E}$ by using column pivoting QR decomposition, and then $\boldsymbol{R}_{\mathrm{b}}$ is decomposed as $\boldsymbol{R}_{\mathrm{b}} = \boldsymbol{U}_{\mathrm{b}}\boldsymbol{D}_{\mathrm{b}}\boldsymbol{V}_{\mathrm{b}}$ by using the ESVD. Let $\boldsymbol{Z} = \boldsymbol{Q}_{\mathrm{b}}\boldsymbol{U}_{\mathrm{b}}\boldsymbol{D}_{\mathrm{b}}^{-1}$.

Step 2 $\mathrm{Z}^{\mathrm{T}}\boldsymbol{S}_{\mathrm{w}}\boldsymbol{Z}$ is decomposed as $\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{S}_{\mathrm{w}}\boldsymbol{Z} = \boldsymbol{U}_{\mathrm{w}}\boldsymbol{D}_{\mathrm{w}}\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}$ by using the ESVD, and the projection matrix of the DLDA/QR-ESVD algorithm is $\boldsymbol{W} = \boldsymbol{D}_{\mathrm{w}}^{-1/2}\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}} = \boldsymbol{D}_{\mathrm{w}}^{-1/2}\boldsymbol{U}_{\mathrm{w}}^{\mathrm{T}}\boldsymbol{D}_{\mathrm{b}}^{-1}\boldsymbol{U}_{\mathrm{b}}^{\mathrm{T}}\boldsymbol{Q}_{\mathrm{b}}^{\mathrm{T}}$.

3 Experiments

The experiments are used to verify the efficiency of the proposed two algorithms and the performance of the DLDA/QR-ESVD is better than that of the DLDA/ESVD by processing a high-dimensional low rank matrix. First, experiments for the DLDA/EVD, DLDA/ESVD and DLDA/QR-ESVD algorithms are conducted on ORL[8], FERET[9] and YALE[10] face databases. Secondly, the comparison testing between the DLDA/ESVD and the DLDA/QR-ESVD are conducted on random matrices. The experiments are tested on the PC with Core™2 Duo 2.99 GHz processor with 1.96 GB of RAM using Matlab 7.0 software.

3.1 Experiments on face databases

Tab. 1 introduces three face databases in experiments, where Size stands for the number of all images in each database; Dimensions are the dimensionalities of image vectors; and Classes are the number of persons.

Tab. 1 Description of three face databases

Database	Size	Dimensions	Classes
ORL	400	10 304	40
FERET	490	10 304	10
YALE	165	45 045	11

In each face database, the recognition rates and the training time of the DLDA/EVD, DLDA/ESVD and DLDA/QR-ESVD algorithms are tested. The recognition rates are used to evaluate the accuracy of the three algorithms. The training time is used to measure the computation time of each algorithm for dimension reduction and feature extraction, and the difference of the execution time in databases is mainly caused by the training time using different algorithms.

There are three main steps for testing the aforementioned algorithms. First, training sets are randomly selected from the face database, and the rest forms testing sets. Secondly, the training sets are trained to achieve dimension reduction and feature extraction using the above three algorithms under the same conditions, and the training time of each algorithm is recorded. Finally, both the training sets and the testing sets are projected into the optimal LDA subspace, and the nearest neighbor classifier based on the Euclidean distance is adopted to be the final classifier[11]. The final result we take is an average result of classification for 40 times based on cross-validation experiments.

Fig. 1 shows the recognition rates on ORL, FERET and

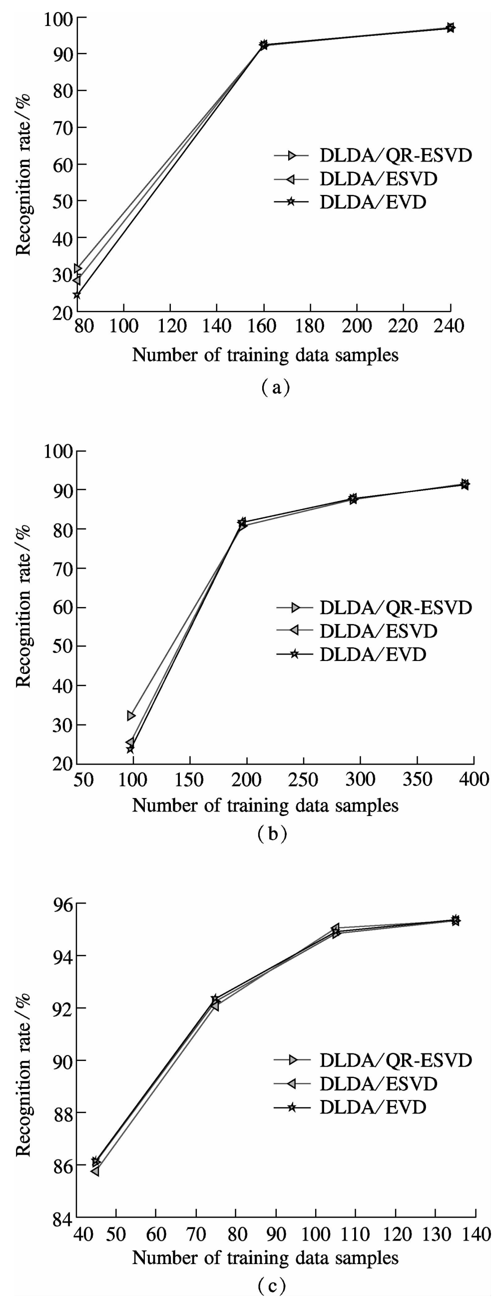

Fig. 1 Recognition rates on different databases. (a) ORL face database; (b) FERET face database; (c) YALE face database

YALE face databases by using the DLDA/EVD, DLDA/ESVD and DLDA/QR-ESVD algorithms, respectively. It can be seen that the three algorithms achieve almost the same recognition rates on the three face databases under different numbers of training samples.

Fig. 2 shows the training time on ORL, FERET and YALE face databases by using three algorithms respectively. It can be seen that the training times of the DLDA/ESVD algorithm and the DLDA/QR-ESVD algorithm are distinctly lower than those of the DLDA/EVD algorithm on the three face databases. The proposed two algorithms consume almost the same training time; the reason is that the rank of between-class matrix $\boldsymbol{S}_{\mathrm{b}}$ is approximately equal to the number of training sample classes ($c \approx r$) on the three face databases.

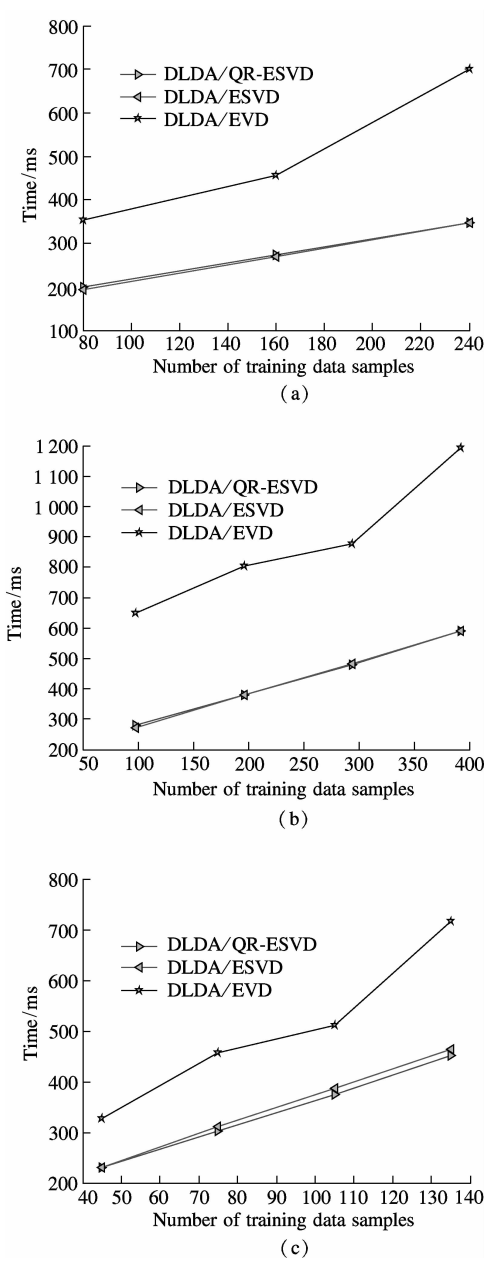

Fig. 2 Computation time on different databases. (a) ORL face database; (b) FERET face database; (c) YALE face database

3.2 Experiments on random data matrices

As it is difficult to find a public database with high-dimensional low rank data matrices to test the DLDA/ESVD and DLDA/QR-ESVD algorithms. Random data matrix $\boldsymbol{H} \in \mathbf{R}^{m \times c}(\operatorname{rank}(\boldsymbol{H}) = r)$ with variable dimensions m from 5 000 to 10 000 are generated to verify the proposed two algorithms. Fig. 3(a) shows that the proposed two algorithms can achieve similar computation time by processing high-dimensional full rank matrices ($c = r = 500$). Fig. 3(b) shows that the computation time of the DLDA/QR-ESVD algorithm is distinctly lower than that of the DLDA/ESVD algorithm by processing high-dimensional low rank matrices ($r \ll c$, $c = 800$, $r = 200$).

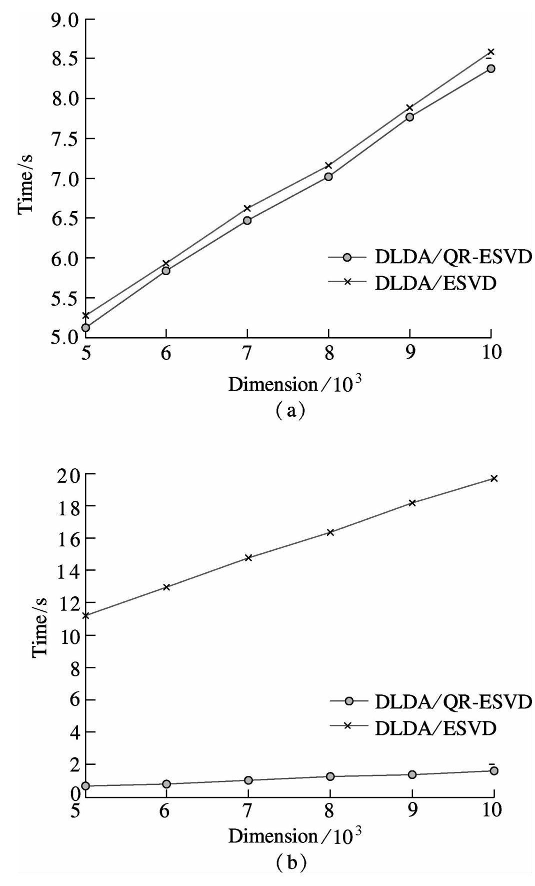

Fig. 3 Computation time on random data matrices. (a) High-dimensional full rank matrices; (b) High-dimensional low rank matrices

4 Conclusion

In this paper, the DLDA/ESVD algorithm is proposed, which directly uses the ESVD to reduce dimension and extract eigenvectors corresponding to nonzero eigenvalues. Then we further propose the DLDA/QR-ESVD algorithm that uses high-performance column pivoting QR decomposition to reduce dimension and ESVD to extract eigenvectors corresponding to nonzero eigenvalues. The proposed two algorithms outperform the DLDA/EVD algorithm in terms of computational complexity and training time. The proposed two algorithms consume almost similar computation time by processing a high-dimensional full rank matrix ($r=c$). But the computation time of the DLDA/QR-ESVD algorithm is distinctly lower than that of the DLDA/ESVD algorithm by processing a high-dimensional low rank matrix ($r\ll c$).

It is worth exploring in two directions. First, since a computationally efficient way of reducing dimension is crucial in many fields of research, a number of applications of the DLDA/ESVD and DLDA/QR-ESVD algorithms should be envisaged. Secondly, the theoretical analysis of the proposed two algorithms should be further studied.

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基于列选主 QR 分解和节约型 SVD 的直接线性鉴别分析

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摘要:针对传统 DLDA 算法计算复杂的问题,提出了 DLDA/ESVD 算法,该算法直接使用 ESVD 降维和提取非零特征值对应的特征向量. 然后,为了提高 DLDA/ESVD 算法处理高维低秩矩阵的性能,提出了 DLDA/QR-ESVD 算法,该算法使用列选主 QR 分解降维,使用 ESVD 提取非零特征值对应的特征向量. 在 ORL,FERET 和 YALE 数据库上的实验结果表明,所提出的 2 种算法具有几乎相同的性能,并在计算复杂性和训练时间方面优于传统的 DLDA 算法. 另外,在随机数据矩阵上的实验结果表明,DLDA/QR-ESVD 算法处理高维低秩矩阵的性能优于 DLDA/ESVD 算法.

关键词:直接线性鉴别分析;列选主的正交三角分解;节约型奇异值分解;降维;特征提取
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