

Constant angle surfaces constructed on curves

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Abstract: The Frenet-Serret formula is used to characterize the constant angle ruled surfaces in \mathbf{R}^3 . When the surfaces are the tangent developmental and normal surfaces, that is, $r(s, v) = \sigma(s) + v(\cos\alpha(s) \cdot t(s) + \sin\alpha(s) \cdot n(s))$, it is shown that each of these surfaces is locally isometric to a piece of a plane or a certain special surface. When the surfaces are normal and binormal surfaces, that is, $r(s, v) = \sigma(s) + v(\cos\alpha(s) \cdot n(s) + \sin\alpha(s) \cdot b(s))$, it is shown that each of these surfaces is locally isometric to a piece of a plane or a cylindrical surface.

Key words: ruled surface; constant angle surface; tangent surface; normal surface; binormal surface

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Since the 19th century, the ruled surfaces have captured the attention of mathematicians and the main properties of these surfaces have been illuminated in almost all the monographs and books on differential geometry. Motivated again by their flatness property, in this paper we classify some ruled surfaces endowed with the constancy angle property. The constant angle surfaces was initially studied in the setting of the product space $S^2 \times \mathbf{R}$ (see Ref. [1]). Then the surfaces with this property in other ambient spaces, namely $H^2 \times \mathbf{R}$ and \mathbf{R}^3 , were investigated^[2-4]. One may also find other relative references in Refs. [5–7].

1 Preliminaries

Let $\sigma: I \rightarrow \mathbf{R}^3$ be a curve parameterized by arc length; i. e., $|\sigma'(s)| = 1$. Throughout this paper, s is the arc length parameter.

Let us denote $t(s) = \sigma'(s)$ the (unit) tangent to the curve. The curvature of $\sigma(s)$ is defined to be $\kappa(s) = |\sigma''(s)|$. If $\kappa \neq 0$, then the (unit) normal of $\sigma(s)$ can be obtained from $\sigma''(s) = \kappa(s) \times n(s)$. Moreover, $b(s) = t(s) \times n(s)$ is called the (unit) binormal to $\sigma(s)$. With these considerations, t , n and b define an

orthonormal basis. We recall the Frenet-Serret formula:

$$\begin{aligned} t'(s) &= \kappa(s)n(s) \\ n'(s) &= -\kappa(s)t(s) + \tau(s)b(s) \\ b'(s) &= -\tau(s)n(s) \end{aligned} \quad (1)$$

where $\tau(s)$ is the torsion of curve σ at s .

2 Main Theorems and Their Proof

First, we consider ruled surface S

$$r(s, v) = \sigma(s) + vl(s) \quad (2)$$

where

$$l(s) = \cos\alpha(s) \cdot t(s) + \sin\alpha(s) \cdot n(s)$$

Set

$$m(s) = -\sin\alpha(s) \cdot t(s) + \cos\alpha(s) \cdot n(s)$$

It is easy to know that $m(s)$ is orthogonal to both $l(s)$ and $b(s)$. Moreover, we have

$$\begin{bmatrix} l' \\ m' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa + \alpha' & \tau \sin\alpha \\ -(\kappa + \alpha') & 0 & \tau \cos\alpha \\ -\tau \sin\alpha & -\tau \cos\alpha & 0 \end{bmatrix} \begin{bmatrix} l \\ m \\ b \end{bmatrix}$$

A direct computation shows that

$$\begin{aligned} r_s &= \sigma'(s) + vl'(s) = (1 - v\alpha'\sin\alpha - v\kappa\sin\alpha)t + \\ &\quad v(\kappa + \alpha')\cos\alpha \cdot n + v\tau\sin\alpha \cdot b \end{aligned}$$

$$r_v = l(s) = \cos\alpha \cdot t + \sin\alpha \cdot n$$

The normal of the surface is

$$N = \frac{r_s \times r_v}{\sqrt{\Delta}} = \frac{(\sin\alpha - v(\alpha' + \kappa))b + v\tau\sin\alpha \cdot m}{\sqrt{\Delta}} \quad (3)$$

where $\Delta = (\sin\alpha - v(\alpha' + \kappa))^2 + v^2\tau^2\sin^2\alpha$.

If surface S is a constant angle surface; i. e., the normal N makes a constant angle θ with the fixed direction k , namely $\angle(N, k) = \theta$; equivalently, $\langle N, k \rangle = \cos\theta$. Substituting (3) into this expression, we obtain a vanishing polynomial expression of the second order in v . So all the coefficients must be identically zero, that is

$$\sin^2\alpha(\langle b, k \rangle^2 - \cos^2\theta) = 0 \quad (4)$$

$$\sin\alpha(\alpha' + \kappa)(\langle b, k \rangle^2 - \cos^2\theta) - \tau\sin^2\alpha\langle b, k \rangle\langle m, k \rangle = 0 \quad (5)$$

$$(\alpha' + \kappa)^2(\langle b, k \rangle^2 - \cos^2\theta) - 2\tau\langle b, k \rangle\sin\alpha(\alpha' + \kappa)\langle m, k \rangle +$$

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$$\tau^2 \sin^2 \alpha \langle m, k \rangle^2 - \tau^2 \sin^2 \alpha \cos^2 \theta = 0 \quad (6)$$

If $\sin \alpha \equiv 0$, then S is a tangent developmental surface which is given by^[8]

$$r(u, v) = (u \cos \theta (\cos v, \sin v) + \gamma(v), u \sin \theta) \quad (7)$$

with

$$\gamma(v) = \cos \theta \left(- \int_0^v \eta(\tau) \sin \tau d\tau, \int_0^v \eta(\tau) \cos \tau d\tau \right) \quad (8)$$

and

$$\eta(\tau) = -\lambda^{-1} \left(\frac{\pi}{2} - \tau \right) \quad (9)$$

where λ is a nonzero constant.

If $\sin \alpha \neq 0$, from Eq. (4) we have

$$\langle b, k \rangle = \pm \cos \theta \quad (10)$$

Then Eq. (5) results in

$$\tau \langle b, k \rangle \langle m, k \rangle = 0 \quad (11)$$

When $\tau \equiv 0$, σ is a plane curve whose binormal coincides with the normal of the plane, so S is a ruled surface with plane curve σ as the generating curve and $l(s)$ as the rulings. Obviously, S is a piece of a plane.

When $\tau \neq 0$, we claim that $\langle b, k \rangle \langle m, k \rangle = 0$. Indeed, if $\langle b, k \rangle = 0$ at somewhere of S , from Eq. (10), we obtain that $\cos \theta = 0$ at the same point. Then Eq. (6) results in

$$\langle m, k \rangle = 0 \quad (12)$$

On the other hand, if $\langle m, k \rangle = 0$ at somewhere of S , then from Eq. (6) we obtain that $\cos \theta = 0$ at the same point; i. e., $\langle b, k \rangle = 0$. Thus, the claim is true. Now differentiating in $\langle b, k \rangle = 0$ yields $\langle n, k \rangle = 0$. Combining $\langle m, k \rangle = 0$, we can obtain that $\langle t, k \rangle = 0$. Now we deduce that k is orthogonal to all t , n and b , which is impossible.

Thus we prove our first main theorem.

Theorem 1 Assume that the ruled surface $S: r(s, v) = \sigma(s) + v(\cos \alpha(s) \cdot t(s) + \sin \alpha(s) \cdot n(s))$ is a constant angle surface. Then S is locally isometric to one of the following surfaces:

1) A surface is given by

$$r(u, v) = (u \cos \theta (\cos v, \sin v) + \gamma(v), u \sin \theta)$$

where $\gamma(v) = \cos \theta \left(- \int_0^v \eta(\tau) \sin \tau d\tau, \int_0^v \eta(\tau) \cos \tau d\tau \right)$,

$$\eta(\tau) = -\lambda^{-1} \left(\frac{\pi}{2} - \tau \right).$$

2) A piece of a plane.

Remark 1 We have known that the tangent developmental constant angle surfaces are generated by cylindrical helices and the normal constant angle surfaces are generated by planar curves (see Theorem 1 and Theorem 5 in

Ref. [8]). From the proof of Theorem 1, the generating curves of the constant angle surfaces $r(s, v) = \sigma(s) + v(\cos \alpha(s) \cdot t(s) + \sin \alpha(s) \cdot n(s))$ are also planar curves provided $\sin \alpha(s) \neq 0$.

In the sequel we consider another kind of ruled surface S , which is defined by

$$r(s, v) = \sigma(s) + vl(s) \quad (13)$$

where

$$l(s) = \cos \alpha(s) \cdot n(s) + \sin \alpha(s) \cdot b(s)$$

Set

$$m(s) = -\sin \alpha(s) \cdot n(s) + \cos \alpha(s) \cdot b(s)$$

It is easy to know that $m(s)$ is orthogonal to both $l(s)$ and $t(s)$. We have

$$\begin{bmatrix} t' \\ l' \\ m' \end{bmatrix} = \begin{bmatrix} 0 & \kappa \cos \alpha & -\kappa \sin \alpha \\ -\kappa \cos \alpha & 0 & \alpha' + \tau \\ \kappa \sin \alpha & -(\alpha' + \tau) & 0 \end{bmatrix} \begin{bmatrix} t \\ l \\ m \end{bmatrix}$$

Now we compute the normal to S . A routine computation shows

$$r_s = \sigma'(s) + vl'(s) = (1 - \kappa v \cos \alpha)t - v \sin \alpha(\tau + \alpha')n + v \cos \alpha(\tau + \alpha')b$$

$$r_v = l(s) = \cos \alpha \cdot n + \sin \alpha \cdot b$$

Then the normal of the surface is

$$N = \frac{r_s \times r_v}{\sqrt{\Delta}} = \frac{(1 - \kappa v \cos \alpha)m - v(\tau + \alpha')t}{\sqrt{\Delta}} \quad (14)$$

where $\Delta = (1 - \kappa v \cos \alpha)^2 + v^2(\tau + \alpha')^2$.

Assume that surface S is a constant angle surface. Then there is a fixed direction k and a constant angle θ such that $\langle N, k \rangle = \cos \theta$. Substituting Eq. (14) into this expression and comparing the coefficients in the vanishing polynomial expression of the second order in v , we find the following relationships:

$$\langle m, k \rangle^2 = \cos^2 \theta \quad (15)$$

$$\kappa \sin \alpha (\langle m, k \rangle^2 - \cos^2 \theta) + (\tau + \alpha') \langle t, k \rangle \langle m, k \rangle = 0 \quad (16)$$

$$\kappa^2 \cos^2 \alpha (\langle m, k \rangle^2 - \cos^2 \theta) + (\tau + \alpha')^2 (\langle t, k \rangle^2 - \cos^2 \theta) + 2\kappa \cos \alpha \langle m, k \rangle \langle t, k \rangle (\tau + \alpha') = 0 \quad (17)$$

We claim that $\tau + \alpha' \equiv 0$. Suppose that $\tau + \alpha' \neq 0$. A contradiction will be deduced. From Eq. (15), we have

$$\langle m, k \rangle = \pm \cos \theta \quad (18)$$

From Eqs. (16) and (17), we obtain

$$(\tau + \alpha')^2 (\langle t, k \rangle^2 - \cos^2 \theta) = 0$$

and thus $\langle t, k \rangle^2 = \cos^2 \theta = \langle m, k \rangle^2$.

At the same time, Eq. (16) shows that $\langle t, k \rangle \langle m, k \rangle = 0$, which implies that $\langle t, k \rangle = \langle m, k \rangle = 0$. Taking the derivative in Eq. (18) and using $\langle t, k \rangle = 0$, we obtain that $\langle l, k \rangle = 0$. This is impossible since k cannot be orthogonal to all t, l, m . Thus, we prove that $\tau + \alpha' \equiv 0$.

Our aim is to show S is locally isometric to a plane or a cylinder. If $\kappa \equiv 0$ locally, then $\tau \equiv 0$ and α is a constant, which means that S is locally a piece of a plane. So we only need to consider the case of $\kappa \neq 0$. Taking the derivative into Eq. (18), we have

$$\kappa \langle t, k \rangle \sin \alpha = (\alpha' + \tau) \langle l, k \rangle = 0$$

When $\sin \alpha = 0$, $l \equiv n$ and S is a piece of a plane by Ref. [8]. When $\sin \alpha \neq 0$, $\langle t, k \rangle = 0$ locally. We have the following computations:

$$\begin{aligned} \langle t, k \rangle = 0 &\Rightarrow \langle t', k \rangle = 0 \Rightarrow \kappa \langle l, k \rangle \cos \alpha = \kappa \langle m, k \rangle \sin \alpha \Rightarrow \\ \kappa \langle \cos \alpha \cdot n + \sin \alpha \cdot b, k \rangle \cos \alpha &= \\ \kappa \langle -\sin \alpha \cdot n + \cos \alpha \cdot b, k \rangle \sin \alpha &\Rightarrow \\ \langle n, k \rangle = 0 &\Rightarrow \langle n', k \rangle = 0 \Rightarrow \tau \langle b, k \rangle = 0 \end{aligned}$$

where $\langle b, k \rangle$ cannot be zero since k cannot be orthogonal to all t, n, b . Then $\tau \equiv 0$ locally. Recall that $\tau + \alpha' \equiv 0$. We know that α is a constant and S is locally a cylindrical surface.

In the end, our second main theorem reads as follows.

Theorem 2 Assume that the ruled surface $S: r(s, v)$

$= \sigma(s) + v(\cos \alpha(s) \cdot n(s) + \sin \alpha(s) \cdot b(s))$ is a constant angle surface. Then S is locally isometric to a plane or a cylindrical surface.

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曲线上的定常角曲面

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摘要:利用 Frenet-Serret 公式来讨论 \mathbf{R}^3 中定常角的直纹面, 给出了它们的特征分类. 如果定常角曲面是具有 $r(s, v) = \sigma(s) + v(\cos \alpha(s) \cdot t(s) + \sin \alpha(s) \cdot n(s))$ 形式的切线面和法向曲面, 则它们局部等距于平面或一类特殊的曲面. 如果定常角曲面是具有 $r(s, v) = \sigma(s) + v(\cos \alpha(s) \cdot n(s) + \sin \alpha(s) \cdot b(s))$ 形式的法向曲面和副法向曲面, 则它们局部等距于平面或柱面.

关键词:直纹面; 定常角曲面; 切线面; 法向曲面; 副法向曲面

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