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Construction of semisimple category over generalized Yetter-Drinfeld modules

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Abstract: Let H be a commutative, noetherian, semisimple and cosemisimple Hopf algebra with a bijective antipode over a field k. Then the semisimplicity of $\mathrm{YD}(H)$ is considered, where $\mathrm{YD}(H)$ means the disjoint union of the category of generalized Yetter-Drinfeld modules ${}_H\mathrm{YD}^H(\alpha,\beta)$ for any $\alpha,\beta \in \mathrm{Aut}_{\mathrm{Hopf}}(H)$. First, the fact that $\mathrm{YD}(H)$ is closed under Mor is proved. Secondly, based on the properties of finitely generated projective modules and semisimplicity of H, $\mathrm{YD}(H)$ satisfies the exact condition. Thus each object in $\mathrm{YD}(H)$ can be decomposed into simple ones since H is noetherian and cosemisimple. Finally, it is proved that $\mathrm{YD}(H)$ is a semisimple category.

Key words: semisimple Hopf algebra; semisimple category; generalized Yetter-Drinfeld module

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In 2007, Panaite and Staic[1] introduced the notion of generalized Yetter-Drinfeld modules which covered both Yetter-Drinfeld modules and anti-Yetter-Drinfeld modules. Liu and Wang[2] studied the notion of generalized weak Yetter-Drinfeld modules and made the category of ${}_H\mathrm{WYD}^H(\alpha,\beta)$ into a braided T-category[3]. The fusion category[4-6] plays an important role in classifying the semisimple Hopf algebra. The semisimple category is the first step to construct a fusion category. In this paper, we discuss the following question: how to make the category of generalized Yetter-Drinfeld modules ${}_H\mathrm{YD}^H(\alpha,\beta)$ into a semisimple category.

Throughout this paper, we assume that H is a Hopf algebra[7-8] with a bijective antipode over a field k. Denote the set of all the automorphisms of H by $\mathrm{Aut}_{\mathrm{Hopf}}(H)$. Let $\alpha,\beta \in \mathrm{Aut}_{\mathrm{Hopf}}(H)$.

1 Preliminaries

Definition 1 For any $\alpha,\beta \in \mathrm{Aut}_{\mathrm{Hopf}}(H)$, a (α,β)-Yetter-Drinfeld module is a k-module M, such that M is a left H-module (with notation $h\otimes m \mapsto h\cdot m$) and a right H-comodule (with notation $m \mapsto m_{(0)}\otimes m_{(1)}$) with the following compatibility condition:

$$\rho(h\cdot m) = h_2\cdot m_{(0)}\otimes\beta(h_3)m_{(1)}\alpha(S^{-1}(h_1))$$

for all $h\in H$ and $m\in M$. The category of (α,β)-Yetter-Drinfeld modules and H-linear H-colinear maps is denoted by ${}_H\mathrm{YD}^H(\alpha,\beta)$.

Define the category of the generalized Yetter-Drinfeld module $\mathrm{YD}(H)$ as the disjoint union of all ${}_H\mathrm{YD}^H(\alpha,\beta)$.

Definition 2 Suppose that $M\in\mathrm{YD}(H)$, then M is called simple if it has no proper subobjects. A direct sum of simple objects is called semisimple. If every object $M\in\mathrm{YD}(H)$ is semisimple, we call the category $\mathrm{YD}(H)$ semisimple.

2 Making YD(H) into semisimple

Lemma 1[1] Suppose that $M\in{}_H\mathrm{YD}^H(\alpha,\beta)$ and $N\in{}_H\mathrm{YD}^H(\gamma,\delta)$, then $M\otimes N\in{}_H\mathrm{YD}^H(\alpha\gamma,\delta\gamma^{-1}\beta\gamma)$ with the following structures

$$h\cdot(m\otimes n) = \gamma(h_1)\cdot m\otimes\gamma^{-1}\beta\gamma(h_2)\cdot n$$

$$m\otimes n \mapsto (m\otimes n)_{(0)}\otimes(m\otimes n)_{(1)} = (m_{(0)}\otimes n_{(0)})\otimes n_{(1)}m_{(1)}$$

Lemma 2 If k is a commutative ring, H is a commutative Hopf algebra over k, $M\in{}_H\mathrm{YD}^H(\alpha,\beta)$, $N\in{}_H\mathrm{YD}^H(\gamma,\delta)$, and M is a finitely generated projective H-module, then

1) ${}_H\mathrm{hom}(M,N)$ is an H-comodule, and ${}_H\mathrm{hom}^H(M,N) = {}_H\mathrm{hom}(M,N)^{\mathrm{co}H}$, where the H-coaction is given by $\rho(f)(m) = f_0(\mathrm{m})\otimes f_1 \triangleq f(m_0)_0\otimes f(m_0)_1S(m_1)$.

2) ${}_H\mathrm{hom}(M,N)\in{}_H\mathrm{YD}^H(\alpha,\beta)$, where the H-action is given by $(h\cdot f)(m)\triangleq hf(m) = f(h\cdot m)$.

Proof 1) Define a map π: ${}_H\mathrm{hom}(M,N)\to{}_H\mathrm{hom}(M,N\otimes H)$ by $\pi(f)(m) = f(m_0)_0\otimes f(m_0)_1S(m_1)$.

For any $m\in M$, $h\in H$, we have

$$\begin{aligned}\pi(f)(h\cdot m) &= f((h\cdot m)_0)_0\otimes f((h\cdot m)_0)_1S((h\cdot m)_1) = \\ &f(h_2\cdot m_0)_0\otimes f(h_2\cdot m_0)_1S(\beta(h_3)m_1\alpha(S^{-1}(h_1))) = \\ &h\cdot f(m_0)_0\otimes\beta(h_3)\beta(S(h_4))f(m_0)_1\alpha(S^{-1}(h_2))\cdot \\ &\alpha(h_1)S(m_1) = h\cdot f(m_0)_0\otimes f(m_0)_1S(m_1) = \\ &h\cdot(\pi(f))(m))\end{aligned}$$

Thus, π is well defined. Since M is a finitely generated projective H-module, we have ${}_H\mathrm{hom}(M,N\otimes H)\cong{}_H\mathrm{hom}(M,N)\otimes H$. So we obtain a map:

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$$\rho: {}_H\mathrm{hom}(M,N)\to {}_H\mathrm{hom}(M,N)\otimes H$$

such that $\rho(f)(m)=f(m_0)_0\otimes f(m_0)_1S(m_1)$, and ${}_H\mathrm{hom}(M,N)\in M^H$.

Now for any $f\in {}_H\mathrm{hom}(M,N)$, if f is H-colinear, then

$$\rho(f)(m)=f(m_0)_0\otimes f(m_0)_1S(m_1)= f(m_0)\otimes m_1S(m_2)=f(m)\otimes 1=(f\otimes 1)(m)$$

So f is coinvariant. Conversely, take $f\in {}_H\mathrm{hom}(M,N)^{\mathrm{co}H}$, then we have

$$\rho_N(f(m))=f(m_0)_0\otimes f(m_0)_1\varepsilon(m_1)= f(m_0)_0\otimes f(m_0)_1S(m_1)m_2=f(m_0)\otimes m_1$$

for any $m\in M$, and f is H-linear. Thus, ${}_H\mathrm{hom}^H(M,N)={}_H\mathrm{hom}(M,N)^{\mathrm{co}H}$.

2) For any $m\in M$, $h\in H$, we have

$$((h\cdot f)_0\otimes(h\cdot f)_1)(m)=(h\cdot f(m_0))_0\otimes (h\cdot f(m_0))_1S(m_1)=h_2\cdot f(m_0)_0\otimes \beta(h_3)f(m_0)_1\alpha(S^{-1}(h_1))S(m_1)= (h_2\cdot f_0\otimes\beta(h_3)f_1\alpha(S^{-1}(h_1)))(m)$$

Lemma 3 Let V be a k-module and N be an H-module, then

1) ${}_H\mathrm{hom}(H\otimes V,N)$ and $\mathrm{hom}(V,N)$ are isomorphic as k-modules, where the bijection is given by $\theta: {}_H\mathrm{hom}(H\otimes V,N)\to\mathrm{hom}(V,N)$, $\theta(f)(v)=f(1\otimes v)$.

2) If V is a projective k-module, then $H\otimes V$ is a projective H-module.

Furthermore, if $V\in M^H$, then $H\otimes V$ is an object of ${}_H\mathrm{YD}^H(\alpha,\beta)$ via

$$h\cdot(h'\otimes v)=hh'\otimes v$$

$$\rho(h\otimes v)=h_2\otimes v_0\otimes\beta(h_3)v_1\alpha(S^{-1}(h_1))$$

Similar to Lemma 2, we can obtain the following lemmas.

Lemma 4 Let $V\in M^H$ is a finitely generated projective k-module. Then for any H-comodule N, we have $\mathrm{hom}(V,N)\in M^H$, where the H-coaction is given by $\rho(g)(v)=g(v_0)_0\otimes g(v_0)_1S(v_1)$. If H is commutative, then for any $N\in {}_H\mathrm{YD}^H(\alpha,\beta)$, we can get ${}_H\mathrm{hom}(H\otimes V,N)\in {}_H\mathrm{YD}^H(\alpha,\beta)$.

Lemma 5 Suppose that H is commutative, and $N\in {}_H\mathrm{YD}^H(\alpha,\beta)$.

1) If $V\in M^H$ is a finitely generated projective k-module, then ${}_H\mathrm{hom}(H\otimes V,N)$ and $\mathrm{hom}(V,N)$ are isomorphic as H-comodules.

2) If k is a field, V is a finite-dimensional k-space and a projective right H-comodule, then $H\otimes V$ is a projective object in ${}_H\mathrm{YD}^H(\alpha,\beta)$.

Proof 1) It is straightforward.

2) Obviously, we have

$$ {}_H\mathrm{hom}^H(H\otimes V,N)\cong {}_H\mathrm{hom}(H\otimes V,N)^{\mathrm{co}H}\cong \mathrm{hom}(V,N)^{\mathrm{co}H}\cong\mathrm{hom}^H(V,N)$$

where the last isomorphism is due to the proof of Lemma 2. So the conclusion holds. From the above two lemmas, we have the following facts.

Lemma 6 Let k be a field, and $M\in {}_H\mathrm{YD}^H(\alpha,\beta)$. Then M is a finitely generated H-module if and only if there exists a finite dimensional H-comodule V and an H-linear H-colinear epimorphism $\pi: H\otimes V\to M$.

Let H^* be the linear dual of H. If $M,N\in M^H$, then $\mathrm{hom}_k(M,N)\in {}_{H^*}M$ under the following H^*-action

$$(h^*\cdot f)(m)=h^*(f(m_0)_1S(m_1))\cdot f(m_0)_0$$

Lemma 7 Assume that H is commutative, and $M,N\in {}_H\mathrm{YD}^H(\alpha,\beta)$. Then ${}_H\mathrm{hom}(M,N)$ is a left H^*-submodule of $\mathrm{hom}_k(M,N)$.

Furthermore, $M\in {}_{H^*}M$ is called rational if the left H^*-action on M is induced by a right H-coaction on M.

Proposition 1 Suppose that H is commutative, k is a field, $M,N\in {}_H\mathrm{YD}^H(\alpha,\beta)$, and M is a finitely generated H-module. Then ${}_H\mathrm{hom}(M,N)\in {}_H\mathrm{YD}^H(\alpha,\beta)$.

Proof By Lemma 6, there exists a finite dimensional H-comodule V and an H-linear H-colinear epimorphism $\pi: H\otimes V\to M$. So we obtain an injective k-linear map ${}_H\mathrm{hom}(\pi,N): {}_H\mathrm{hom}(M,N)\to {}_H\mathrm{hom}(H\otimes V,N)$. For any $\phi\in H^*$, $v\in V$, $h\in H$, $f\in {}_H\mathrm{hom}(M,N)$, we have $\pi(h\otimes v)=h\cdot v$, $\rho(1\otimes v)=1\otimes v_0\otimes v_1$, and

$$((\phi\cdot f)\circ\pi)(1\otimes v)=(\phi\cdot f)(v)= \phi(f(v_0)_1S(v_1))f(v_0)_0= \phi(f(\pi(1\otimes v_0))_1S(v_1))f(\pi(1\otimes v_0))_0= \phi(f(\pi(1\otimes v)_0)_1S(1\otimes v)_1)f(\pi(1\otimes v)_0)_0= (\phi\cdot(f\circ\pi))(1\otimes v)$$

It follows that ${}_H\mathrm{hom}(\pi,N)$ is H^*-linear. Then by Lemma 2, ${}_H\mathrm{hom}(H\otimes V,N)$ is an H-comodule, and, therefore, a rational H^*-module. Thus ${}_H\mathrm{hom}(M,N)$ is a rational H^*-submodule of ${}_H\mathrm{hom}(H\otimes V,N)$. This means that ${}_H\mathrm{hom}(M,N)$ is an H-comodule. Then we obtain ${}_H\mathrm{hom}(M,N)\in {}_H\mathrm{YD}^H(\alpha,\beta)$ by Lemma 2.

We say that ${}_H\mathrm{YD}^H(\alpha,\beta)$ satisfies the exact condition if the following property holds: if $M\in {}_H\mathrm{YD}^H(\alpha,\beta)$ is a finitely generated H-module, then the functor ${}_H\mathrm{hom}(M,\_): {}_H\mathrm{YD}^H(\alpha,\beta)\to {}_H\mathrm{YD}^H(\alpha,\beta)$ is exact.

By Proposition 1, if H is commutative and M is a finitely generated H-module, we have ${}_H\mathrm{hom}(M,N)\in {}_H\mathrm{YD}^H(\alpha,\beta)$ for any $N\in {}_H\mathrm{YD}^H(\alpha,\beta)$. Obviously ${}_H\mathrm{YD}^H(\alpha,\beta)$ satisfies the exact condition if H is semisimple.

Proposition 2 Assume that H is commutative, and ${}_H\mathrm{YD}^H(\alpha,\beta)$ satisfies the exact condition and the functor $(-)^{\mathrm{co}H}: {}_H\mathrm{YD}^H(\alpha,\beta)\to {}_kM$ is exact. Then any finitely generated H-module $M\in {}_H\mathrm{YD}^H(\alpha,\beta)$ is a projective object.

Proof we have ${}_H\mathrm{hom}^H(M,\_)\cong {}_H\mathrm{hom}(M,\_)^{\mathrm{co}H}=(-)^{\mathrm{co}H}\circ {}_H\mathrm{hom}(M,\_)$ which implies that ${}_H\mathrm{hom}^H(M,\_)$ is also an exact functor.

Proposition 3 Under the same condition of Proposition 2, suppose that H is noetherian. Then any finitely generated H-module $M \in {}_H\mathrm{YD}^H(\alpha,\beta)$ is a direct sum of a family of simple subobjects which are also finitely generated as H-modules in ${}_H\mathrm{YD}^H(\alpha,\beta)$.

Proof Assume that N is a subobject of M. Then N and M/N are finitely generated H-modules since H is noetherian. Furthermore, N and M/N are projective objects. So we have a split exact sequence in ${}_H\mathrm{YD}^H(\alpha,\beta)$: $0 \to N \to M \to M/N \to 0$.

Thus the conclusion holds.

Take $M \in {}_H\mathrm{YD}^H(\alpha,\beta)$ and an H-subcomodule V of M. We set

$$HV = \{\sum_{i \in I} a_i v_i \mid a_i \in H, v_i \in V\}$$

where I is a finite set. Then HV is a subobject of M in ${}_H\mathrm{YD}^H(\alpha,\beta)$ via:

$$h \cdot (\sum_{i \in I} a_i v_i) = \sum_{i \in I} h a_i v_i$$

$$\rho(\sum_{i \in I} a_i v_i) = \sum_{i \in I} (a_i)_2 (v_i)_0 \otimes \beta((a_i)_3)(v_i)_1 \alpha(S^{-1}(a_i)_1)$$

Theorem 1 Let H be commutative and noetherian, ${}_H\mathrm{YD}^H(\alpha,\beta)$ satisfies the exact condition and the functor $(-)^{\mathrm{co}H}$: ${}_H\mathrm{YD}^H(\alpha,\beta) \to {}_k M$ is exact. Then every $M \in {}_H\mathrm{YD}^H(\alpha,\beta)$ is a direct sum of a family of simple subobjects of M which are finitely generated as H-modules in ${}_H\mathrm{YD}^H(\alpha,\beta)$. Therefore, ${}_H\mathrm{YD}^H(\alpha,\beta)$ is a semisimple category.

Proof For any $m \in M$, m belongs to a finite dimensional H-subcomodule V_m of M. Then V_m is a finitely generated H-module. By Proposition 3, V_m is a direct sum of a family of simple subobjects which are finitely generated. Let Ω be the set of all direct sums $N = \oplus_{i \in I} N_i$ where every N_i is both a finitely generated H-module and a simple subobject of M in ${}_H\mathrm{YD}^H(\alpha,\beta)$. Then the sum of two elements in Ω is also an object in Ω. Thus Ω contains a maximal element M' through Zorn's Lemma. For any $m \in M$, we have $m \in HV_m \in \Omega$. This means that $HV_m + M' = M'$. So $M = M'$. Thus, the conclusion holds.

Corollary 1 Let H be commutative and noetherian (particularly finite dimensional), semisimple and cosemisimple. Then each $M \in {}_H\mathrm{YD}^H(\alpha,\beta)$ is a direct sum of a family of simple subobjects of M which are finitely generated as H-modules in ${}_H\mathrm{YD}^H(\alpha,\beta)$. Hence ${}_H\mathrm{YD}^H(\alpha,\beta)$ is a semisimple category.

Proof Since H is cosemisimple, the functor $(-)^{\mathrm{co}H}$: $M^H \to {}_k M$ is exact. Thus $(-)^{\mathrm{co}H}$: ${}_H\mathrm{YD}^H(\alpha,\beta) \to {}_k M$ is exact. Furthermore, the semisimplicity implies that ${}_H\mathrm{YD}^H(\alpha,\beta)$ satisfies the exact condition. Then by Theorem 1, the conclusion holds.

Theorem 2 As the disjoint union of all ${}_H\mathrm{YD}^H(\alpha,\beta)$, the category of the generalized Yetter-Drinfeld modules $\mathrm{YD}(H)$ is also semisimple.

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广义 Yetter-Drinfeld 模上半单范畴的构造

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摘要:设 H 是域 k 上的可换、诺特、半单、余半单的 Hopf 代数,且具有双射对极. 考虑了其上 YD(H)范畴的半单性,其中 YD(H)是 H 上的广义 Yetter-Drinfeld 模范畴${}_H\mathrm{YD}^H(\alpha,\beta)$(其中 $\alpha,\beta \in \mathrm{Aut}_{\mathrm{Hopf}}(H)$)的无交并. 首先证明了 YD($H$)是一个对态射集封闭的范畴;然后利用有限生成投射模的性质和 H 的半单性,可得 YD(H)是满足正合性条件的;进而由 H 是诺特、余半单的 Hopf 代数,得到 YD(H)中的对象都可分解为单对象的直和. 最终得到 YD(H)是一个半单范畴.

关键词:半单 Hopf 代数;半单范畴;广义 Yetter-Drinfeld 模

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