

# $L(1, 2)$ -edge-labeling for necklaces

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**Abstract:** For a graph  $G$  and two positive integers  $j$  and  $k$ , an  $m$ - $L(j, k)$ -edge-labeling of  $G$  is an assignment from the set  $\{0, 1, \dots, m\}$  to the edges, such that adjacent edges receive labels that differ by at least  $j$ , and edges at distance two receive labels that differ by at least  $k$ . The  $\lambda'_{j,k}$ -number of  $G$ , denoted by  $\lambda'_{j,k}(G)$ , is the minimum integer  $m$  overall  $m$ - $L(j, k)$ -edge-labeling of  $G$ . The necklace is a specific type of Halin graph. The  $L(1, 2)$ -edge-labeling of necklaces is studied and the lower and upper bounds on  $\lambda'_{1,2}$ -number for necklaces are given. Also, both the lower and upper bounds are attainable.

**Key words:** channel assignment;  $L(j, k)$ -edge-labeling; Cartesian product; Halin graph; necklace

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Throughout this paper, standard notations are used from graph theory<sup>[1]</sup>. Let  $\mathbf{Z}_+$  be the set of nonnegative integers and let  $[a, b]$  denote the set  $\{n \in \mathbf{Z}_+ \mid a \leq n \leq b\}$ . For the two positive integers  $j$  and  $k$ , an  $m$ - $L(j, k)$ -labeling of  $G$  is an assignment  $f$  from the set  $[0, m]$  to each vertex of  $G$ , such that the following distance conditions are satisfied:  $|f(u) - f(v)| \geq j$  if  $u$  and  $v$  are adjacent and  $|f(u) - f(v)| \geq k$  if  $u$  and  $v$  are at distance two. The  $\lambda_{j,k}$ -number of a graph  $G$ , denoted by  $\lambda_{j,k}(G)$ , is the minimum integer  $m$  overall  $L(j, k)$ -labeling of  $G$ .

The  $L(j, k)$ -labeling of graphs is inspired by the channel assignment problem introduced by Hale<sup>[2]</sup>. The  $L(2, 1)$ -labeling was formulated and studied by Griggs and Yeh<sup>[3]</sup> in 1992. Since then,  $L(2, 1)$ -labeling and  $L(j, k)$ -labeling ( $j \geq k$ ) of graphs have been studied extensively. Refer to surveys<sup>[4-5]</sup>.

A variation of the channel assignment problem is the code assignment in computer networks<sup>[6]</sup>. The task is to assign integer “control codes” to a network of computer stations with distance restrictions. This is the same as the problem of  $L(j, k)$ -labeling with  $j \leq k$ . Jin and Yeh<sup>[6]</sup> studied the  $L(j, k)$ -labeling for  $(j, k) \in \{(0, 1), (1, 1), (1, 2)\}$ . Calamoneri et al.<sup>[7]</sup> investigated the  $\lambda_{j,k}$ -numbers of trees with  $j \leq k$ . Chen et al.<sup>[8-10]</sup> also studied the  $L(j, k)$ -labeling for  $j \leq k$ .

In this paper, the edge version of  $L(j, k)$ -labeling is

studied. Two edges  $e_1$  and  $e_2$  are adjacent (at distance one) if and only if they meet at a common vertex. Two edges  $e_1$  and  $e_2$  are at distance two if and only if they are nonadjacent but adjacent to a common edge. For two positive integers  $j$  and  $k$ , an  $m$ - $L(j, k)$ -edge-labeling of  $G$  is a function  $f: E(G) \rightarrow [0, m]$  such that  $|f(e_1) - f(e_2)| \geq j$  if  $e_1$  and  $e_2$  are adjacent, and  $|f(e_1) - f(e_2)| \geq k$  if  $e_1$  and  $e_2$  are at distance two. The  $\lambda'_{j,k}$ -number of  $G$ , denoted by  $\lambda'_{j,k}(G)$ , is the minimum integer  $m$  overall  $L(j, k)$ -edge-labeling of  $G$ .

The  $L(j, k)$ -edge-labeling was first investigated by Georges and Mauro<sup>[11]</sup>. They determined the  $\lambda'_{1,1}$ -numbers and  $\lambda'_{2,1}$ -numbers of complete graphs,  $\Delta$ -regular trees for  $\Delta \geq 2$ ,  $n$ -dimensional cubes for small  $n$  and  $n$ -wheels. In addition, the  $L(j, k)$ -edge-labeling was also studied in Refs. [8, 12–13].

The strong chromatic index, i. e.,  $\lambda'_{1,1}$ -number and the  $\lambda'_{2,1}$ -number of the Halin graph have been well studied<sup>[12, 14-15]</sup>. In this paper, the  $\lambda'_{1,2}$ -numbers of necklaces, a specific type of Halin graph are investigated.

Suppose that  $T$  is a tree with no vertex of degree two and at least one vertex of degree three or more. A Halin graph  $G = T \cup C$  is a planar graph, where  $C$  is a cycle connecting the leaves (vertices of degree 1) of  $T$  in the cyclic order determined by a drawing of  $T$ . A caterpillar is a tree such that after the removal of the leaves it becomes a path. For  $h \geq 1$ , suppose that  $T_h$  is a caterpillar with the path  $P_{h+2}$  of length  $h+1$ . The vertices along  $P_{h+2}$  are marked with  $0, 1, \dots, h, h+1$ , and the other vertices are marked with  $1', 2', \dots, h'$  such that  $\{i, i'\} \in E(T_h)$  ( $1 \leq i \leq h$ ). The necklace  $Ne_h$  is a graph obtained from  $T_h$  by adding the edges  $\{0, 1'\}, \{1', 2'\}, \dots, \{h', h+1\}$  and  $\{h+1, 0\}$  (see Fig. 1). Note that necklaces form a specific class of 3-regular Halin graph and  $Ne_1 \cong K_4$ .

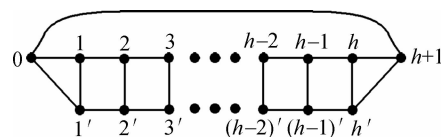


Fig. 1 Necklace  $Ne_h$

Given the two graphs  $G$  and  $H$ , the Cartesian product of these two graphs, denoted by  $G \square H$ , is defined by  $V(G \square H) = V(G) \times V(H)$  and  $E(G \square H) = \{(u, x)(v, y) \mid uv \in E(G) \text{ and } x = y \text{ or } xy \in E(H) \text{ and } u = v\}$ . The Cartesian product of  $P_2$  and  $P_h$  is a subgraph of  $Ne_h$ , induced by the set of vertices  $\{1, 2, \dots, h, 1', 2', \dots, h'\}$ . Thus,

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$$\lambda'_{1,2}(Ne_h) \geq \lambda'_{1,2}(P_2 \square P_h).$$

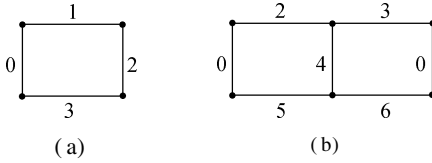
## 1 Cartesian Produce of $P_2$ and $P_h$

Let  $P_h$  be a path with  $h$  vertices. The edges of  $P_2 \square P_h$  consist of horizontal edges and vertical edges. The vertices of  $P_2 \square P_h$  are denoted by the same notations as the vertices of  $Ne_h$  in Fig. 1. For convenience, the horizontal edge,  $(i, i+1)$  (or  $(i', (i+1)')$ ), is denoted by  $h_i$  (or  $h_{i'}$ ), and the vertical edge,  $(j, j')$ , by  $e_j$  ( $1 \leq i \leq h-1$  and  $1 \leq j \leq h$ ). The following theorem gives the  $\lambda'_{1,2}$ -number of  $P_2 \square P_h$ .

**Theorem 1** Let  $P_h$  be a path with  $h \geq 2$  vertices. Then

$$\lambda'_{1,2}(P_2 \square P_h) = \begin{cases} 3 & h = 2 \\ 6 & h = 3 \\ 7 & h \geq 4 \end{cases}$$

**Proof** Since  $P_2 \square P_2$  is a 4-cycle, it is clear that  $\lambda'_{1,2}(P_2 \square P_2) = 3$  (see Fig. 2(a)). Fig. 2(b) gives a 6- $L(1, 2)$ -edge-labeling of  $P_2 \square P_3$ . Thus,  $\lambda'_{1,2}(P_2 \square P_3) \leq 6$ .



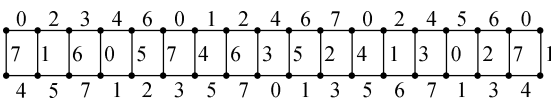
**Fig. 2** Optimal  $L(1, 2)$ -edge-labeling. (a)  $P_2 \square P_2$ ; (b)  $P_2 \square P_3$

It can now be proved that  $\lambda'_{1,2}(P_2 \square P_3) \geq 6$ . Since the distance between any two edges in  $E(P_2 \square P_3) \setminus \{e_1\}$  is at most two, it is clear that  $\lambda'_{1,2}(P_2 \square P_3) \geq 5$ . Suppose that  $\lambda'_{1,2}(P_2 \square P_3) = 5$  and  $f$  is a 5- $L(1, 2)$ -edge-labeling of  $P_2 \square P_3$ .

Since  $|E(P_2 \square P_3)| = 7$  and  $f$  has only six labels to use, the label of  $e_1$  must be equal to the label of some other edge. Due to the distance condition,  $f(e_1) = f(e_3)$ . If  $f(e_1) = f(e_3) = k$ , then  $k \pm 1$  and  $k$  cannot be assigned to the remaining five edges. That is, there are at most four labels left for the remaining five edges, which must obtain distinct labels. This is a contradiction. Thus,  $\lambda'_{1,2}(P_2 \square P_3) = 6$ .

For  $h \geq 4$ , one can obtain a 7- $L(1, 2)$ -edge-labeling of  $P_2 \square P_h$ , whose period is described in Fig. 3. Hence,  $\lambda'_{1,2}(P_2 \square P_h) \leq 7$  ( $h \geq 4$ ).

To complete the proof, it suffices to show that  $\lambda'_{1,2}(P_2 \square P_4) \geq 7$ . Since  $P_2 \square P_3$  is a subgraph of  $P_2 \square P_4$ , it is clear that  $\lambda'_{1,2}(P_2 \square P_4) \geq 6$ . Suppose that  $\lambda'_{1,2}(P_2 \square P_4) = 6$  and  $f$  is a 6- $L(1, 2)$ -edge-labeling of  $P_2 \square P_4$ .



**Fig. 3** A periodic 7- $L(1, 2)$ -edge-labeling of  $P_2 \square P_{18}$

Consider the eight edges in  $E' = E(P_2 \square P_4) \setminus \{e_1, e_4\}$ . Due to the distances among the edges in  $E'$ , if  $f(h_1) \neq f(h_{3'})$  and  $f(h_{1'}) \neq f(h_3)$ , then all edges in  $E'$  must receive different labels. This is a contradiction since  $f$  uses only seven labels. Thus  $f(h_1) = f(h_{3'})$  or  $f(h_{1'}) = f(h_3)$ . By symmetry, it may be assumed that  $f(h_1) = f(h_{3'})$ . Note that each edge in  $E' \setminus \{h_1, h_{3'}\}$  is at distance two from edge  $h_1$  or  $h_{3'}$ . If  $f(h_1) = f(h_{3'}) \notin \{0, 6\}$ , then there are only four labels for the six edges in  $E' \setminus \{h_1, h_{3'}\}$ . This is impossible due to the distance conditions. So,  $f(h_1) = f(h_{3'}) \in \{0, 6\}$ . Similarly,  $f(h_{1'}) = f(h_3) \in \{0, 6\}$ . Therefore, it follows that the edges of the square in the middle of  $P_2 \square P_4$  can only be assigned by the labels in  $\{2, 3, 4\}$ . This is a contradiction. Hence,  $\lambda'_{1,2}(P_2 \square P_4) \geq 7$ , and the theorem holds.

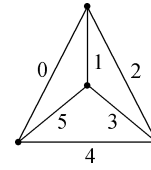
## 2 Necklaces

In this section, we consider the  $\lambda'_{1,2}$ -numbers of necklaces. Also, we give sharp bounds on the  $\lambda'_{1,2}$ -numbers for necklaces.

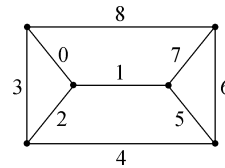
**Theorem 2** Let  $Ne_h$  be a necklace. Then

$$\lambda'_{1,2}(Ne_h) = \begin{cases} 5 & h = 1 \\ 8 & h = 2 \text{ and } 3 \\ 7 & h = 4 \end{cases}$$

**Proof** It is simple to check that  $\lambda'_{1,2}(Ne_1) = 5$  (see Fig. 4). For  $Ne_2$ , it is also straightforward to see that all edges must be assigned distinct colors. Hence,  $\lambda'_{1,2}(Ne_2) = 8$  (see Fig. 5).

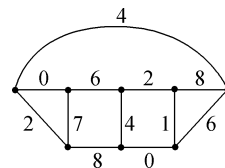


**Fig. 4** A 5- $L(1, 2)$ -edge-labeling of  $Ne_1$



**Fig. 5** An 8- $L(1, 2)$ -edge-labeling of  $Ne_2$

For  $Ne_3$ , Fig. 6 gives an 8- $L(1, 2)$ -edge-labeling of  $Ne_3$  and then  $\lambda'_{1,2}(Ne_3) \leq 8$ . It can be proved that  $\lambda'_{1,2}(Ne_3) \geq 8$ . Suppose that  $\lambda'_{1,2}(Ne_3) \leq 7$  and  $f$  is a 7- $L(1, 2)$ -edge-labeling of  $Ne_3$ . Let  $L_i = \{e \in E(Ne_3) \mid f(e) = i\}$  and  $l_i$  be



**Fig. 6** An 8- $L(1, 2)$ -edge-labeling of  $Ne_3$

the cardinality of  $L_i$ . By the distance conditions and the structure of  $Ne_3$ , it is clear that  $l_i \leq 2$ . Moreover, it is not difficult to verify that if  $l_i = 2$ , then  $l_{i-1} \leq 1$  and  $l_{i+1} \leq 1$ . In particular, if the edges  $(0, 4)$  and  $(2, 2')$  are assigned the same label, say  $k$ , then  $l_{k-1} = l_{k+1} = 0$ .

Since  $|E(Ne_3)| = 12$  and  $f$  has only 8 labels to use, there are at least four labels which are used twice. Note that the labels are from interval  $[0, 7]$ . By the above discussion, then  $l_i + l_{i+1} \leq 3$  for  $i = 0, 1, \dots, 6$ . It follows that there are at most four labels used twice and the other four labels used once. So the edges  $(0, 4)$  and  $(2, 2')$  cannot be assigned to the same label. By the structure of  $Ne_3$ , for any edge  $e \in \{(0, 1), (0, 1'), (1, 2), (1, 1'), (1', 2')\}$ , if there is an edge  $e'$  such that  $f(e) = f(e') = i$ , then  $e' \in \{(2, 3), (2', 3'), (3, 3'), (3, 4), (3', 4)\}$ . Due to the distance conditions,  $f((0, 4)) \neq i \pm 1$  and  $f((2, 2')) \neq i \pm 1$ . So, the edges  $(0, 4)$  and  $(2, 2')$  cannot be labeled. This is a contradiction and  $\lambda'_{1,2}(Ne_3) \geq 8$ . Therefore,  $\lambda'_{1,2}(Ne_3) = 8$ .

For  $Ne_4$ , since  $P_2 \square P_4$  is a subgraph of  $Ne_4$ , then  $\lambda'_{1,2}(Ne_4) \geq 7$  by Theorem 1. On the other hand, Fig. 7 gives a 7- $L(1, 2)$ -edge-labeling of  $Ne_4$ . Thus,  $\lambda'_{1,2}(Ne_4) = 7$  and Theorem 2 holds.

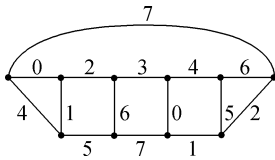


Fig. 7 A 7- $L(1, 2)$ -edge-labeling of  $Ne_4$

In the following, the bounds for  $\lambda'_{1,2}$ -number of  $Ne_h$  with  $h \geq 5$  are discussed. In order to obtain an upper bound, it suffices to provide  $L(1, 2)$ -edge-labelings for necklaces. First, we give the following lemma:

**Lemma 1** Suppose that  $G_1$  is a graph with  $\lambda'_{1,2}(G_1) \leq 8$ . Let  $u$  and  $v$  be two adjacent vertices of  $G_1$  with  $d(u) = d(v) = 3$  and  $N(u) \cap N(v) = \emptyset$ . Let  $x, y$  and  $z, w$  be the other two neighbors of  $u$  and  $v$ , respectively. Suppose that  $f$  is an 8- $L(1, 2)$ -edge labeling of  $G_1$ , in which  $f(xu) = 0$ ,  $f(uy) = 5$ ,  $f(uv) = 4$ ,  $f(zv) = 2$  and  $f(vw) = 3$ . Let  $G_2$  be a graph obtained from  $G_1$  by replacing  $uv$  with  $P_2 \square P_7$  (see Fig. 8). Then  $f$  can be extended to a 8- $L(1, 2)$ -edge labeling of  $G_2$ .

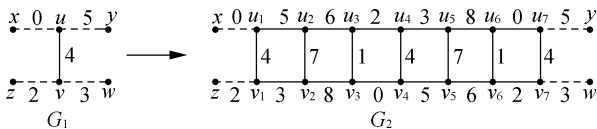


Fig. 8 The extension of 8- $L(1, 2)$ -edge-labeling of  $G_1$

**Proof** Fig. 8 shows the extension of  $f$  to a 8- $L(1, 2)$ -edge labeling of  $G_2$ .

**Theorem 3** For  $h \geq 5$ ,  $7 \leq \lambda'_{1,2}(Ne_h) \leq 8$ .

**Proof** Since  $P_2 \square P_h$  is a subgraph of  $Ne_h$  and  $h \geq 5$ , we have  $\lambda'_{1,2}(Ne_h) \geq 7$  by Theorem 1. Thus, the lower

bound holds. To prove the upper bound, it suffices to give an 8- $L(1, 2)$ -edge-labeling of  $Ne_h$ . It is shown in Fig. 9 that  $\lambda'_{1,2}(Ne_h) \leq 8$  for  $h \in \{4, 5, 6, 7, 8, 9\}$ . Applying Lemma 1 repeatedly we can obtain  $\lambda'_{1,2}(Ne_h) \leq 8$  for all  $h \geq 5$ . Thus, Theorem 3 holds.

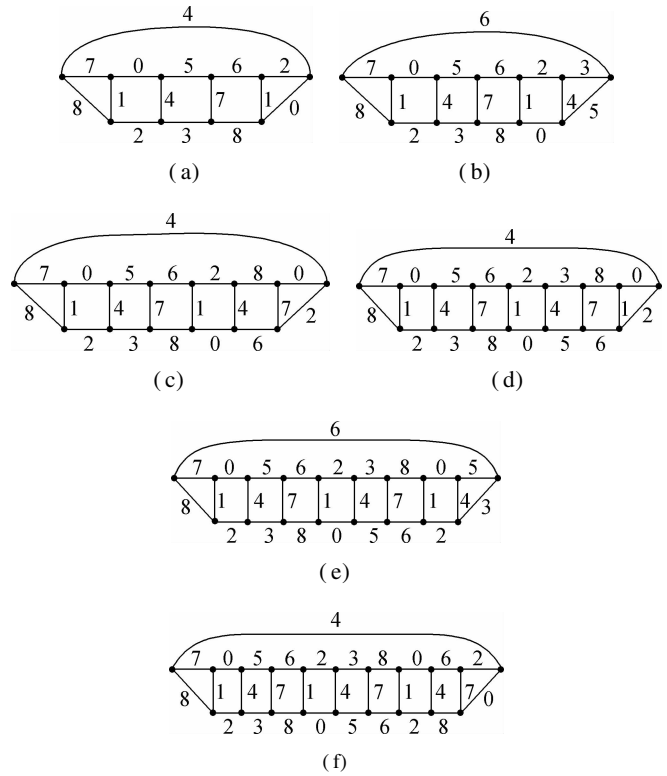


Fig. 9 8- $L(1, 2)$ -edge-labelings of  $Ne_k$  ( $k \in [4, 9]$ ). (a)  $Ne_4$ ; (b)  $Ne_5$ ; (c)  $Ne_6$ ; (d)  $Ne_7$ ; (e)  $Ne_8$ ; (f)  $Ne_9$

We conclude this section by raising the following theorem, in which we obtain some type of necklaces whose  $\lambda'_{1,2}$ -numbers are the lower bound in Theorem 4.

**Theorem 4** If  $h = 16k + 4$  for some  $k \in \mathbb{Z}_+$ , then  $\lambda'_{1,2}(Ne_h) = 7$ .

**Proof** By Theorem 3, it suffices to give a 7- $L(1, 2)$ -edge-labeling of  $Ne_h$  with  $h = 16k + 4$ . When  $k = 0$ , we have  $\lambda'_{1,2}(Ne_4) = 7$  by Theorem 2. So, we consider  $k \geq 1$ . Fig. 7 gives a 7- $L(1, 2)$ -edge-labeling of  $Ne_4$ . Similar to Lemma 1, we extend this labeling function of  $Ne_4$  to a 7- $L(1, 2)$ -edge-labeling of  $Ne_h$  with  $h = 16k + 4$  by repeatedly replacing the edge  $(2, 2')$  of  $Ne_4$  with  $P_2 \square P_{17}$  (as shown in Fig. 10). Thus, Theorem 4 holds.

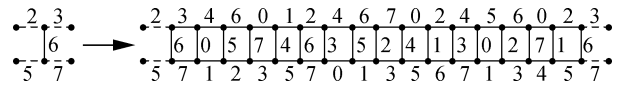


Fig. 10 The extension of 7- $L(1, 2)$ -edge-labeling of  $Ne_4$  given in Fig. 7

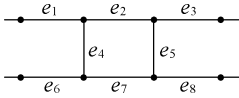
### 3 $\lambda'_{1,2}$ -number of $Ne_{10}$

In this section,  $\lambda'_{1,2}(Ne_{10}) = 8$  is proved, which shows that the upper bound in Theorem 3 is sharp.

For some  $i \in [2, h-3]$ , let  $H_i$  be the subgraph of  $Ne_h$  induced by the vertices in  $\{i+k, (i+k)'\mid k \in [0, 3]\}$ , and let  $H'_i = H_i \setminus \{(i, i'), (i+3, (i+3)')\}$ . Then  $|E(H'_i)| = 8$ . Let  $h \geq 5$ . Suppose that  $\lambda'_{1,2}(Ne_h) = 7$  and  $f$  is a 7- $L(1, 2)$ -edge-labeling of  $Ne_h$ . We first prove the following two properties of  $f$ .

**Property 1** The labels assigned to the edges of  $H'_i$  must be distinct.

**Proof** For convenience, we denote the edges of  $H'_i$  with  $e_1, e_2, \dots, e_8$  as indicated in Fig. 11. Suppose that there are two edges, say  $e_i$  and  $e_j$  that received the same label. Then the set  $\{i, j\}$  can only be  $\{1, 8\}$  or  $\{3, 6\}$ .



**Fig. 11** Locations of the edges in  $H'_i$

**Claim 1**  $f(e_1) = f(e_8)$  if and only if  $f(e_3) = f(e_6)$ .

**Proof** Suppose that  $f(e_1) = f(e_8) = k$ . If  $f(e_3) \neq f(e_6)$ , then the edges in  $E(H'_i) \setminus \{e_1, e_8\}$  must obtain distinct labels. Due to the distance conditions, the labels  $k \pm 1$  cannot be assigned to the remaining six edges of  $H'_i$ . If  $k \notin \{0, 7\}$ , then there are only five labels in  $[0, 7]$  assigned to the remaining six edges, which must obtain distinct labels. It is a contradiction. So, by the symmetry of labels, it can be assumed that  $f(e_1) = f(e_8) = 0$ . Then, consider the cases of  $f(e_2) = 2, f(e_4) = 2$  and  $f(e_6) = 2$ , respectively. Each case should be divided into several subcases. For each subcase, it is not difficult to find contradiction (we omit the proofs). Thus,  $f(e_3) = f(e_6)$ . By the symmetry of  $H'_i$ , the claim holds.

**Claim 2** If  $f(e_1) = f(e_8) = a$  and  $f(e_3) = f(e_6) = b$ , then  $\{a, b\} \cap \{0, 7\} \neq \emptyset$ .

**Proof** Suppose that  $\{a, b\} \cap \{0, 7\} = \emptyset$ . Then the labels in  $A = \{a-1, a, a+1, b-1, b, b+1\}$  cannot be used for the remaining four edges of  $E(H'_i)$ . Due to the distance conditions,  $a$  and  $b$  cannot be the consecutive integers. So  $|A \cap [0, 7]| \leq 3$ . It is a contradiction since the labels assigned to the remaining four edges must be distinct. Thus, the claim holds.

Without loss of generality, we can assume that  $a = 0$ . Then by Claim 1, Claim 2 and the symmetry of labels, we only need to consider the following three cases:

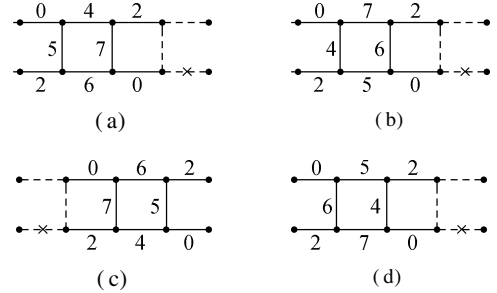
**Case 1**  $a = 0$  and  $b \in [3, 6]$ .

**Proof** In this case, the labels in  $F = \{0, 1, b-1, b, b+1\}$  are forbidden for the edges  $e_2, e_4, e_5$  and  $e_7$ . Since  $b \in [3, 6]$ ,  $|F \cap [0, 7]| = 5$ . Then there are only three labels for edges  $e_2, e_4, e_5$  and  $e_7$ . It is a contradiction.

**Case 2**  $a = 0$  and  $b = 2$ .

**Proof** In this case, it is clear that  $\{f(e_2), f(e_4), f(e_5), f(e_7)\} = \{4, 5, 6, 7\}$ . Note that  $e_2, e_3, e_7$  and  $e_4$  form a 4-cycle. Due to the distance condition, the four labels 4, 5, 6 and 7 must be assigned to the above four edges

in clockwise or counterclockwise order along the cycle. By symmetry, we consider the four cases of  $f(e_2) = 4, f(e_4) = 4, f(e_7) = 4$  and  $f(e_5) = 4$ , respectively. For each case, we can obtain a contradiction as indicated in Fig. 12 (In the figures, the edges marked with a ford can not be assigned by the labels in  $[0, 7]$ ).



**Fig. 12** The contradictions occurred in case 2. (a)  $f(e_2) = 4$ ; (b)  $f(e_4) = 4$ ; (c)  $f(e_7) = 4$ ; (d)  $f(e_5) = 4$

**Case 3**  $a = 0$  and  $b = 7$ .

**Proof** The proof of this case is similar to the argument of case 2.

By the above arguments, Property 1 holds.

**Property 2** For  $3 \leq i \leq h-3$ , the labels assigned to the edges of  $(i, i')$  and  $(i+2, (i+2)')$  must be distinct.

**Proof** Suppose that there is some  $i \in [3, h-3]$  with  $f((i, i')) = f((i+2, (i+2)')) = k$ . Similar to the graph  $P_2 \square P_h$ , the horizontal edge of  $Ne_h$ ,  $(i, i+1)$  (or  $(i', (i+1)')$ ), is denoted by  $h_i$  (or  $h_{i'}$ ). Due to the distance condition, labels  $k \pm 1$  cannot be assigned to the edges in  $\{h_i, h_{i+1}, h_{i'}, h_{(i+1)'}\} \cup \{(i+1, (i+1)')\}$ . By Property 1, the labels assigned to the edges in subgraph  $H'_{i-1}$  are distinct, and so in subgraph  $H'_i$ . It follows that  $\{f(h_{i-1}), f(h_{(i-1)'})\} = \{f(h_{i+2}), f(h_{(i+2)'})\}$ . If  $f(h_{i-1}) = f(h_{i+2}) = a$ , then  $a \pm 1$  cannot be used on the edges in  $H'_{i-1}$  and  $H'_i$ , contradicting to Property 1. So, we have  $f(h_{i-1}) = f(h_{(i+2)'})$  and  $f(h_{(i-1)'}) = f(h_{i+2})$ .

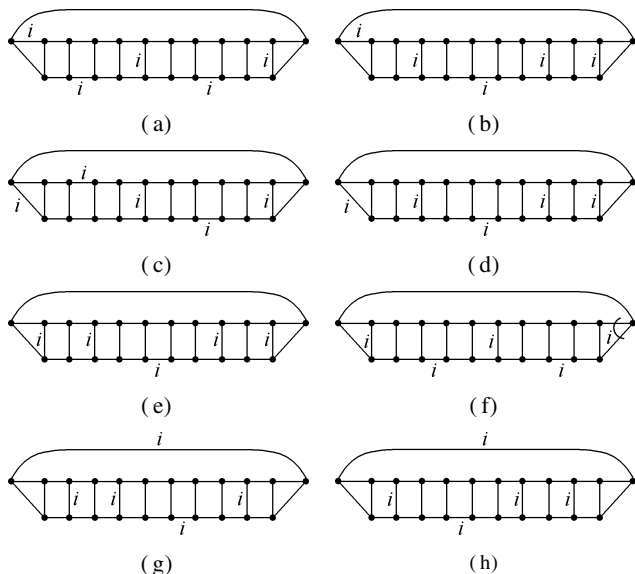
By the symmetry of labels, we can assume that  $k = 0, 1, 2$  and  $3$ . When  $k = 0$ , without loss of generality, let  $f(h_{i-1}) = f(h_{(i+2)'}) = 1$ . Then we only need to consider the cases of  $f(h_{(i-1)'}) = f(h_{i+2}) = 4, 5, 6$  and  $7$ . For each case, we can obtain a similar contradiction as the proofs in case 2 of Property 1. The situation when  $k = 1, 2$  and  $3$  are similar. Thus, Property 2 holds.

By the above two properties, we obtain the following result.

**Theorem 5**  $\lambda'_{1,2}(Ne_{10}) = 8$ .

**Proof** By Theorem 3, it suffices to prove  $\lambda'_{1,2}(Ne_{10}) \geq 8$ . Suppose that  $\lambda'_{1,2}(Ne_{10}) \leq 7$  and  $f$  is a 7- $L(1, 2)$ -edge-labeling of  $Ne_{10}$ . Let  $L_i = \{e \in E(Ne_{10}) \mid f(e) = i\}$  and  $l_i$  be the cardinality of  $L_i$ . Then it is easy to check that  $l_i \leq 5$  for all  $i \in [0, 7]$  by Properties 1 and 2. Fig. 13 gives all the possibilities of the label  $i$  used five times, which are not symmetrical. Since  $|E(Ne_{10})| = 33$ , there is at least one label used five times. It is straightforward.

ward to check that if label  $i$  is used five times, then  $i \pm 1$  can be used at most three times. Therefore, the edges of  $Ne_{10}$  cannot be assigned to only eight labels. Hence, a contradiction is obtained. Thus, Theorem 5 holds.



**Fig. 13** The possibilities of label  $i$  used five times. (a)  $f((0, 1)) = f((2', 3')) = f((5, 5')) = f((7', 8')) = f((10, 10')) = i$ ; (b)  $f((0, 1)) = f((3, 3')) = f((5', 6')) = f((8, 8')) = f((10, 10')) = i$ ; (c)  $f((0, 1')) = f((2, 3)) = f((5, 5')) = f((7', 8')) = f((10, 10')) = i$ ; (d)  $f((0, 1')) = f((3, 3')) = f((5', 6')) = f((8, 8')) = f((10, 10')) = i$ ; (e)  $f((1, 1')) = f((3, 3')) = f((5', 6')) = f((8, 8')) = f((10, 10')) = i$ ; (f)  $f((1, 1')) = f((3', 4')) = f((6, 6')) = f((8', 9')) = f((10, 11))$  (or  $f(10', 11)) = i$ ; (g)  $f((2, 2')) = f((4, 4')) = f((6', 7')) = f((9, 9')) = f((11, 0)) = i$ ; (h)  $f((2, 2')) = f((4', 5')) = f((7, 7')) = f((9, 9')) = f((11, 0)) = i$

By Properties 1 and 2, this paper is completed by raising the following conjecture:

**Conjecture 1** For necklaces  $Ne_h$ ,  $\lambda'_{1,2}(Ne_h) = 8$  except for  $h = 16k + 4$  ( $k \in \mathbb{Z}_+$ ).

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## 项链的 $L(1, 2)$ -边标号

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**摘要:** 给定一个图  $G$  和 2 个正整数  $j$  和  $k$ , 图  $G$  的一个  $m$ - $L(j, k)$ -边标号是从图的边集到非负整数集合  $\{0, 1, \dots, m\}$  的一个映射, 该映射满足相邻的边所对应的整数相差至少为  $j$ , 距离为 2 的边所对应的整数相差至少为  $k$ . 在图  $G$  的所有  $m$ - $L(j, k)$ -边标号中, 最小的整数  $m$  称为图  $G$  的  $L(j, k)$ -边标号数, 记为  $\lambda'_{j,k}(G)$ . 项链是一类特殊的 Halin 图, 研究了项链的  $L(1, 2)$ -边标号, 给出了项链的  $L(1, 2)$ -边标号数的上界和下界, 并且此上界和下界都是可达的.

**关键词:** 频道分配;  $L(j, k)$ -边标号; 笛卡尔乘积; Halin 图; 项链  
中图分类号: O157.5