Journal of Southeast University (English Edition) Vol. 30, No. 3, pp. 391 - 395 Sept. 2014 ISSN 1003—7985

Hom-dimodules and FRT theorem of Hom type

Chen Xiuli[1] Li Fang[2] Chen Jianlong[1]

([1]Department of Mathematics, Southeast University, Nanjing 211189, China)

([2]Department of Mathematics, Zhejiang University, Hangzhou 310028, China)

Abstract: In order to study the deformation of algebras, the notions of Hom-algebras are introduced. The Hom-algebra is a generalization of the classical associative algebra. First, the Hom-type generalization of dimodules, which is called the Hom-dimodule, is introduced, and its properties are discussed Moreover, the category of Hom-dimodules in connection with the Hom $\boldsymbol{D}$-equation $R^{12}R^{23}=R^{23}R^{12}$ for $R\in \mathrm{End}_k(M\otimes M)$ and a Hom-module M is investigated. Some solutions of the Hom $\boldsymbol{D}$-equation from Hom-dimodules over Hom-bialgebras are given, and the FRT-type theorem is constructed in the category of Hom-dimodules. The results generalize and improve the FRT-type theorem in the category of dimodules.

Key words: Hom-bialgebra; Hom-dimodule; Hom $\boldsymbol{D}$-equation

doi: 10.3969/j. issn. 1003 - 7985.2014.03.025

Throughout this paper, all the modules are left modules without specification and all the spaces are k-spaces for a fixed field k.

A (long) dimodule over a bialgebra H is a vector space M with a left H-action ρ_M and a right H-coaction ρ^M so that the following compatibility condition holds: $(hm)_{(0)}\otimes(hm)_{(1)}=\sum hm_{(0)}\otimes m_{(1)}$ for all $h\in H$ and $m\in M$. We denote this category with ${}_H L^H$, which is also a special case of the Doi-Hopf module category. This category ${}_H L^H$ was first defined by Long[1] for a commutative and cocommutative H and was studied in connection with the construction of the Brauer group of an H-dimodule algebra. It is interesting to note that for a commutative and cocommutative H, ${}_H\mathrm{YD}^H$ (the category of Yetter-Drinfel'd modules) is precisely ${}_H L^H$[2]. Naturally, for an arbitrary H, ${}_H\mathrm{YD}^H$ and ${}_H L^H$ are fundamentally different. ${}_H\mathrm{YD}^H$ plays a determinant role in describing the solutions of the quantum Yang-Baxter equation. It is natural to ask which equation will play a key role in ${}_H L^H$. In Ref. [3], considering that ${}_H\mathrm{YD}^H$ is deeply involved in solving the quantum Yang-Baxter equation, the author studied ${}_H L^H$ in connection with the $\boldsymbol{D}$-equation which is described presently. Given a vector space M, and $R\in \mathrm{End}_k(M\otimes M)$, R is said to be a solution of the $\boldsymbol{D}$-equation if $R^{12}R^{23}=R^{23}R^{12}$ in $\mathrm{End}_k(M\otimes M\otimes M)$ where $R^{12}=R\otimes id$, and $R^{23}=id\otimes R$.

The concept of a Hom-algebra was introduced by Larsson et al[4-5]. It is a special class in the deformation of algebras. Recently, the Hom-algebra has been studied by several authors. The further development of the Hom-algebra theory led Makhlouf et al.[6-8] consequently to Hom-associative algebra, Hom-coassociative coalgebra and Hom-bialgebra, and described many of the extending properties of classical associative algebras, coalgebras, bialgebras and Hopf algebra structures. In fact, the notion of Hom-associative algebras generalize associative algebras to a situation, where associativity law is twisted by a linear map. From Ref. [5], it is clear that the commutator bracket multiplication is defined using the multiplication in Hom-associative algebra and it leads naturally to Hom-Lie algebras. Following the patterns of Hom-Lie and Hom-associative algebras, one can define Hom-bialgebras as nonassociative and non-coassociative[9]. It is a generalization of the bialgebra in which the non(co)associativity is controlled by the twisted map. In this paper, we only investigate the case of Hom-associative algebras. Hom-versions of the Yang-Baxter equation was studied in Refs.[9 - 12]. Many classes of the solutions for the Hom-Yang-Baxter equation are constructed. Moreover, Hom-type generalizations of FRT quantum groups, including quantum matrices and related quantum groups, are obtained in Ref.[13]. The above motivates us to study the generalization of the $\boldsymbol{D}$-equation.

In this paper, we mainly study Hom-versions of (Long) dimodules and $\boldsymbol{D}$-equations of the Hom type, and investigate the category of Hom-dimodules in connection with the Hom $\boldsymbol{D}$-equation.

1 Preliminaries

In this section, we recall the definition of a Hom-bialgebra and (comodules) modules over an Hom-algebra (coalgebra).

A Hom-module[14] is a pair (V,α) in which V is a vector space and $\alpha: V\to V$ is a linear map. A morphism $(V,\alpha)\to(V',\alpha')$ of Hom-modules is a linear map $f: V\to V'$ such that $\alpha'\circ f=f\circ\alpha$. The tensor product of the Hom-modules (V,α_v) and (W,α_w) consists of the vector space

Received 2013-03-28.

Biographies: Chen Xiuli (1980—), female, doctor; Chen Jianlong (corresponding author), male, doctor, professor, jlchen@seu.edu.cn.

Foundation items: The National Natural Science Foundation of China (No. 11371089), the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20120092110020), the Postdoctoral Innovation Funds of Southeast University (No. 3207013601).

Citation: Chen Xiuli, Li Fang, Chen Jianlong. Hom-dimodules and FRT theorem of Hom type[J]. Journal of Southeast University (English Edition), 2014, 30(3): 391 - 395. [doi: 10.3969/j. issn. 1003 - 7985.2014.03.025]

$V\otimes W$ and the linear self-map $\alpha_V\otimes\alpha_W$.

Definition 1 A Hom-associative algebra[5] is a triple (A,μ,α) in which (A,α) is a Hom-module and $\mu: A\otimes A\to A$ is a bilinear map such that 1) $\alpha\circ\mu=\mu\circ(\alpha\otimes\alpha)$ (multiplicativity); and 2) $\mu\circ(\alpha\otimes\mu)=\mu\circ(\mu\otimes\alpha)$ (Hom-associativity). In the following, we also write $\mu(a\otimes b)$ as ab.

The Hom-associative algebra is said to be unital[7] if a homomorphism exists, $\eta: k\to A$ satisfying $\mu\circ(\eta\otimes id)=id$ and $\mu\circ(id\otimes\eta)=id$.

A Hom-coassociative coalgebra[6–7] is a triple (C,Δ,α) in which (C,α) is a Hom-module and $\Delta: C\to C\otimes C$ is a linear map such that 1) $\alpha^{\otimes 2}\circ\Delta=\Delta\circ\alpha$ (comultiplicativity); and 2) $(\alpha\otimes\Delta)\circ\Delta=(\Delta\otimes\alpha)\circ\Delta$ (Hom-coassociativity).

A Hom-coassociative coalgebra is said to be counital[7] if there is a map $\varepsilon: C\to k$ satisfying $(\varepsilon\otimes id)\circ\Delta=id$ and $(id\otimes\varepsilon)\circ\Delta=id$.

A Hom-bialgebra[5,8] is a quadruple $(A,\mu,\alpha,\eta,\Delta,\alpha,\varepsilon)$, in which (A,μ,α,η) is a hom-associative algebra, $(A,\Delta,\alpha,\varepsilon)$ is a Hom-coassociative coalgebra, and Δ is a morphism of Hom-associative algebras, i. e., $\Delta(xy)=\sum_{(x)(y)} x_1y_1\otimes x_2y_2=\Delta(x)\Delta(y)$.

Remark 1 In Ref. [7], the definition of a Hom-bialgebra is also written as follows: A Hom-bialgebra is a seven-uple $(A,\mu,\alpha,\eta,\Delta,\alpha,\varepsilon)$ where (A,μ,α,η) is a unital hom-associative algebra and $(A,\Delta,\alpha,\varepsilon)$ is a counital Hom-coassociative coalgebra, and the following condition holds: $\Delta(xy)=\sum_{(x)(y)} x_1y_1\otimes x_2y_2=\Delta(x)\Delta(y)$, $\varepsilon(xy)=\varepsilon(x)\varepsilon(y)$ and $\varepsilon\circ\alpha(x)=\varepsilon(x)$.

Example 1 (classical structures) 1) If (A,μ) is an associative algebra and $\alpha: A\to A$ is an algebra morphism, then $A_\alpha=(A,\mu_\alpha,\alpha)$ is a Hom-associative algebra with the twisted multiplication $\mu_\alpha=\alpha\circ\mu$. Indeed, the Hom-associativity axiom $\mu_\alpha\circ(\alpha\otimes\mu_\alpha)=\mu_\alpha\circ(\mu_\alpha\otimes\alpha)$ is equal to α^2 when applied to the associativity axiom of μ. Likewise, both sides of the multiplicativity axiom $\alpha\circ\mu_\alpha=\mu_\alpha\otimes(\alpha\otimes\alpha)$ are equal to $\alpha^2\circ\mu$.

2) Dually, if (C,Δ) is a coassociative coalgebra and $\alpha: C\to C$ is a coalgebra morphism, then $C_\alpha=(C,\Delta_\alpha,\alpha)$ is a Hom-coassociative coalgebra with the twisted comultiplication $\Delta_\alpha=\Delta\circ\alpha$.

3) A bialgebra is an exact Hom-bialgebra with $\alpha=id$. More generally, combining the previous two cases, if (A,μ,Δ) is a bialgebra and $\alpha: A\to A$ is a bialgebra morphism, then $A_\alpha=(A,\mu_\alpha,\Delta_\alpha,\alpha)$ is a Hom-bialgebra.

Definition 2 1) Let (A,μ_A,α_A) be a Hom-associative algebra, and (M,α_M) be a Hom-module. A left A-module structure on M[12] consists of a morphism $\rho: A\otimes M\to M$ of Hom-modules such that $\rho\circ(\alpha_A\otimes\rho)=\rho\circ(\mu_A\otimes\alpha_M)$ and $\alpha_M\circ\rho_M=\rho_M\circ(\alpha_H\otimes\alpha_M)$ ($*$). We also write $\rho(a\otimes m)$ as am for $a\in A$ and $m\in M$. In this notation, ($*$) can be written as $\alpha_A(a)(bm)=(ab)\alpha_M(M)$ and $\alpha_M(ax)=\alpha_A(a)\alpha_M(x)$.

2) Dually, let (C,Δ,α_C) be a Hom-coassociative coalgebra, A left C-comodule structure on M[14] consists of a Hom-module (M,α_M) together with a linear map $\rho^M: M\to C\otimes M$ such that $(\Delta\otimes\alpha_M)\circ\rho^M=(\alpha_C\otimes\rho^M)\circ\rho^M$ and $(\alpha_C\otimes\alpha_M)\circ\rho^M=\rho^M\circ\alpha_M$. Similarly, we can define a right C-comodule.

If M and N are A-modules, then a morphism of A-modules $f: M\to N$ is a morphism of the Hom-modules such that $f\circ\rho_M=\rho_N\circ(id_A\otimes f)$. Similarly, morphisms of C-comodule are defined in a clear way.

2 Hom-Dimodules

Definition 3 Let $(H,\mu_H,\Delta_H,\alpha_H)$ be a Hom-bialgebra, and (M,α_M) be a Hom-module. A Hom-dimodule is a triple (M,ρ_M,ρ^M) such that 1) (M,ρ_M) is a left H-module; 2) (M,ρ^M) is a right H-comodule; 3) $\rho_M\circ\rho^M=(\rho_M\otimes\alpha_M)\circ(\alpha_H\otimes\rho^M)$.

We may also write 3) as $(hm)_{(0)}\otimes(hm)_{(1)}=\sum\alpha_H(h)m_{(0)}\otimes\alpha_H(m_{(1)})$.

The category of Hom-dimodules over a Hom-bialgebra H with H-module morphisms and H-comodule morphisms are denoted by ${}_H\mathrm{HL}^H$.

If H is a bialgebra, i. e., $\alpha_H=id_H$ and $\alpha_M=id_M$, then the above definition of the Hom-dimodule coincides with the usual definition of the dimodule. The following result proves that dimodules deform into Hom-dimodules by endomorphisms.

Proposition 1 Let (H,μ_H,Δ_H) be a bialgebra, and (M,ρ_M,ρ^M) be an H-dimodule. Assume that $\alpha_H: H\to H$ is a bialgebra morphism, and $\alpha_M: M\to M$ is a k-linear map such that

$$\alpha_M\circ\rho_M=\rho_M\circ(\alpha_H\otimes\alpha_M)\qquad(1)$$

$$\rho^M\circ\alpha_M=(\alpha_H\otimes\alpha_M)\circ\rho^M\qquad(2)$$

Define the maps $\rho_{\alpha,M}=\alpha_M\circ\rho_M: H\otimes M\to M$ and $\rho^M_\alpha=\rho^M\circ\alpha_M: M\to H\otimes M$. Then

1) $(H,\mu_\alpha,\Delta_\alpha,\alpha_H)$ is a Hom-bialgebra, where $\mu_\alpha=\alpha_H\circ\mu_H$ and $\Delta_\alpha=\Delta_H\circ\alpha_H$;

2) $(M,\rho_{\alpha,M},\rho^{\alpha,M})$ is a Hom-dimodule over a Hom-bialgebra H, where H is a Hom-bialgebra as 1).

Proof 1) It follows from 1) of Example 1.

2) First, by Definition 1 and Eq. (2), it is easy to prove that $\rho_{\alpha,M}$ gives (M,α_M) the structure of a left H-module, i. e., $\rho_{\alpha,M}\circ(\alpha_H\otimes\rho_{\alpha,M})=\rho_{\alpha,M}\circ(\mu_H\otimes\alpha_M)$.

Similarly, by Definition 2, we can prove that $\rho^{\alpha,M}$ gives (M,α_M) the structure of a right H-comodule if and only if $(\rho^{\alpha,M}\otimes\alpha_H)\circ\rho^{\alpha,M}=(\alpha_M\otimes\Delta_H)\circ\rho^{\alpha,M}$.

Finally, it is only needed to check that $\rho^{\alpha,M}\circ\rho_{\alpha,M}=(\rho_{\alpha,M}\otimes\alpha_H)\circ(\alpha_H\otimes\rho^{\alpha,M})$. In fact, we have

$$\rho^{\alpha,M}\circ\rho_{\alpha,M}=(\rho^M\circ\alpha_M)\circ(\alpha_M\circ\rho_M)=$$
$$(\alpha_M\otimes\alpha_H)\circ\rho^M\circ\rho_M\circ(\alpha_H\otimes\alpha_M)=$$

$$(\alpha_M\otimes\alpha_H)\circ(\rho_M\otimes id_H)\circ(id_H\otimes\rho^M)\circ(\alpha_H\otimes\alpha_M) = (\rho_{\alpha,M}\otimes\alpha_H)\circ(\alpha_H\otimes\rho^{\alpha,M})$$

where the second equality holds by Eqs. (1) and (2). Because M is a H-dimodule, the third equality holds.

Proposition 2 Let $(H,\mu_H,\Delta_H,\alpha_H)$ be a Hom-bialgebra, (M,α_M) and (N,α_N) be Hom-dimodules. Then $M\otimes N$ is a Hom-dimodule with the action and coaction, for all $m\in M$, $n\in N$, $h\in H$,

$$\rho_{M\otimes N}(h\otimes m\otimes n)=\alpha_H(h_{(1)})\cdot m\otimes\alpha_H(h_{(2)})\cdot n$$
$$\rho^{M\otimes N}(m\otimes n)=m_{(0)}\otimes n_{(0)}\otimes\alpha_H(m_{(1)}n_{(1)})$$

Proof We check that $\rho_{M\otimes N}$ and $\rho^{M\otimes N}$ is well defined. For any $h,l\in H$, $m\in M$ and $n\in N$,

$$\begin{aligned}&\rho_{M\otimes N}\circ(\alpha_H\otimes\rho_{M\otimes N})(h\otimes l\otimes m\otimes n)=\\&\rho_{M\otimes N}(\alpha_H(h)\otimes\alpha_H(l_{(1)})\cdot m\otimes\alpha_H(l_{(2)})\cdot n)=\\&\alpha_H((\alpha_H(h))_{(1)})\alpha_H(l_{(1)})\cdot m)\otimes\alpha_H(\alpha_H(h))_{(2)})(\alpha_H(l_{(2)})\cdot n)=\\&\alpha_H^2(h_{(1)})(\alpha_H(l_{(1)})\cdot m)\otimes\alpha_H^2(h_{(2)})(\alpha_H(l_{(2)})\cdot n)=\\&(\alpha_H(h_{(1)})\alpha_H(l_{(1)})\cdot\alpha_M(m)\otimes(\alpha_H(h_{(2)})\alpha_H(l_{(2)}))\alpha_N(n)=\\&a_H((hl)_{(1)})\alpha_M(m)\otimes\alpha_H((hl)_{(2)})\alpha_N(n)=\\&\rho_{M\otimes N}(hl\otimes\alpha_M(m)\otimes\alpha_N(n))=\\&p_{M\otimes N}\circ(\mu_H\otimes\alpha_{M\otimes N})(h\otimes l\otimes m\otimes n)\end{aligned}$$

The forth equality holds because M and N are H-modules. Similarly, we can prove that $(M\otimes N,\rho^{M\otimes N})$ is a right H-comodule. It remains to be checked that the compatibility condition holds, i. e., $\rho_{M\otimes N}\circ\rho^{M\otimes N}=(\rho_{M\otimes N}\otimes\alpha_H)\circ(\alpha_H\otimes\rho^{M\otimes N})$. Since M and N are H-dimodules, $(\alpha_H(h)m)_{(0)}\otimes(\alpha_H(h)m)_{(1)}=\alpha_H^2(h)m_{(0)}\otimes\alpha_H(m_{(1)})$, $(\alpha_H(h)n)_{(0)}\otimes(\alpha_H(h)n)_{(1)}=\alpha_H^2(h)n_{(0)}\otimes\alpha_H(n_{(1)})$. For any $m\in M$, $n\in N$, $h\in H$,

$$\begin{aligned}&\rho^{M\otimes N}\circ\rho_{M\otimes N}(h\otimes m\otimes n)=\rho^{M\otimes N}(\alpha_H(h_{(1)})\cdot m\otimes\alpha_H(h_{(2)})\cdot n)=\\&(\alpha_H(h_{(1)})\cdot m)_{(0)}\otimes(\alpha_H(h_{(2)})\cdot n)_{(0)})\otimes\alpha_H((\alpha_H(h_{(1)})\cdot m)_{(1)}\\&(\alpha_H(h_{(2)})\cdot n)_{(1)})=\alpha_H^2(h)\cdot m_{(0)}\otimes\alpha_H^2(h)\cdot n_{(0)}\otimes\\&\alpha_H(\alpha_H(m_{(1)})\alpha_H(n_{(1)}))=(\rho_{M\otimes N}\otimes\alpha_H)(\alpha_H(h)\otimes m_{(0)}\otimes\\&n_{(0)}\otimes\alpha_H(m_{(1)})\alpha_H(n_{(1)}))=(\rho_{M\otimes N}\otimes\alpha_H)\circ\\&(\alpha_H\otimes\rho^{M\otimes N})(h\otimes m\otimes n)\end{aligned}$$

3 Solutions of Hom *D*-Equation

First we introduce the definition of the Hom ***D***-equation. Moreover, we construct many solutions to the Hom ***D***-equation by Hom-dimodules over the Hom-bialgebra.

Definition 4 Let (M,α_M) be a Hom-module and B: $M\otimes M\to M\otimes M$ be a morphism of Hom-modules, i. e., $B\circ\alpha_M^{\otimes 2}=\alpha_M^{\otimes 2}\circ B$. We call B a solution of the Hom ***D***-equation if $B^{12}B^{23}=B^{23}B^{12}$, where $B^{12}=B\otimes\alpha_M$, $B^{23}=\alpha_M\otimes B$.

The ***D***-equation[3] is a special case of the Hom ***D***-equation when $\alpha=id$.

Proposition 3 Let B be a solution of the Hom ***D***-equation for the Hom-module (M,α_M).

1) If $\lambda\in k$, then λB is also a solution of the Hom ***D***-equation for (M,α_M).

2) If both α and B are invertible, then B^{-1} is a solution of the Hom ***D***-equation for (M,α_M^{-1}).

Proof 1) First, $\lambda B\circ(\alpha_M\otimes\alpha_M)=\lambda(B\circ(\alpha_M\otimes\alpha_M))=\lambda((\alpha_M\otimes\alpha_M)\circ B)=(\alpha_M\otimes\alpha_M)\circ\lambda B$, and $\lambda B\otimes\alpha=\lambda(B\otimes\alpha)$, $\alpha\otimes\lambda B=\lambda(\alpha\otimes B)$, so we have $(\lambda B)^{12}(\lambda B)^{23}=(\lambda B)^{23}(\lambda B)^{12}$.

2) It follows from $B^{-1}\circ(\alpha_M^{-1}\otimes\alpha_M^{-1})=((\alpha_M\otimes\alpha_M)\circ B)^{-1}=(B\circ(\alpha_M\otimes\alpha_M))^{-1}=(\alpha_M^{-1}\otimes\alpha_M^{-1})\circ B^{-1}$, and $B^{-1}\otimes\alpha_M^{-1}=(B\otimes\alpha_M)^{-1}$, $\alpha_M^{-1}\otimes B^{-1}=(\alpha_M\otimes B)^{-1}$.

Next, we will show that a Hom-bialgebra gives rise to many solutions of the Hom ***D***-equation by its Hom-dimodules. The following shows that ${}_H\mathrm{HL}^H$ plays a role in solving the Hom ***D***-equation.

Proposition 4 Let $(H,\mu_H,\Delta_H,\alpha_H)$ be a Hom-bialgebra, and (M,α_M) be a Hom-module. Assume that $(M,\rho_{\alpha,M},\rho^{\alpha,M})$ is Hom-dimodule and α_M is an H-module morphism. Then the map

$$B':M\otimes M\to M\otimes M,\quad m\otimes n\mapsto\alpha_H(n_{(1)})\cdot m\otimes\alpha_M(n_{(0)})\tag{3}$$

is a solution of the Hom ***D***-equation.

Proof First, it is easy to check that $B'\circ(\alpha_M\otimes\alpha_M)(m\otimes n)=(\alpha_M\otimes\alpha_M)\circ B'(m\otimes n)$.
For $l,m,n\in M$, we have

$$\begin{aligned}&B'^{12}B'^{23}(l\otimes m\otimes n)=(B'\otimes\alpha_M)(\alpha_M\otimes B')(l\otimes m\otimes n)=\\&(B'\otimes\alpha_M)(\alpha_M(l)\otimes\alpha_H(n_{(1)})\cdot m\otimes\alpha_M(n_{(0)}))=\\&\alpha_H((\alpha_H(n_{(1)})\cdot m)_{(1)})\cdot\alpha_M(l)\otimes\alpha_M((\alpha_H(n_{(1)})\cdot m)_{(0)})\otimes\\&\alpha_M^2(n_{(0)})=\alpha_H(\alpha_H(m_{(1)}))\cdot\alpha_M(l)\otimes\alpha_M(\alpha_H(\alpha_H(n_{(1)}))\cdot m_{(0)})\\&\otimes\alpha_M^2(n_{(0)})=\alpha_M(\alpha_H(m_{(1)})\cdot l)\otimes\alpha_H(\alpha_H(n_{(1)}))\cdot\alpha_M(m_{(0)})\\&\otimes\alpha_M^2(n_{(0)})=B'^{23}(\alpha_H(m_{(1)})\cdot l\otimes\alpha_M(m_{(0)})\otimes\alpha_M(n))=\\&B'^{23}B'^{12}(l\otimes m\otimes n)\end{aligned}$$

The fourth equality holds because (M,α_M) is a Hom-dimodule. Since α_M is H-module morphism, the fifth equality is true. By 2) of Definition 2, we have $\alpha_M(m_{(0)})\otimes\alpha_H(m_{(1)})=(\alpha_M(m))_{(0)}\otimes(\alpha_M(m))_{(1)}$, so the sixth equality holds by (3). Therefore, $B'^{12}B'^{23}=B'^{23}B'^{12}$, i. e., B' is a solution of the Hom ***D***-equation.

Corollary 1 Let (H,μ_H,Δ_H) be a bialgebra, and $\alpha_H:H\to H$ be a bialgebra morphism. Assume that (M,ρ_M,ρ^M) is a dimodule and $\alpha_M:M\to M$ a k-linear map such that $\alpha_M\circ\rho_M=\rho_M\circ(\alpha_H\otimes\alpha_M)$, and $\rho^M\circ\alpha_M=(\alpha_M\otimes\alpha_H)\circ\rho^M$. If α_M is H-module morphism. Then the map (3) is a solution of the Hom ***D***-equation.

In Corollary 1, if $\alpha_H=id:H\to H$, then the conditions $\alpha_M\circ\rho_M=\rho_M\circ(id\otimes\alpha_M)$ and $\rho^M\circ\alpha_M=(\alpha_M\otimes id_H)\circ\rho^M$ mean that α_M is H-module morphism and H-comodule a morphism. So we have the following corollary.

Corollary 2 Let (H,μ_H,Δ_H) be a bialgebra. Assume that (M,ρ_M,ρ^M) is a dimodule and $\alpha_M:M\to M$ is H-module morphism and H-comodule morphism. Then the map (3) is a solution of the Hom $\boldsymbol{D}$-equation.

4 FRT Theorem of Hom Type

In Ref.[7], the authors give a FRT type theorem: if M is finite dimensional, then any solution R of the $\boldsymbol{D}$-equation has the form $R=R_{(M,\rho_M,\rho^M)}$, where (M,ρ_M,ρ^M) is a $D(R)$-dimodule over a bialgebra $D(R)$ and $R_{(M,\rho_M,\rho^M)}$ is the special map $R_{(M,\rho_M,\rho^M)}(m\otimes n):=\sum n_{(1)}m\otimes n_{(0)}$.

Next, we recall the details of the FRT type construction from Theorem 3.6 in Ref. [3].

Let M be a finite dimensional vector space. Assume that $\{m_1,\cdots,m_n\}$ is a basis for M and $(x^{ji}_{uv})_{i,j,u,v}$ is a family of scalars of k such that

$$R(m_v\otimes m_u) = \sum_{i,j} x^{ji}_{uv}\, m_i\otimes m_j \tag{4}$$

for all $u,v=1,2,\cdots,n$.

Let $T(C)$ be an algebra generated by (C_{ij}). Define the bialgebra

$$D(R) = \frac{T(C)}{I},\ I = \langle \sum_v (x^{ji}_{uv}\, C_{vl} - x^{jv}_{kl}\, C_{iv})\rangle \tag{5}$$

with the comultiplication $\Delta(C_{ij}) = \sum_{k=1}^{n} C_{ik}\otimes C_{kj}$.

1) The left $D(R)$-module structure $\rho_M: D(R)\otimes M\to M$ given by $\rho_M(C_{ju}\otimes m_v) = \sum_i x^{ji}_{uv}\, m_i$;

2) The right $D(R)$-comodule structure $\rho^M: M\to M\otimes D(R)$ given by $\rho^M(m_l) = \sum_{v=1}^{n} m_v\otimes C_{vl}$.

For m_v, m_u the elements of the given basis, $R_{(M,\rho_M,\rho^M)} = R(m_v\otimes m_u) = \sum_{v=1}^{n} x^{ji}_{uv}\, m_i\otimes m_j$. So we obtain that $(M,\rho_M,\rho^M)\in {}_{D(R)}L^{D(R)}$ and $R=R_{(M,\rho_M,\rho^M)}$.

Lemma 1 Let $D(R)$ be the bialgebra with $R: M\otimes M\to M\otimes M$ a solution of the D-equation, and assume that (M,ρ_M,ρ^M) has the structure of the object in ${}_{D(R)}L^{D(R)}$, where $R=R_{(M,\rho_M,\rho^M)}$. Let $\lambda_i\in k$ be invertible scalars such that

$$\lambda_i\lambda_j x^{ji}_{kv} = \lambda_k\lambda_v x^{ji}_{kv} \tag{6}$$

for all $i,j,k,v=1,2,\cdots,n$. Then there is a bialgebra morphism $\alpha: D(R)\to D(R)$ determined by $\alpha(C_{ij})=\lambda_i^{-1}\lambda_j C_{ij}$ for all i and j.

Proof It is to show that α_A is an algebra morphism, we only need to prove that $\alpha(\sum_v x^{ji}_{kv}\, C_{vl}) = \alpha(\sum_v x^{jv}_{kl}\, C_{iv})$.

In fact,

$$\alpha(\sum_v x^{ji}_{kv}\, C_{vl}) = \sum_v x^{ji}_{kv}\,\lambda_l\lambda_v^{-1}\, C_{vl} =$$
$$\sum_v x^{ji}_{kv}\,\lambda_i\lambda_v^{-1}\,\lambda_l\lambda_i^{-1}\, C_{vl} =$$
$$\sum_v x^{ji}_{kv}\,\lambda_j^{-1}\,\lambda_k\lambda_l\lambda_i^{-1}\, C_{vl} =$$
$$\sum_v x^{jv}_{kl}\,\lambda_j^{-1}\,\lambda_k\lambda_l\lambda_i^{-1}\, C_{iv} =$$
$$\sum_v x^{jv}_{kl}\,\lambda_i^{-1}\,\lambda_v C_{iv} = \alpha(\sum_v x^{jv}_{kl}\, C_{iv})$$

The third equality is right by (6), the fourth equality follows from (5), and the fifth equality also follows from (6).

Lemma 2 The linear map $\alpha_M: M\to M$ can be defined as $\alpha_M(m_i)=\lambda_i m_i$ for all i. Then we have

$$\alpha_M\circ\rho_M = \rho_M\circ(\alpha\otimes\alpha_M)$$
$$(\alpha_M\otimes\alpha)\circ\rho^M = \rho^M\circ\alpha_M$$
$$(\alpha_M\otimes\alpha_M)\circ R = R\circ(\alpha_M\otimes\alpha_M)$$

where $\rho_M: D(R)\otimes M\to M$ is the left $D(R)$-module structure map and $\rho^M: M\to M\otimes D(R)$ is the right $D(R)$-comodule structure map.

Proof First, since Δ is an algebra morphism, we have

$$\Delta(\alpha(C_{ij})) = \sum(\lambda_i^{-1}\lambda_j C_{ij}) = \sum\lambda_i^{-1}\lambda_j C_{ik}\otimes C_{kj} =$$
$$\sum\lambda_i^{-1}\lambda_k C_{ik}\otimes\lambda_k^{-1}\lambda_j C_{kj} = (\alpha\otimes\alpha)(\Delta(C_{ij}))$$

This shows that α is a bialgebra morphism. Next,

$$\alpha_M\circ\rho_M(C_{ju}\otimes m_v) = \alpha_M(C_{ju}\cdot m_v) =$$
$$\alpha(\sum_i x^{ji}_{uv}\, m_i) = \sum_i x^{ji}_{uv}\,\lambda_i m_i$$

and

$$\rho_M\circ(\alpha\otimes\alpha_M)(C_{ju}\otimes m_v) = \alpha(C_{ju})\cdot\alpha_M(m_v) =$$
$$\lambda_j^{-1}\lambda_u C_{ju}\cdot\lambda_v m_v = \lambda_j^{-1}\lambda_u\lambda_v\sum_i x^{ji}_{uv}\, m_i =$$
$$\sum_i \lambda_j^{-1}\lambda_j\lambda_i x^{ji}_{uv}\, m_i$$

so $\alpha_M\circ\rho_M=\rho_M\circ(\alpha\otimes\alpha_M)$.

Since

$$(\alpha_M\otimes\alpha)\circ\rho^M(m_v) = (\alpha_M\otimes\alpha)(\sum_i^n m_i\otimes C_{iv}) =$$
$$\alpha_M(m_i)\otimes\alpha(C_{iv}) = \sum_{i=1}^{n}\lambda_i m_i\otimes\lambda_i^{-1}\lambda_v C_{iv} =$$
$$\sum_{i=1}^{n}\lambda_v m_i\otimes C_{iv} = \rho^M\circ\alpha_M(m_v)$$

$(\alpha_M\otimes\alpha)\circ\rho^M=\rho^M\circ\alpha_M$ holds.

Finally,

$$(\alpha_M\otimes\alpha_M)\circ R(m_v\otimes m_u) = (\alpha_M\otimes\alpha_M)(\sum_j c_{ju}\cdot m_v\otimes m_j) =$$
$$\sum_{i,j}(\alpha_M\otimes\alpha_M)(x^{ji}_{uv}\, m_i\otimes m_j) = \sum_{i,j} x^{ji}_{uv}\,\lambda_i\lambda_j m_i\otimes m_j =$$
$$\sum_{i,j} x^{ji}_{uv}\,\lambda_u\lambda_v m_i\otimes m_j = R\circ(\alpha_M\otimes\alpha_M)(m_v\otimes m_u)$$

So $(\alpha_M\otimes\alpha_M)\circ R=R\circ(\alpha_M\otimes\alpha_M)$ is true.

Theorem 1 Let (M,α_M) be a finitely dimensional Hom-module, and assume that $(D(R),\mu,\Delta)$ is a bialgebra. With the same hypotheses as Lemma 1 and Lemma 2

There is a Hom-bialgebra $D(R)_\alpha=(D(R),\mu_\alpha,\Delta_\alpha)$, where $\mu_\alpha=\alpha\circ\mu$, $\Delta_\alpha=\Delta\circ\alpha$ such that $(M,\rho_{\alpha,M},\rho^{\alpha,M})$ is a Hom $D(R)$-dimodule, where $\rho_{\alpha,M}=\alpha_M\circ\rho_M$, $\rho_\alpha^M=\rho^M\circ\alpha_M$.

Proof It follows with Proposition 1, Lemma 1 and Lemma 2.

Example 2 Let us first recall the example considered in Ref. [3]. Let $a,b,c\in k$ and M be two dimensional vector spaces with $\{m_1,m_2\}$ a basis. Let $f,g\in\mathrm{End}(M)$ such that it gives the basis $f\in\begin{bmatrix}a&1\\0&a\end{bmatrix}$, $g\in\begin{bmatrix}b&c\\0&b\end{bmatrix}$. Then $R=f\otimes g$, with respect to the denoted basis $\{m_1\otimes m_1,m_1\otimes m_2,m_2\otimes m_1,m_2\otimes m_2\}$ of $M\otimes M$, is given by $R=\begin{bmatrix}ab&ac&b&c\\0&ab&0&b\\0&0&ab&ac\\0&0&0&ab\end{bmatrix}$. Then R is a solution for the D-equation.

Let $C_{11},C_{12},C_{21},C_{22}$ be the four generators of $D(R)$. By the Example 3. 8. 1 in Ref. [3], we know that $C_{21}=0$ and $C_{22}=C_{11}$. If we assume that $C_{11}=x$, $C_{12}=y$, then $D(R)=k\langle x,y\rangle$, the free algebra generated by x and y, and $\Delta(x)=x\otimes x$, $\Delta(y)=x\otimes y+y\otimes x$, $\varepsilon(x)=1$, $\varepsilon(y)=0$. Let λ be an invertible scalar in k, and set $\lambda_1=\lambda_2=\lambda$. We check that (5) holds, i. e., $\lambda_i\lambda_j x_{kv}^{ji}=\lambda_k\lambda_v x_{kv}^{ji}$. Indeed, by Eq. (4), there are only the following non-zero elements $x_{11}^{11}=ab$, $x_{21}^{11}=ac$, $x_{21}^{21}=ab$, $x_{12}^{11}=b$, $x_{12}^{12}=ab$, $x_{21}^{11}=c$, $x_{22}^{21}=b$, $x_{22}^{12}=ac$, $x_{22}^{22}=ab$. So the corresponding (6) are true. Define the maps $\alpha:D(R)\to D(R)$ where $\alpha(x)=x,\alpha(y)=y$, and $\alpha_M:M\to M$, where $\alpha_M(m_1)=\lambda m_1,\alpha_M(m_2)=\lambda m_2$.

Therefore, by Theorem 1, we obtain the Hom-bialgebra $D(R)_\alpha=(D(R),\mu_\alpha=\alpha\circ\mu,\Delta_\alpha=\Delta\circ\alpha)$ such that $(M,\rho_{\alpha,M}=\alpha_M\circ\rho_M,\rho^{\alpha,M}=\alpha_M\circ\rho^M)$ is a Hom $D(R)$-dimodule.

Remark 2 Example 2 also gives an example of Hom-dimodules.

References

[1] Long F W. The Brauer group of dimodule algebras [J]. *Journal of Algebra*, 1974, **31**(1/2/3): 559－601.

[2] Caenepeel C, Militaru G, Zhu S L. Crossed modules and Doi-Hopf modules [J]. *Israel Journal of Mathematics*, 1997, **100**(4): 221－247.

[3] Militaru G. The long dimodules category and nonlinear equations [J]. *Algebras and Representation Theory*, 1999, **2**(2): 177－200.

[4] Larsson D, Silvestrov S D. Quasi-Hom-Lie algebras, central extensions and 2-cocycle-like identities [J]. *Journal of Algebra*, 2005, **288**(2): 321－344.

[5] Makhlouf A, Silvestrov S. Hom-algebra structures [J]. *Journal of Generalized Lie Theory and Applications*, 2008, **2**(2): 51－64.

[6] Makhlouf A, Silvestrov S. Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras [M]//*Generalized Lie theory in mathematics, physics and beyond*. Berlin: Springer, 2009:189－206.

[7] Makhlouf A, Silvestrov S. Hom-algebras and Hom-coalgebras [J]. *Journal of Algebra and Its Applications*, 2010, **9**(4): 1－37.

[8] Yau D. Hom-bialgebras and comodule Hom-algebras [J]. *International Electronic Journal of Algebra*, 2010, **8**: 45－64.

[9] Yau D. The classical Hom-Yang-Baxter equation and Hom-Lie bialgebras [EB/OL]. (2009-05-12) [2013-02-27]. http://arxiv. org/abs/0905. 1890.

[10] Yau D. The Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras [J]. *Journal of Physics A*, 2009, **42**(16):165202-1－165202-12.

[11] Yau D. The Hom-Yang-Baxter equation and Hom-Lie algebras [J]. *Journal of Mathematical Physics*, 2011, **52**(5): 053502-1－053502-19.

[12] Yau D. Hom-quantum groups Ⅰ: quasi-triangular Hom-bialgebras [J]. *Journal of Physics A*, 2012, **45**(6): 065203-1－065203-23.

[13] Yau D. Hom-quantum groups Ⅱ: cobraided Hom-bialgebras and Hom-quantum geometry [EB/OL]. (2009-07-10)[2013-02-27]. http://arxiv. org/abs/0907. 1880.

[14] Yau D. Enveloping algebras of Hom-Lie algebras [J]. *Journal of Generalized Lie Theory and Applications*, 2008, **2**(2): 95－108.

Hom-dimodules 与 Hom 型的 FRT 定理

陈秀丽[1]　李　方[2]　陈建龙[1]

([1] 东南大学数学系，南京 211189)

([2] 浙江大学数学系，杭州 310028)

摘要:为了研究代数形变理论,引入了 Hom-代数的概念. 事实上 Hom-代数是经典结合代数的推广. 首先介绍了 dimodule 的 Hom 型推广,即 Hom-dimodule,并对其相关性质进行讨论. 进一步研究了 Hom-dimodule 范畴与 Hom ***D***-方程 $R^{12}R^{23}=R^{23}R^{12}$ 的关系,其中 $R\in\mathrm{End}_k(M\otimes M)$ 且 M 为 Hom 模. 针对 Hom 双代数上的 Hom-dimodule 给出了 Hom ***D***-方程的一些解,并在 Hom-dimodules 范畴中构造 FRT-型定理. 这些结果推广并改进了 dimodule 范畴中的 FRT-型定理.

关键词: Hom 双代数; Hom-dimodule; Hom ***D***-方程

中图分类号:O153