Algorithm for reconstructing compressed sensing color imaging using the quaternion total variation

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Abstract: A new method for reconstructing the compressed sensing color image by solving an optimization problem based on total variation in the quaternion field is proposed, which can effectively improve the reconstructing ability of the color image. First, the color image is converted from RGB (red, green, blue) space to CMYK (cyan, magenta, yellow, black) space, which is assigned to a quaternion matrix. Meanwhile, the quaternion matrix is converted into the information of the phase and amplitude by the Euler form of the quaternion. Secondly, the phase and amplitude of the quaternion matrix are used as the smoothness constraints for the compressed sensing (CS) problem to make the reconstructing results more accurate. Finally, an iterative method based on gradient is used to solve the CS problem. Experimental results show that by considering the information of the phase and amplitude, the proposed method can achieve better performance than the existing method that treats the three components of the color image as independent parts.

Key words: total variation; compressed sensing; quaternion; sparse reconstruction; color image restoration

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A well-known Nyquist/Shannon sampling theorem states that the sampling rate must be at least twice the maximum frequency of the signal, which is the golden rule used in signal and image processing. In many applications such as digital images and video cameras, the Nyquist rate is so high that we obtain enormous sample results. Signal compression is one of the ways to solve this problem. However, it is actually a waste of resources, since much insignificant or redundant information is discarded during the compression process.

Recently, compressed sensing (CS), a new method to capture and represent compressible signals at a rate significantly below the Nyquist rate, provides us with a powerful tool to improve imaging efficiency. It has been explored in depth and extensively reported on in Refs. [1 − 4]. Besides, the quaternion or hyper-complex algebraic theory and its applications in signal and image processing, especially in the field of color image processing[5], have also attracted much attention. The compressed sensing can be used in color images, but the three components of the color image are generally treated as three independent parts, leading to the correlation among the three channels being ignored[6-7]. To correct this, Yu et al. [8] took the sparsity information of the phase into consideration and proposed a new strategy for the CS problem. Moreover, the reconstruction results perform better when the total variation (TV) strategy is used in the CS problem[9-11].

In this paper, we present a new algorithm to reconstruct the compressed sensing color images on the basis of the previous work[8-9]. First, we convert the color image from RGB space to CMYK space so that the four components can be assigned to a quaternion matrix. Then, we propose a new model based on the model proposed in ref. [9] by adding the amplitude and phase information of the quaternion matrix as the smoothness constraints. Finally, an iterative method based on gradient is used to solve the CS problem.

1 TV Model of CS Problem

Let $\boldsymbol{f} \in \mathbf{R}^N$ be the signal that is sampled. It has a sparse representation over a transform $\boldsymbol{\Psi} \in \mathbf{R}^{N\times N}$. This means that we have a sparse signal $\boldsymbol{x}$ and $\boldsymbol{x} = \boldsymbol{\Psi f}$. Let $\boldsymbol{\Phi} \in \mathbf{R}^{M\times N}$ ($M \ll N$) be the measurement matrix and we obtain the measured signal $\boldsymbol{Y} = \boldsymbol{\Phi x} = \boldsymbol{\Phi\Psi f}$, which is a linear combination of the elements of $\boldsymbol{f}$. The basic model of compressed sensing is given as follows:

$$\hat{f} = \arg\min \| \boldsymbol{Y} - \boldsymbol{\Phi\Psi f} \|_2^2 + \lambda \| \boldsymbol{\Psi f} \|_1 \tag{1}$$

The model proposed in Eq. (1) is the basic model used in the CS problem. While in the two-dimensional image reconstruction, the TV strategy is used so as to make full use of the structure information of the image gradient. The TV model of signal $\boldsymbol{f}$ is the summation for the magnitude of the gradient of each pixel given as follows:

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$$\mathrm{TV}(\boldsymbol{f}) = \sum_{j,k} \sqrt{(D^{\mathrm{v}}_{j,k}\boldsymbol{f})^2 + (D^{\mathrm{h}}_{j,k}\boldsymbol{f})^2} = \sum_{j,k} |\nabla_{j,k}\boldsymbol{f}| \quad (2)$$

where $D^{\mathrm{h}}\boldsymbol{f}$ and $D^{\mathrm{v}}\boldsymbol{f}$ are the derivatives of $\boldsymbol{f}$ along the horizontal and vertical directions, respectively, and they are defined as

$$D^{\mathrm{h}}_{j,k}\boldsymbol{f} = \begin{cases} f_{j,k} - f_{j,k+1} & 1 \leqslant k < N \\ 0 & k = N \end{cases}$$
$$D^{\mathrm{v}}_{j,k}\boldsymbol{f} = \begin{cases} f_{j,k} - f_{j+1,k} & 1 \leqslant j < N \\ 0 & j = N \end{cases} \quad (3)$$

The TV model of the CS problem for the two-dimensional image[9] is given as

$$\hat{f} = \arg\min \| \boldsymbol{Y} - \boldsymbol{\Phi\Psi f} \|_2^2 + \lambda \mathrm{TV}(\boldsymbol{f}) \quad (4)$$

The first term is data fidelity, which means the solution we obtain is close to the observation. The second term is the smoothness constraint about the structure information of the image gradient.

2 The Proposed Method

Traditionally, the three components of the color image are treated as three independent parts [7], which do not take the correlation of the three channels into consideration. Also, the hypercomplex model of the color image in the RGB model [5] is given as

$$\boldsymbol{F}(x,y) = \boldsymbol{f}_{\mathrm{R}}(x,y)\mathrm{i} + \boldsymbol{f}_{\mathrm{G}}(x,y)\mathrm{j} + \boldsymbol{f}_{\mathrm{B}}(x,y)\mathrm{k} \quad (5)$$

where $\boldsymbol{f}_{\mathrm{R}}(x,y)$, $\boldsymbol{f}_{\mathrm{G}}(x,y)$, $\boldsymbol{f}_{\mathrm{B}}(x,y)$ represents the R, G, B components of the image, respectively; and i, j, k are the three imaginary units of a quaternion.

Let $\boldsymbol{q} \in \mathbf{Q}^N$ be a quaternion signal, which means that $\boldsymbol{q} = \boldsymbol{A} + \boldsymbol{B}\mathrm{i} + \boldsymbol{C}\mathrm{j} + \boldsymbol{D}\mathrm{k}$, $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} \in \mathbf{R}^N$. In order to obtain the phase information, we can use the Euler formula of quaternion:

$$\theta_{ij} = \arccos \frac{a_{ij}}{\sqrt{a_{ij}^2 + b_{ij}^2 + c_{ij}^2 + d_{ij}^2}} \quad (6)$$

The reconstruction results perform better because we have taken the phase information into consideration[8]. But the Euler expression of Eq. (5) shows that the phase of the RGB model is a constant, $\pi/2$, which means that we cannot use the phase as a constrain condition of the CS problem. We can convert the color image of size $N \times N$ from RGB space to CMYK space by

$$\left.\begin{aligned} \boldsymbol{K} &= \min(1 - \boldsymbol{f}_{\mathrm{R}}(x,y), 1 - \boldsymbol{f}_{\mathrm{G}}(x,y), 1 - \boldsymbol{f}_{\mathrm{B}}(x,y)) \\ \boldsymbol{C} &= (1 - \boldsymbol{f}_{\mathrm{R}}(x,y) - \boldsymbol{K})/(1 - \boldsymbol{K}) \\ \boldsymbol{M} &= (1 - \boldsymbol{f}_{\mathrm{G}}(x,y) - \boldsymbol{K})/(1 - \boldsymbol{K}) \\ \boldsymbol{Y} &= (1 - \boldsymbol{f}_{\mathrm{B}}(x,y) - \boldsymbol{K})/(1 - \boldsymbol{K}) \end{aligned}\right\} \quad (7)$$

Then, the converted result is assigned to a $N \times N$ quaternion matrix:

$$\boldsymbol{q} = \boldsymbol{C} + \boldsymbol{M}\mathrm{i} + \boldsymbol{Y}\mathrm{j} + \boldsymbol{K}\mathrm{k} \quad \boldsymbol{C}, \boldsymbol{M}, \boldsymbol{Y}, \boldsymbol{K} \in \mathbf{R}^{N \times N} \quad (8)$$

In this way, we can obtain the phase information of the color image and use it as the smoothness constraint for the CS problem. Finally, an improved quaternion TV model of the CS problem is proposed by adding the phase and amplitude of the color image as the smoothness constraints. The improved model is given as

$$\hat{q} = \arg\min J(\boldsymbol{q})$$
$$\text{s.t. } J(\boldsymbol{q}) = \| \boldsymbol{Y} - \boldsymbol{\Phi q} \|_2^2 + \lambda_1 \mathrm{TV}(|\boldsymbol{q}|) + \lambda_2 \mathrm{TV}(\boldsymbol{\theta}) \quad (9)$$

where $|\boldsymbol{q}|$ represents the amplitude and $\boldsymbol{\theta}$ represents the phase.

In order to solve Eq. (9), we first take the gradient of $J(\boldsymbol{q})$ with respect to $\boldsymbol{q}$ and arrange the result in a compact form as follows:

$$\nabla_{j,k} J(\boldsymbol{q}) = 2\boldsymbol{\Phi}^{\mathrm{H}}(\boldsymbol{\Phi q}_{j,k} - \boldsymbol{Y}) + \lambda_1 \boldsymbol{\Lambda}_1 \frac{\boldsymbol{q}_{j,k}}{\sqrt{\boldsymbol{q}_{j,k}^* \boldsymbol{q}_{j,k} + \varepsilon}} + \lambda_2 \boldsymbol{\Lambda}_2 \frac{2}{\boldsymbol{\eta} \bar{\boldsymbol{q}}_{j,k}} \quad (10)$$

where ε is a small constant and $\bar{\boldsymbol{q}}$ is the conjugate of $\boldsymbol{q}$.

$$\left.\begin{aligned} \boldsymbol{\eta} &= \frac{\boldsymbol{M}\mathrm{i} + \boldsymbol{Y}\mathrm{j} + \boldsymbol{K}\mathrm{k}}{\sqrt{\boldsymbol{M}^2 + \boldsymbol{Y}^2 + \boldsymbol{K}^2}} \\ \boldsymbol{\Lambda}_1 &= \frac{D^{\mathrm{v}}_{j,k}\boldsymbol{q}}{|\nabla_{j,k}\boldsymbol{q}|} + \frac{D^{\mathrm{h}}_{j,k}\boldsymbol{q}}{|\nabla_{j,k}\boldsymbol{q}|} - \frac{D^{\mathrm{v}}_{j-1,k}\boldsymbol{q}}{|\nabla_{j-1,k}\boldsymbol{q}|} - \frac{D^{\mathrm{h}}_{j,k-1}\boldsymbol{q}}{|\nabla_{j,k-1}\boldsymbol{q}|} \\ \boldsymbol{\Lambda}_2 &= \frac{D^{\mathrm{v}}_{j,k}\boldsymbol{\theta}}{|\nabla_{j,k}\boldsymbol{\theta}|} + \frac{D^{\mathrm{h}}_{j,k}\boldsymbol{\theta}}{|\nabla_{j,k}\boldsymbol{\theta}|} - \frac{D^{\mathrm{v}}_{j-1,k}\boldsymbol{\theta}}{|\nabla_{j-1,k}\boldsymbol{\theta}|} - \frac{D^{\mathrm{h}}_{j,k-1}\boldsymbol{\theta}}{|\nabla_{j,k-1}\boldsymbol{\theta}|} \end{aligned}\right\} \quad (11)$$

The following quasi-Newton iteration based on gradient can be used to solve the CS problem in Eq. (9).

$$\boldsymbol{q}_{j,k}^{i+1} = \boldsymbol{q}_{j,k}^{i} - \gamma \boldsymbol{u}_k^i \nabla_{j,k} J(\boldsymbol{q}) \quad (12)$$

where γ is the step size and $\boldsymbol{u}$ is given as

$$\boldsymbol{u}_k^i = \left(2\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi} + \lambda_1 \boldsymbol{\Lambda}_1 \frac{1}{\sqrt{\boldsymbol{q}_{j,k}^* \boldsymbol{q}_{j,k} + \varepsilon}} \right)^{-1} \quad (13)$$

When $\dfrac{\| \boldsymbol{q}_{j,k}^{i+1} - \boldsymbol{q}_{j,k}^{i} \|^2}{\| \boldsymbol{q}_{j,k}^{i} \|^2} < \varepsilon$, the iteration stops.

3 Experimental Results

In this section, we compare the method[7] whereby the three components of the color image are treated as three independent parts with the proposed method on several experiments. The experiments are conducted on the 512 $\times$ 512 Lena image, Baboon image and Peppers image, respectively.

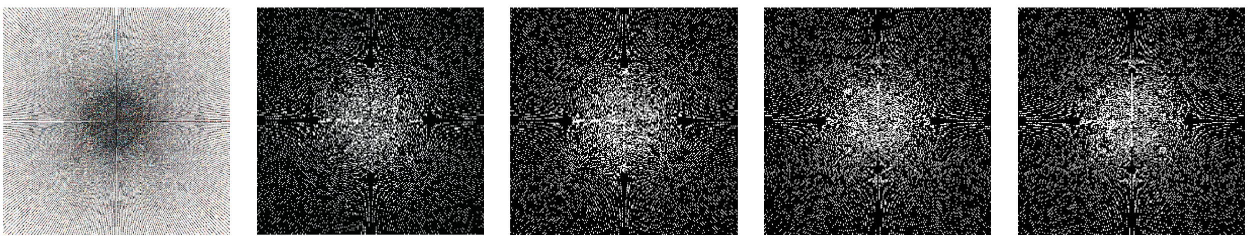
First, we convert the three color images from RGB space to CMYK space. The converted result of Lena is shown in Fig. 1. Then, we assign the four parts of CMYK to a quaternion matrix $\boldsymbol{Q} = \boldsymbol{C} + \boldsymbol{M}\mathrm{i} + \boldsymbol{Y}\mathrm{j} + \boldsymbol{K}\mathrm{k}$, $\boldsymbol{Q} \in \mathbf{Q}^{512 \times 512}$. In order to obtain the sparse signal of the quaternion matrix, we translate it into the quaternion Fourier domain[5] by using qfft2 in qtfm (quaternion toolbox for Matlab) and use ifftshift to rectify the asymmetric caused by the qfft2 transform. The sparse signal of the quaternion matrix is given in Fig. 2.

Next, we use the ray sampling model in the Fourier domain as the sampling matrix $\boldsymbol{\Phi}$ for the sparse quaternion signal to obtain the measurement matrix.

The values of λ_1 and λ_2 are fixed at 0.02 and 0.01 in the model of Eq. (9). Using the sampling rate of 36.5%, the experimental results of the proposed method are shown in Figs. 3, 4 and 5, respectively.

(a) (b) (c) (d) (e)

Fig. 1 Lena in CMYK space. (a) Original image; (b) C part; (c) M part; (d) Y part; (e) K part

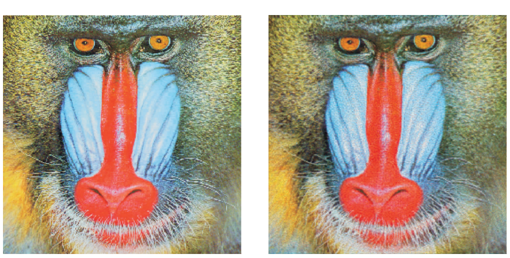


(a) (b) (c) (d) (e)

Fig. 2 Lena in sparse domain. (a) Sparse image; (b) C part; (c) M part; (d) Y part; (e) K part

(a) (b)

Fig. 3 Lena image. (a) Original image; (b) Recovery image



(a) (b)

Fig. 4 Baboon image. (a) Original image; (b) Recovery image

(a) (b)

Fig. 5 Peppers image. (a) Original image; (b) Recovery image

From Tab. 1, we can see that the proposed method has a better performance than the method that treats the three components of the color image as independent parts[7].

Tab. 1 The PSNR values of different methods using the sampling rate of 36.5%

Images	Method in Ref. [7]			The proposed method		
	Lena	Baboon	Peppers	Lena	Baboon	Peppers
PSNR/dB	30.20	27.68	30.79	31.52	28.78	31.58
Error rate/%	2.38	9.04	1.07	0.81	5.25	0.82

4 Conclusion

In this paper, we proposed a new algorithm for the compressed sensing color imaging in an efficient manner. We translated the color images to the quaternion domain and gave a new TV model combined with the amplitude and phase information as the smoothness constraints for the CS problem. The experimental results demonstrate that the proposed method achieves better performance compared to the method proposed in Ref. [7] in both the PSNR and the error rate. Thus, the proposed method can be used for improving the reconstruction results of color images.

In the future, we will consider further improvements in the phase reconstruction. Moreover, the existing convex optimization algorithms such as the interior point method can also be applied to our strategy for the compressed color image CS problem.

References

[1] Candes E J, Tao T. Decoding by linear programming

[J]. *IEEE Transactions on Information Theory*, 2005, **51**(12): 4203−4215.
[2] Donoho D L. Compressed sensing [J]. *IEEE Transactions on Information Theory*, 2006, **52**(4): 1289−1306.
[3] Candès E J, Romberg J, Tao T. Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information [J]. *IEEE Transactions on Information Theory*, 2006, **52**(2): 489−509.
[4] Baraniuk R. Compressive sensing [J]. *IEEE Signal Processing Magazine*, 2007, **24**(4): 118−121.
[5] Ell T A, Sangwine S J. Hypercomplex Fourier transforms of color images [J]. *IEEE Transactions on Image Processing*, 2007, **16**(1): 22−35.
[6] Mairal J, Elad M, Sapiro G. Sparse representation for color image restoration [J]. *IEEE Transactions on Image Processing*, 2008, **17**(1): 53−69.
[7] Majumdar A, Ward R K. Compressed sensing of color images [J]. *Signal Processing*, 2010, **90**(12): 3122−3127.
[8] Yu S, Khwaja A S, Ma J. Compressed sensing of complex-valued data [J]. *Signal Processing*, 2012, **92**(2): 357−362.
[9] Dai Q, Sha W. The physics of compressive sensing and the gradient-based recovery algorithms [EB/OL]. (2009-06-08) [2014-04-14]. http://arxiv.org/abs/0906.1487.
[10] Poon C. On the role of total variation in compressed sensing-structure dependence [EB/OL]. (2014-01-22) [2014-04-14]. http://arxiv.org/pdf/1407.5337.
[11] Oliveri G, Anselmi N, Massa A. Compressive sensing imaging of non-sparse 2D scatterers by a total-variation approach within the born approximation[J]. *IEEE Transactions on Antennas and Propagation*, 2014, **62**(10): 5157−5170.

基于四元数域总变差方法的压缩感知彩色图像重建算法

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摘要:提出了一种基于四元数域总变差方法的彩色图像压缩感知重建算法,该算法可有效提高彩色图像的重建能力.首先,将彩色图像从RGB空间转换到CMYK空间,并将CMYK空间的各个分量赋值给一个四元数矩阵.同时通过四元数的欧拉形式,将四元数矩阵转换为幅度和相位的信息.然后,为了完善重建的结果,将四元数矩阵的幅度和相位作为压缩感知优化方程新的平滑约束项.最后,用基于梯度的迭代算法来求解压缩感知优化方程.实验结果表明,所提出的算法考虑了幅度和相位的信息,比现有的将彩色图像的3个分量当作独立分量的算法效果好.

关键词:总变差;压缩感知;四元数;稀疏重建;彩色图像恢复
中图分类号:TP391