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A decision model of optimal production reliability and warranty length in an imperfect production system

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Abstract: A decision model to maximize the total profit of manufacturers in an imperfect production system is constructed. In this model, the production reliability and the warranty length are jointly used as decision variables for the case that products are sold with a warranty; i. e., the demand is dependent on the warranty length and sale price. Also, all the non-conforming quality (defective) items in the production process are refurbished to conform to quality ones at a cost. The existence and uniqueness of the optimal values of production reliability and the warranty length are proved by using the Euler-Lagrange method in analyzing the model. A numerical example is also provided to illustrate the effectiveness of the decision model. The sensitivity analysis of the key parameters of the optimal solution and objective value is presented in addition.

Key words: imperfect production system; warranty length; production reliability; Euler-Lagrange method

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This paper intends to construct a decision model to maximize the total profit for manufacturers. To reach the goal of profit maximization, manufacturers need to increase the revenue and reduce the costs. To increase the revenue, product quality is an important factor for manufacturers to attract more buyers. Therefore, producers often provide a longer product warranty as a guarantee of the good quality of the products.

Some researchers studied the optimal production policy under the condition that products are sold with a warranty in an imperfect production system[1−4]. Other researchers focus their attention on the sale policy under the condition when products are sold with warranty. They presented a model to determine the optimal sale price and the warranty length with profit maximization[5−7]. However, one of the problems for these studies is that the rework cost of imperfect items is ignored.

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On the cost control side, manufacturers need to reduce the defects. Ideally, the production system should work under perfect conditions and have zero defects by drawing maintenance policy for production machines[8]. In reality, however, the production system can have different problems for different reasons, such as machine aging, workers' mistakes, etc.; i. e., the production system is imperfect[9]. Under these conditions, the production system will produce non-conforming quality (defective) items. The defective products are usually refurbished so that materials can be reused and the value of the defective items be recovered. Obviously, the manufacturers should minimize the rework cost.

The rate of defective items is related to the reliability of the production process. There are two different approaches for studying the reliability of production processes in the literature. The first approach proposes that the production system is required to produce 100% conforming quality items when appropriate advanced machineries and technologies are used. However, malfunctions of the production system do occur from time to time. The reliability of the production system is the rate of defects and the total products. This rate is denoted by θ which is between 0 and 1, i. e., $0 < \theta \leqslant 1$. Manufacturers will determine the optimal value of θ to maximize the profit or minimize the cost[10−11]. The second approach contends that θ is a number which is between a maximum value and a minimal value due to the production machines constraint and basic product quality required, i. e., $\theta \in [\theta_{\min}, \theta_{\max}]$. The production reliability is a decision variable which can be adjusted to the optimal value to maximize profit[12−14]. However, either of these two approaches considers the product warranty policy.

To remedy these problems, we try to develop a model by optimizing production reliability and warranty length simultaneously for an imperfect production with rework items.

1 Problem Description and Assumptions

All the manufacturers want to gain a competitive edge in the market and achieve a higher profit. So, in the production process, they must make an effective investment decision concerning the reliability of the production process; i. e., they need to determine the optimal production reliability. For the same type product, a higher reliability in the production process can help the firm to obtain more profit in a competitive market. At the product sales stage, the manufacturers need to make an effective sale policy. The product warranty policy is widely used as an incentive to attract more customers. Thus, the manufacturers need to determine an optimal product warranty length. Both the reliability of the production process and warranty length are related to the manufacturers' profits. Therefore, the manufacturers need to determine the optimal reliability of the production process and optimal warranty length in order to maximize the total profits of the production system.

The decision model is built with the following assumptions:

1) The imperfect production system produces only two types of items for a single product, conforming quality items and nonconforming quality (defective) items. A defective item can be refurbished at the cost of making a conforming item.

2) The production rate is greater than the demand of the product; i. e., shortage is not allowed.

3) The failure rate of post-sale products follows an exponential failure distribution.

4) The length of the production cycle T is finite.

2 Formulation and Analysis of the Model

2.1 Formulation of the model

The total cost consists of the development cost, the unit production cost, rework cost for defective items, inventory holding cost and warranty cost. The effects of inflation and time value of money also should be considered.

The development cost is a function of product reliability θ. It can be expressed as[15]

$$D_C(\theta) = A + B\mathrm{e}^{k(\theta_{\max} - \theta)/(\theta - \theta_{\min})} \tag{1}$$

where $\theta = \dfrac{\text{Number of failures}}{\text{Total number of operating hours}}$. θ has the lower bound $\theta_{\min}$ and the upper bound $\theta_{\max}$, i. e., $\theta \in [\theta_{\min}, \theta_{\max}]$. Smaller values of production process reliability indicates more conforming to quality products and a higher production process reliability. The parameters A, B, k are positive constants.

The unit production cost is a function of product reliability θ and dynamic production rate $p(t)$. It is expressed as[6]

$$P_C(t, \theta) = M_C + \frac{D_C(\theta)}{p(t)} + \alpha p(t) \tag{2}$$

where M_C is the material cost, which partly depends on θ, i. e., $M_C = M_{C_0} - M_{C_1}\theta$, and $M_{C_0} > 0, M_{C_1} \geqslant 0$. The parameter α is a variation constant of tool/die cost. It is clear that $P_C(t, \theta)$ is the minimum at $p(t) = \sqrt{D_C(\theta)/\alpha}$. Let R_1 denote the rework costs for each unit of defective items, and then the rework cost for these defective items is $R_1\theta p(t)$

The demand function depends log-linearly on the sale price S_P and the warranty length W. It can be expressed as[7]

$$D = D(S_P, W) = k_1 S_P^{-a}(k_2 + W)^b \tag{3}$$

where $k_1 > 0, k_2 \geqslant 0, a > 1$ and $0 < b < 1$. The parameter k_1 is an amplitude factor and k_2 is a time displacement constant. The parameters a and b represent the price elasticity and the displaced warranty length elasticity, respectively.

In this paper, we assume that the product failure rate is exponentially distributed. When the production reliability is denoted by θ and the repair cost per unit item is denoted by R_2. It is easy to obtain the expected warranty cost during the warranty period $R_2\theta WD$.

We assume that the on-hand inventory $Q(t)$ is zero at $t = 0$ and $t = T$, i. e., $Q(0) = Q(T) = 0$. The change of the inventory level can therefore be expressed as the following differential equation:

$$\dot{Q} = \frac{\partial Q(t)}{\partial t} = p(t) - D(S_P, W) \tag{4}$$

Assume that H_C denotes the inventory holding cost per item per unit time. Hence, the inventory holding cost during $[0, T]$ is $\int_0^T H_C Q \mathrm{d}t$.

Inflation exists in every economy, especially, in developing economies and the inflation rate will in turn affect the interest rate. To incorporate the effects of the interest rate and the inflation rate, we use r and i to denote the interest rate and the inflation rate respectively, and thus $\delta = r - i$.

From the previous analysis, the total profit function with the effect of inflation and time-value of money during $[0, T]$ can be expressed as

$$\begin{aligned}\pi = &\int_0^T \mathrm{e}^{-\delta t}\{[S_P - P_C(\theta, t)]p(t) - R_1\theta p(t) - \\ &R_2\theta WD(S_P, W) - H_C Q\}\mathrm{d}t = \\ &\int_0^T \mathrm{e}^{-\delta t}\{(S_P - M_C - R_1\theta)(\dot{Q} + D) - D_C(\theta) - \\ &\alpha(\dot{Q} + D)^2 - R_2\theta WD - H_C Q\}\mathrm{d}t = \\ &\int_0^T \psi(\dot{Q}, Q, t)\,\mathrm{d}t\end{aligned} \tag{5}$$

where

$$\begin{aligned}\psi(\dot{Q}, Q, t) = \mathrm{e}^{-\delta t}\{&(S_P - M_C - R_1\theta)(\dot{Q} + D) - D_C(\theta) - \\ &\alpha(\dot{Q} + D)^2 - R_2\theta WD - H_C Q\}\end{aligned} \tag{6}$$

2.2 Analysis of the model

Lemma 1 The total profit function π has a maximum value with $Q = Q(t)$ in the interval $[0, T]$.

Proof We denote

$$Q_\varepsilon(t) = Q(t) + \varepsilon\zeta(t) \qquad t \in [0, T] \tag{7}$$

where ε is a sufficiently small variable; $\zeta(t)$ is a first continuous differentiable function of t and $\zeta(t) \geqslant 0$ for all values of t. From $Q_\varepsilon(0) = Q_\varepsilon(T) = 0$, it is clear that $\zeta(0) = \zeta(T) = 0$. The value of the total profit function π for $Q_\varepsilon(t)$ is given by

$$\pi(\varepsilon) = \int_0^T \psi_\varepsilon \mathrm{d}t \tag{8}$$

where $\psi_\varepsilon = \psi(\dot{Q}(t) + \varepsilon\dot{\zeta}(t), Q(t) + \varepsilon\zeta(t), t)$.

To find the maximum value of $\pi(\varepsilon)$, we must have $\partial\pi(\varepsilon)/\partial\varepsilon = 0$ and $\partial^2\pi(\varepsilon)/\partial\varepsilon^2 < 0$. Since ε is a sufficiently small variable, we can let $\varepsilon = 0$; i. e., we must have $\left.\frac{\partial\pi(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}$ and $\left.\frac{\partial^2\pi(\varepsilon)}{\partial\varepsilon^2}\right|_{\varepsilon=0} < 0$. Calculating the first order and the second order derivative of $\pi(\varepsilon)$ with respect to ε, we obtain

$$\begin{aligned}\left.\frac{\partial\pi(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} &= \int_0^T\left\{\zeta(t)\frac{\partial\psi}{\partial Q} + \dot{\zeta}(t)\frac{\partial\psi}{\partial\dot{Q}}\right\}\mathrm{d}t = \\ &\int_0^T \zeta(t)\left\{\frac{\partial\psi}{\partial Q} - \frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial\dot{Q}}\right)\right\}\mathrm{d}t\, \left.\frac{\partial\pi(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} = \\ &\int_0^T\left\{\zeta(t)\frac{\partial\psi}{\partial Q} + \dot{\zeta}(t)\frac{\partial\psi}{\partial\dot{Q}}\right\}\mathrm{d}t = \\ &\left[\zeta(t)\frac{\partial\psi}{\partial\dot{Q}}\right]_0^T + \int_0^T \zeta(t)\left\{\frac{\partial\psi}{\partial Q} - \frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial\dot{Q}}\right)\right\}\mathrm{d}t\, \left.\frac{\partial\pi(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} = \\ &\int_0^T\left\{\zeta(t)\frac{\partial\psi}{\partial Q} + \dot{\zeta}(t)\frac{\partial\psi}{\partial\dot{Q}}\right\}\mathrm{d}t = \\ &\int_0^T \zeta(t)\left\{\frac{\partial\psi}{\partial Q} - \frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial\dot{Q}}\right)\right\}\mathrm{d}t\end{aligned} \tag{9}$$

$$\left.\frac{\partial^2\pi(\varepsilon)}{\partial\varepsilon^2}\right|_{\varepsilon=0} = \int_0^T\left\{\zeta^2\frac{\partial^2\psi}{\partial Q^2} + 2\zeta\dot{\zeta}\frac{\partial^2\psi}{\partial Q\partial\dot{Q}} + \dot{\zeta}^2\frac{\partial^2\psi}{\partial\dot{Q}^2}\right\}\mathrm{d}t \tag{10}$$

Therefore, the necessary condition for the extreme value existence of π is $\left.\frac{\partial\pi(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} = 0$, i. e., $\frac{\partial\psi}{\partial Q} - \frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial\dot{Q}}\right) = 0$.

Differentiating the function ψ with respect to Q and $\dot{Q}$, we obtain

$$\frac{\partial\psi}{\partial Q} = -H_{\mathrm{C}}\mathrm{e}^{-\delta t},\ \frac{\partial\psi}{\partial\dot{Q}} = [-2\alpha(\dot{Q} + D) + (S_{\mathrm{P}} - M_{\mathrm{C}} - R_1\theta)]\mathrm{e}^{-\delta t} \tag{11}$$

$$\frac{\partial^2\psi}{\partial Q^2} = 0,\ \frac{\partial^2\psi}{\partial Q\partial\dot{Q}} = 0,\ \frac{\partial^2\psi}{\partial\dot{Q}^2} = -2\alpha\mathrm{e}^{-\delta t} \tag{12}$$

Substituting Eq. (11) and Eq. (12) into Eq. (10), we obtain

$$\left.\frac{\partial^2\pi(\varepsilon)}{\partial\varepsilon^2}\right|_{\varepsilon=0} = -2\alpha\int_0^T \mathrm{e}^{-\delta t}\dot{\zeta}^2\mathrm{d}t < 0 \tag{13}$$

Therefore, by the differential and integral calculus, we find that the profit function π has a maximum value. The proof is completed.

The Euler-Lagrange equation for the maximum value of π is

$$\frac{\partial\psi}{\partial Q} - \frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial\dot{Q}}\right) = 0 \tag{14}$$

Substituting Eq. (6) into Eq. (14), we obtain

$$\ddot{Q} - \delta\dot{Q} = \frac{H_{\mathrm{C}} - \delta N}{2\alpha} + \delta D \triangleq L \tag{15}$$

where $N = S_{\mathrm{P}} - M_{\mathrm{C}_0} + w_0\theta > 0$ and $w_0 = M_{\mathrm{C}_1} - R_1 < 0$.

Solving the second order differential equation, we obtain that the general solution of $Q(t)$ is

$$Q = E + F\mathrm{e}^{\delta t} - \frac{L}{\delta}t \tag{16}$$

where E and F are the abstract functions. Now, using the boundary conditions, $Q(0) = 0 = Q(T)$, the values of E and F are

$$E = -F = -\frac{TL}{\delta(\mathrm{e}^{\delta T} - 1)} = -v_1L \tag{17}$$

where $v_1 = \frac{T}{\delta(\mathrm{e}^{\delta T} - 1)} > 0$.

Using Eq. (4), the optimal dynamic production rate is obtained as

$$p(t) = \delta v_1 L\mathrm{e}^{\delta t} - \frac{L}{\delta} + D \tag{18}$$

Substituting Eq. (18) into Eq. (5), the total profit function can be expressed as

$$\begin{aligned}\pi = &\int_0^T \mathrm{e}^{-\delta t}\left\{\left(\delta F\mathrm{e}^{\delta t} - \frac{L}{\delta} + D\right)N - D_{\mathrm{C}}(\theta) - \left(\delta F\mathrm{e}^{\delta t} - \frac{L}{\delta} + D\right)^2\alpha - \right. \\ &\left. R_2\theta WD - \left(E + F\mathrm{e}^{\delta t} - \frac{L}{\delta}t\right)H_{\mathrm{C}}\right\}\mathrm{d}t = \\ &u_1LN - v_2DN + v_2D_{\mathrm{C}}(\theta) + \frac{\alpha u_1}{\delta}L^2 + \alpha v_2D^2 - \\ &2\alpha u_1DL + R_2v_2\theta WD - \frac{u_1H_{\mathrm{C}}}{\delta}L\end{aligned} \tag{19}$$

where $v_2 = (\mathrm{e}^{-\delta T} - 1)/\delta < 0$, $u_1 = \delta v_1T + v_2/\delta = -\sum_{k=1}^{\infty}\frac{(\delta T)^{2k+2}}{(2k+2)!}/[\delta^2(\mathrm{e}^{\delta T} - 1)] < 0$.

Theorem 1 If $k' \leqslant k \leqslant k''$, the equation $\frac{\partial\pi}{\partial\theta} = 0$ has a unique solution θ^* in $\left[\theta_{\min} + \frac{1}{2}k(\theta_{\max} - \theta_{\min}), \theta_{\max}\right]$, where

$$k' = -1 + \sqrt{1 + \frac{u_1\delta\omega_0^2(\theta_{\max} - \theta_{\min})^2}{2\alpha Bv_2}},\ k'' = \min\{1, K_0\}$$

$K_0 =$

$$\frac{\left(\frac{u_1\delta w_0}{2\alpha}N(\theta_{\max})-(\delta u_1-v_2)w_0D-R_2v_2W-\frac{u_1w_0H_C}{2\alpha}\right)(\theta_{\min}-\theta_{\max})}{v_2B}$$

Proof Let π be a function of θ, i. e. $\pi=\pi(\theta)$. Calculating the first order derivative of $\pi(\theta)$ with respect to θ and simplification, we obtain

$$\frac{\partial\pi}{\partial\theta}=-\frac{u_1\delta w_0}{2\alpha}N+(\delta u_1-v_2)w_0D+v_2\frac{\partial D_C(\theta)}{\partial\theta}+R_2v_2WD+\frac{u_1w_0H_C}{2\alpha} \tag{20}$$

$$\frac{\partial^2\pi}{\partial\theta^2}=-\frac{u_1\delta w_0^2}{2\alpha}+v_2\frac{\partial^2D_C(\theta)}{\partial\theta^2} \tag{21}$$

where

$$\frac{\partial D_C(\theta)}{\partial\theta}=\frac{Bk(\theta_{\min}-\theta_{\max})}{(\theta-\theta_{\min})^2}e^{k(\theta_{\max}-\theta)/(\theta-\theta_{\min})} \tag{22}$$

$$\frac{\partial^2D_C(\theta)}{\partial\theta^2}=\left(-\frac{2}{\theta-\theta_{\min}}+\frac{k(\theta_{\min}-\theta_{\max})}{(\theta-\theta_{\min})^2}\right)\cdot\frac{Bk(\theta_{\min}-\theta_{\max})}{(\theta-\theta_{\min})^2}e^{k(\theta_{\max}-\theta)/(\theta-\theta_{\min})} \tag{23}$$

and $\frac{\partial D_C(\theta)}{\partial\theta}<0$, $\frac{\partial^2D_C(\theta)}{\partial\theta^2}>0$ for all values of $\theta\in\left[\theta_{\min}+\frac{1}{2}k(\theta_{\max}-\theta_{\min}),\theta_{\max}\right]$.

Let $\varphi(\theta)=\frac{\partial\pi}{\partial\theta}$ by $v_2<0$ and Eq. (22), and we have $\lim_{\theta\to\theta_{\min}}\varphi(\theta)=+\infty$. By Eq. (20) and Eq. (22), we obtain

$$\varphi(\theta_{\max})=-\frac{u_1\delta w_0}{2\alpha}N(\theta_{\max})+(\delta u_1-v_2)w_0D+R_2v_2WD+\frac{u_1w_0H_C}{2\alpha}+\frac{v_2Bk}{\theta_{\min}-\theta_{\max}} \tag{24}$$

Therefore, $\varphi(\theta_{\max})<0\Rightarrow k\leqslant\left(\frac{u_1\delta w_0}{2\alpha}N(\theta_{\max})-(\delta u_1-v_2)w_0D-R_2v_2W-\frac{u_1w_0H_C}{2\alpha}\right)(\theta_{\min}-\theta_{\max})/v_2B=K_0$. Let $k''=\min\{1,K_0\}$. The equation $\varphi(\theta)=\frac{\partial\pi}{\partial\theta}=0$ at least has a solution θ^* in $\left[\theta_{\min}+\frac{1}{2}k(\theta_{\max}-\theta_{\min}),\theta_{\max}\right]$, when $0\leqslant k\leqslant k''$. By $v_2<0$, Eq. (21) and Eq. (23), we have

$$\varphi'(\theta)=\frac{\partial^2\pi}{\partial\theta^2}\leqslant\varphi'(\theta_{\max})=\frac{1}{(\theta_{\max}-\theta_{\min})^2}\cdot\left[(v_2B)k^2+(2v_2B)k-\frac{u_1\delta\omega_0^2}{2\alpha}(\theta_{\max}-\theta_{\min})^2\right]$$

Let $\phi(k)=(v_2B)k^2+(2v_2B)k-\frac{u_1\delta\omega_0^2}{2\alpha}(\theta_{\max}-\theta_{\min})^2$.

By solving the equation $\phi(k)=0$, we obtain $k'=-1+\sqrt{1+u_1\delta\omega_0^2(\theta_{\max}-\theta_{\min})^2/2\alpha Bv_2}$. By $v_2B<0$, we know that $\phi(k)\leqslant0$ when $k\geqslant k'$. Hence,

$$\varphi'(\theta)=\frac{\partial^2\pi}{\partial\theta^2}=\frac{1}{(\theta_{\max}-\theta_{\min})}\phi(k)\leqslant0$$

So, the equation $\varphi(\theta)=\frac{\partial\pi}{\partial\theta}=0$ has a unique solution θ^* in $\left[\theta_{\min}+\frac{1}{2}k(\theta_{\max}-\theta_{\min}),\theta_{\max}\right]$ when $k'\leqslant k\leqslant k''$. The proof is completed.

Theorem 2 If $\theta\in[\tilde{\theta},\theta_{\max}]$ the equation $\frac{\partial\pi}{\partial W}=0$ has a unique solution W^* in $[0,\infty)$, where $\tilde{\theta}=\max\{\theta_{\min},\theta_0\}$ and

$\theta_0=$

$$\frac{-(u_1\delta-v_2)(S_P-M_{C_0})+u_1H_C-2\alpha(v_2-\delta u_1)k_1S_P^{-a}k_2^b}{(u_1\delta-v_2)w_0+\frac{k_2R_2v_2}{b}}<\theta_{\max}$$

Proof Let π be a function of W, i. e. , $\pi=\pi(W)$. Calculating the first order and second order derivative of $\pi(W)$ with respect to W and simplification, we obtain

$$\frac{\partial\pi}{\partial W}=\left[(u_1\delta-v_2)N-u_1H_C+2\alpha(v_2-\delta u_1)D+R_2v_2\theta\left(\frac{k_2+W}{b}+W\right)\right]D_W=\eta(W)D_W \tag{25}$$

$$\frac{\partial^2\pi}{\partial W^2}=\eta(W)D_{WW}+\left[2\alpha(v_2-\delta u_1)D_W+R_2v_2\theta\left(\frac{1}{b}+1\right)\right]D_W \tag{26}$$

where

$$\eta(W)=(u_1\delta-v_2)N-u_1H_C+2\alpha(v_2-\delta u_1)D+R_2v_2\theta\left(\frac{k_2+W}{b}+W\right) \tag{27}$$

$$D_W=\frac{\partial D}{\partial W}=bk_1S_P^{-a}(k_2+W)^{b-1}=\frac{b}{k_2+W}D \tag{28}$$

$$D_{WW}=\frac{\partial^2D}{\partial W^2}=b(b-1)k_1S_P^{-a}(k_2+W)^{b-2} \tag{29}$$

It can be clearly demonstrated that $D_W>0, D_{WW}<0$. By $v_2-\delta u_1=-\delta^2v_1T<0$, $\eta(W)$ is a strictly decreasing function of W and $\lim_{W\to\infty}\eta(W)\to-\infty$. Therefore, $\lim_{W\to\infty}\frac{\partial\pi}{\partial W}\to-\infty$. On the other hand,

$$\eta(0)>0\Rightarrow\theta>\frac{-(u_1\delta-v_2)(S_P-M_{C_0})+u_1H_C-2\alpha(v_2-\delta u_1)k_1S_p^{-a}k_2^b}{(u_1\delta-v_2)w_0+\frac{k_2R_2v_2}{b}}=\theta_0$$

Let $\tilde{\theta}=\max\{\theta_{\min},\theta_0\}$. We obtain that $\left.\frac{\partial\pi}{\partial W}\right|_{W=0}>0$ for all values of $\theta\in[\tilde{\theta},\theta_{\max}]$. Therefore, the equation $\frac{\partial\pi}{\partial W}=0$ has the unique solution W^* in $[0,\infty)$ when $\theta\in[\tilde{\theta},\theta_{\max}]$. The proof is completed.

Theorem 3 The joint equation $\frac{\partial\pi}{\partial W}=0$ and $\frac{\partial\pi}{\partial\theta}=0$ has a unique solution (W^*,θ^*) for W in $[0,+\infty)$ and θ in $[\theta',\theta_{\max}]$ where $\theta'=\max\left\{\tilde{\theta},\theta_{\min}+\frac{1}{2}(\theta_{\max}-\theta_{\min})\right\}$.

Proof By Eq. (25), we obtain

$$\theta=\theta(W)=\frac{-(u_1\delta-v_2)(S_P-M_{C_0})+u_1H_C-2\alpha(v_2-\delta u_1)D}{(u_1\delta-v_2)w_0+R_2v_2\left[\frac{(k_2+W)}{b}+W\right]} \tag{30}$$

Therefore, the value of θ is determined by W solely. Substituting Eq. (30) into Eq. (20), we obtain

$$\chi(W)=-\frac{u_1\delta w_0}{2\alpha}[S_p-M_{C_0}+w_0\theta(W)]+(\delta u_1-v_2)w_0D+R_2v_2WD+\frac{u_1w_0H_C}{2\alpha}+\frac{v_2Bk(\theta_{\min}-\theta_{\max})}{(\theta(W)-\theta_{\min})^2}\mathrm{e}^{k(\theta_{\max}-\theta(W))/(\theta(W)-\theta_{\min})}$$

By the proof process of Theorem 2, we know that the solution of equation $\chi(W)=0$ exists and is unique in $[0,+\infty)$ when $\theta\in[\theta',\theta_{\max}]$, where

$$\theta'=\max\left\{\tilde{\theta},\theta_{\min}+\frac{1}{2}(\theta_{\max}-\theta_{\min})\right\} \tag{31}$$

Hence, the joint equation $\frac{\partial\pi}{\partial W}=0$ and $\frac{\partial\pi}{\partial\theta}=0$ has a unique solution (W^*,θ^*) for W in $[0,+\infty)$ and θ in $[\theta',\theta_{\max}]$. The proof is completed.

3 Numerical Example

We assume that the values of parameters in this model are as follows: $\theta_{\min}=0.1$, $\theta_{\max}=0.9$, $\delta=0.01$, $A=100$ dollar, $B=150$ dollar, $k=0.15$, $S_P=95$ dollar, $M_{C_0}=15$ dollar, $M_{C_1}=1$ dollar, $R_1=8$ dollar, $R_2=10$ dollar, $k_1=1\,500$, $a=1.2$, $k_2=2.8$ month, $b=0.6$, $\alpha=0.02$, and $T=2$ month. To find the impact of reliability of the production process on the development cost and the total profit, we take the production reliability as decision variability. By Eq. (1) and Eq. (19), we let $W=12$ month and use the previous values of parameters, we obtain the optimal product $\theta^*=0.25$ with maximal profit $\pi^*=2\,669$ dollar by Matlab 7.0 (see Fig. 1). By Theorem 1, the optimal solution is the global optimal solution.

From Fig. 1, when manufacturers take some measures to improve the reliability of the production process (i. e., they lower the value of production reliability, even though that will result in a higher development cost), the manufacturers achieve higher profits and reach the maximal profit at $\theta^*=0.25$. However, after that, if manufacturers continue to increase investment in the reliability of production process, the development cost will directly increase and the total profit will decrease. Consequently, the model can be used to help manufacturers draw up an effective investment policy regarding the reliability of production process.

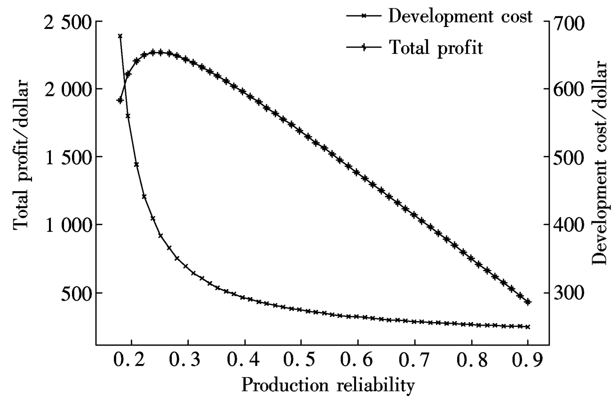

Fig. 1 Production reliability vs. total profit and development cost

To assess the impact of the warranty length on both the warranty cost and the total profit, we take the warranty length as decision variability. By the expected warranty cost $R_2\theta WD$ and Eq. (19), we let $\theta=0.30$ and use the previous values for the parameters. We obtain the optimal warranty length $W^*=18.6$ month with the maximum profit $\pi^*=2\,399$ dollar by Matlab 7.0 (see Fig. 2). By Theorem 2, the optimal solution is the global optimal solution.

From Fig. 2, we can see that the warranty cost increases when the warranty length increases. In the interval $[0,18.6]$, the total profit increases when warranty length increases. The total profit is maximized at $W^*=18.6$ month. The manufacturers often increase the sale volume by giving a longer warranty length. But when the warranty length is greater than 18.6 month, the warranty cost will increase and the total profit will decrease. Therefore, this model can be used to help the manufacturers determine an optimal product warranty length with maximum profit.

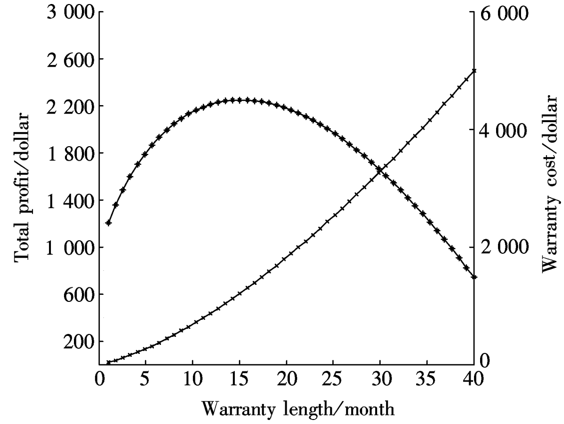

Fig. 2 The warranty length vs. the total profit and warranty cost

The manufacturers often take some measures to obtain more profit, and those measures not only impact the optimal product reliability but also impact the optimal warranty length. To understand the phenomenon clearly, we take both the product reliability and warranty length as decision variables. By Eq. (19), and using the previous values of parameters, we obtain the maximum profit $\pi^* = 2\ 247$ dollar at $\theta^* = 0.22$ and $W^* = 21.58$ month by Matlab 7.0 (see Fig. 3).

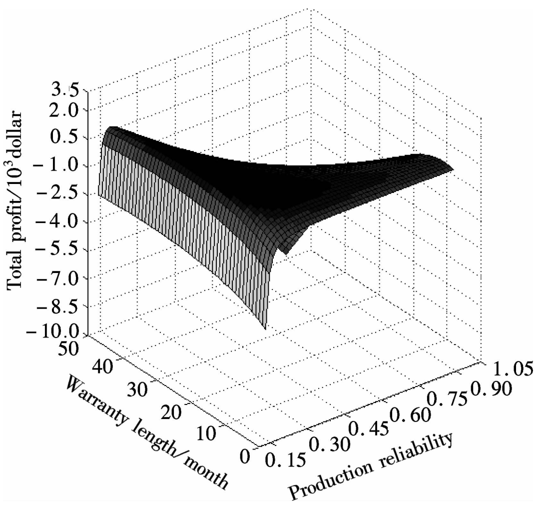

Fig. 3 Total profit vs. warranty length and production reliability

From Fig. 3, we know that the optimal solution (θ^*, W^*) is the global optimal solution. Hence, the model can be used to help the manufacturers create an effective policy to improve their profits when both the reliability of the production process and warranty length are considered. The sensitivity analysis of key parameters is in Tab. 1.

Tab. 1 Sensitivity analysis of key parameters %

Parameters	+/−	θ^*	W^*	π^*
k	+50	+21.11	−18.16	−11.77
	+25	+7.04	−4.54	−6.42
	−25	−14.07	+18.16	+8.46
	−50	−21.11	+31.79	+19.10
k_1	+50	−7.04	+9.08	+69.52
	+25	−7.04	+9.08	+34.79
	−25	+7.04	−4.54	−33.91
	−50	+14.07	−9.08	−66.66
a	+50	+316.67	−81.74	−115.23
	+25	+49.72	−36.33	−96.47
	−25	−21.11	+22.71	−48.66
	−50	−28.15	+9.08	+93.40
k_2	+50	0	−4.54	+3.85
	+25	0	0	+1.93
	−25	0	+4.54	−1.90
	−50	0	+4.54	−3.81
b	+50	−21.11	+68.12	+200.41
	+25	−14.07	+40.87	+74.25
	−25	+14.07	−31.79	−42.46
	−50	+56.30	−68.12	−66.08

From Tab. 1, we can see clearly that the impact of the parameters change value on the optimal production reliability, optimal warranty length and the maximum total profit. For instance, the change of parameter k_2 has impacts on the optimal warranty length W^* but not on the optimal production reliability θ^*. As the parameter k decreases, W^* increases and can attract more consumers.

As a result, θ^* must decrease (implying a more reliable production process) to reduce the warranty cost and increase profit. Consequently, the model can be used for the manufacturers to improve the production process and make an optimal warranty sale policy with profit maximization.

4 Conclusion

In an imperfect production process, the manufacturers need to decide what is their optimal production reliability and product warrant length. The model constructed in this paper can be used for the manufacturers to make effective investment policies to succeed when competing with similar products. The sale price is fixed for a fixed period. So, the product quality is very important in the competitive market. Product quality is related to the reliability of the production process. Thus, the model is useful for determining an optimal production reliability for the manufacturers who sell their products with warranties. Longer warranty length is an effective strategy to improve the sale volume, because it often implies higher product quality. However, too long warranty length results in higher expected warranty costs and lower profits. Therefore, it is very important to make an optimal warranty sale policy with profit maximization. The model can be used to solve the proposed problems. Both the reliability of production process and warranty length have impact on the total profit. From Tab. 1, we can see that this model can also guide the manufacturers in their strategic planning to maximize their profits.

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不完美生产系统中最优生产可靠性和产品保证销售期决策模型

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摘要：在不完美生产系统中，以制造商利润最大化为目标，建立了一个决策模型. 该模型以生产可靠性和保证期为联合决策变量，考虑了一种产品保证销售策略，即需求依赖于产品保证期和销售价格. 并且在不完美生产过程中产生的所有不合格品均以一定的成本返工成合格品. 利用 Euler-Lagrange 方法对模型进行分析，证明了最优生产可靠性和保证期的存在唯一性. 通过数值实例证实了模型的有效性，并就关键参数对最优解和最优目标值影响进行了敏感性分析.

关键词：不完美生产系统；保证期；生产可靠性；Euler-Lagrange 方法

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