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KAM tori for generalized Boussinesq equation

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Abstract: One-dimensional generalized Boussinesq equation $u_{tt} - u_{xx} + (f(u) + u_{xx})_{xx} = 0$ with periodic boundary condition is considered, where $f(u) = u^3$. First, the above equation is written as a Hamiltonian system, and then by choosing the eigenfunctions of the linear operator as bases, the Hamiltonian system in the coordinates is expressed. Because of the intricate resonance between the tangential frequencies and normal frequencies, some quasi-periodic solutions with special structures are considered. Secondly, the regularity of the Hamiltonian vector field is verified and then the fourth-order terms are normalized. By the Birkhoff normal form, the non-degeneracy and non-resonance conditions are obtained. Applying the infinite dimensional Kolmogorov-Arnold-Moser (KAM) theorem, the existence of finite dimensional invariant tori for the equivalent Hamiltonian system is proved. Hence many small-amplitude quasi-periodic solutions for the above equation are obtained.

Key words: generalized Boussinesq equation; quasi-periodic solution; Hamiltonian system; invariant tori

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In 1870s, Boussinesq[1] derived some model equations for the propagation of small amplitude long waves of the surface of water and first mathematically explained the existence of Scott Russell's solitary wave phenomenon. One of the most well-known Boussinesq's equations is

$$u_{tt} - u_{xx} - \frac{3}{2}\varepsilon(u^2)_{xx} - \frac{\varepsilon}{3}u_{xxxx} = 0 \qquad t>0, x \in \mathbf{R} \tag{1}$$

As the initial value problem is linearly ill posed, we are interested in the following equation connected with Boussinesq's work for water waves, which has the advantage of being linearly well posed and is just a variant of Eq. (1):

$$u_{tt} - u_{xx} + u_{xxxx} + f(u)_{xx} = 0 \qquad t>0, x \in \mathbf{R} \tag{2}$$

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when $f(u) = u^2$. Eq. (2) is derived from a model of a nonlinear string.

There are many significant results on Eq. (2). Bona et al. [2] proved the global existence of smooth solutions and the stability of solitary waves. Liu et al.[3-4] obtained instable solitary waves, instable and blow-up solutions, and the strong instability of solitary-wave solutions. Through the variational iteration method, Yusufoǧlu[5] showed blow-up solutions.

In this paper, we consider Eq. (2) with nonlinearity $f(u) = u^3$ under the periodic boundary condition:

$$u(t, x+2\pi) = u(t, x) \tag{3}$$

Both the KAM method and the CWB method can be applied to prove the existence of quasi-periodic solutions for Hamiltonian partial differential equations. Compared to the CWB method, the KAM method can give more information on the stability and dynamics. For Hamiltonian PDEs under the Dirichlet boundary condition, the operator corresponding to the linear part of equation only possesses simple eigenvalues, which is a simple case for the KAM theorem. However, under the periodic boundary condition, the presence of multiple normal frequencies brings great difficulty to handle more complicated resonance between the frequencies. In 2000, Chierchia and You[6] developed the KAM skill for the case of multiple normal frequencies. It is required that the perturbed vector field has good regularity which ensures some decay property for the shift of normal frequencies. If the vector field has no such regularity, the compact form[7] or Töplitz-Lipschitz property[8] of the perturated Hamiltonian can be used to overcome this difficulty. But the perturated Hamiltonian system does not have these properties; thus, the previous methods are invalid for our problem. However, by observation, the vector field in our problem has some special property, which motivates us to seek some special solutions.

1 Hamiltonian Structure and Main Result

Define phase space $P = H^1([0, 2\pi]) \times L^2([0, 2\pi])$. Let $\boldsymbol{w} = (u, v) \in P$ with $u_t = v_x$. Then Eq. (2) is equivalent to the following Hamiltonian system:

$$\left.\begin{aligned} u_t &= v_x \\ v_t &= \partial_x(u - u_{xx} - f(u)) \end{aligned}\right\} \tag{4}$$

with the Hamiltonian function

$$H(\boldsymbol{w}) = \int_0^{2\pi} \frac{u^2}{2} + \frac{v^2}{2} + \frac{u_x^2}{2} + g(u)\,\mathrm{d}x$$

where $g(u) = -\int_0^u f(s)\,\mathrm{d}s$. Let

$$\boldsymbol{J} = \begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix}$$

be the weak derivative operator with respect to the L^2 inner product on the space $L^2([0,2\pi]) \times L^2([0,2\pi])$ in the following sense:

$$(\boldsymbol{J}\boldsymbol{w}, z) = -\int_0^{2\pi} \langle \boldsymbol{w}, \boldsymbol{J}z\rangle \mathrm{d}x = -\int_0^{2\pi} (v(x)\varphi'(x) + u(x)\psi'(x))\,\mathrm{d}x$$

where $\boldsymbol{w}(x) = (u(x), v(x)) \in L^2([0,2\pi]) \times L^2([0,2\pi])$, $z(x) = (\varphi(x), \psi(x)) \in C_0^\infty(0,2\pi) \times C_0^\infty(0,2\pi)$.

Eq. (4) can be written as

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = \boldsymbol{J}\,\nabla_w H \tag{5}$$

where $\nabla_w H$ denotes the weak derivative of H with respect to the L^2-inner product.

Note that $\boldsymbol{J}$ is an anti-self-adjoint operator in $P \subset L^2([0,2\pi]) \times L^2([0,2\pi])$. Eq. (5) is a Hamiltonian system with the Hamiltonian H and the phase space P. The corresponding Poisson structure is defined by

$$\{F,G\} = \int_0^{2\pi} \nabla \boldsymbol{F}^{\mathrm{T}} \boldsymbol{J}\, \nabla \boldsymbol{G}\,\mathrm{d}x = \int_0^{2\pi} \left(\frac{\partial F}{\partial u}\partial_x \frac{\partial G}{\partial v} + \frac{\partial F}{\partial v}\partial_x \frac{\partial G}{\partial u} \right)\mathrm{d}x$$

where $F(w), G(w) \in C^\infty(P)$. Note that ∂_x is still understood to be a weak derivative operator.

Now we consider the linear part of Eq. (5) and define a linear operator by L: $\boldsymbol{w} \in D = H^3 \times H^1 \cap P \to L^2 \times L^2$, L: $\boldsymbol{w} = (u,v) \mapsto L\boldsymbol{w} = (v_x,\ u_x - u_{xxx})$. It is easy to see that under the periodic boundary condition (3), L has eigenvalues μ_j and the corresponding eigenfunctions $\boldsymbol{\phi}_j$, $\boldsymbol{\psi}_j$, $j \in \mathbf{Z}$, where $\mu_0 = 0$, $\boldsymbol{\phi}_0 = (0,1)$, $\boldsymbol{\psi}_0 = (1,0)$ and for $j \geqslant 1$,

$$\mu_{\pm j} = \mp \mathrm{i}\sqrt{j^4 + j^2},\quad \boldsymbol{\phi}_{\pm j} = \begin{bmatrix} \alpha_j \sin jx \\ \pm \mathrm{i}\beta_j \cos jx \end{bmatrix}$$

$$\boldsymbol{\psi}_{\pm j} = \begin{bmatrix} \alpha_j \cos jx \\ \mp \mathrm{i}\beta_j \sin jx \end{bmatrix}$$

with coefficients

$$\alpha_j = \sqrt{\frac{1}{\pi}\frac{1}{j^2+2}},\quad \beta_j = \sqrt{\frac{1}{\pi}\frac{j^2+1}{j^2+2}}$$

$P_0 = \{(c_1, c_2) \mid c_1, c_2 \in C\}$ denotes null space of the operator L. Let $\Sigma = P/P_0$ be the quotient space of P over P_0. Then the Poisson product is non-degenerate on Σ and L has eigenvalues $\mu_{\pm j}$ with eigenfunctions $\boldsymbol{\phi}_{\pm j}$, $\boldsymbol{\psi}_{\pm j}$ on Σ, $j \geqslant 1$.

Since $\Sigma = \mathrm{span}\{\boldsymbol{\phi}_{\pm j}, \boldsymbol{\psi}_{\pm j}\}_{j\geqslant 1}$, for $\boldsymbol{w} \in \Sigma$, we have $\boldsymbol{w} = \sum_{j\neq 0} (r_j q_j \boldsymbol{\phi}_j + r_j p_j \boldsymbol{\psi}_j)$, where $r_j = r_{-j}$ is the weight to be decided later. Let $\boldsymbol{q} = \{q_j\}$, $\boldsymbol{p} = \{p_j\}_{j\neq 0}$. Then $(\boldsymbol{q}, \boldsymbol{p})$ are the coordinates of $\boldsymbol{w}$ with respect to the bases $\{\boldsymbol{\phi}_{\pm j}, \boldsymbol{\psi}_{\pm j}\}_{j\geqslant 1}$.

Let $a > 0$, $s > 0$ and $\boldsymbol{q} = (q_1, q_2, \cdots, q_n, \cdots)$. Define $\|\boldsymbol{q}\|_{a,s}^2 = \sum_{j=1}^{\infty} |q_j|^2 j^{2s} \mathrm{e}^{2ja}$. Then $l^{a,s} = \{\boldsymbol{q} \mid \|\boldsymbol{q}\|_{a,s} < \infty\}$ is a Hilbert space. Suppose that $\boldsymbol{q} = (\cdots, q_{-n}, \cdots q_{-1}, q_1, \cdots, q_n, \cdots) = \{q_j\}_{j\neq 0}$ and $\boldsymbol{q}_{\pm} = (q_{\pm 1}, \cdots, q_{\pm n}, \cdots)$. Define $\|\boldsymbol{q}\|_{a,s}^2 = \|\boldsymbol{q}_+\|_{a,s}^2 + \|\boldsymbol{q}_-\|_{a,s}^2$. Let $l_b^{a,s} = \{\boldsymbol{q} = \{q_j\}_{j\neq 0} \mid \|q\|_{a,s} < \infty\}$, then it is also a Hilbert space.

For convenience of notation, let $q_{-j} = \bar{q}_j$, $p_{-j} = \bar{p}_j$, $j \geqslant 1$. We endow $l_b^{a,s} \times l_b^{a,s}$ with symplectic structure $\omega = \mathrm{i}\sum_{j\geqslant 1} (d\bar{q}_j \wedge dq_j + d\bar{p}_j \wedge dp_j)$. If $\bar{q}_j$, $\bar{p}_j$ are the conjugate of q_j, p_j, respectively, then the Hamiltonian $H(\boldsymbol{w})$ is a real function.

To consider the Hamiltonian system (5) in the coordinates $\boldsymbol{q}$, $\bar{\boldsymbol{q}}$, $\boldsymbol{p}$, $\bar{\boldsymbol{p}}$, we first compute the Poisson bracket. To make the Poisson bracket in a standard form, $r_j = r_{-j}$ is set to be $\sqrt{\dfrac{j}{2\pi\alpha_j\beta_j}}$, $j \geqslant 1$, then

$$\{F,G\} = \mathrm{i}\sum_{j\geqslant 1} \left(\frac{\partial F}{\partial q_{-j}}\frac{\partial G}{\partial q_j} - \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial q_{-j}} + \frac{\partial F}{\partial p_{-j}}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial p_{-j}} \right)$$

One has the standard symplectic structure $\mathrm{i}\sum_{j\geqslant 1} (dq_{-j} \wedge dq_j + dp_{-j} \wedge dp_j)$. In these coordinates, the Hamiltonian Eq. (5) is written as

$$\dot{q}_j = -\mathrm{i}\frac{\partial H}{\partial q_{-j}},\quad \dot{q}_{-j} = \mathrm{i}\frac{\partial H}{\partial q_j},\quad \dot{p}_j = -\mathrm{i}\frac{\partial H}{\partial p_{-j}},\quad \dot{p}_{-j} = \mathrm{i}\frac{\partial H}{\partial p_j}\quad \forall j \geqslant 1 \tag{6}$$

with the Hamiltonian

$$H = \Lambda + G = \sum_{j\geqslant 1} \mu_j (q_j q_{-j} + p_j p_{-j}) + \int_0^{2\pi} g(u)\,\mathrm{d}x \tag{7}$$

where $\mu_j = j\sqrt{1 + j^2}$.

Lemma 1 Let $a, s > 0$. If $t \in I \to (\boldsymbol{q}(t), \boldsymbol{p}(t)) \in l_b^{a,s} \times l_b^{a,s}$ is a solution of (6), then

$$u(t,x) = \sum_{j\geqslant 1} (\gamma_j (q_j(t) + q_{-j}(t)) \sin jx + \gamma_j (p_j(t) + p_{-j}(t)) \cos jx)$$

is a solution of Eq. (2), where $\gamma_j = \dfrac{1}{\sqrt{2\pi}}\sqrt[4]{\dfrac{j^2}{1+j^2}}$.

In the coordinates $\boldsymbol{q}$, $\bar{\boldsymbol{q}}$, $\boldsymbol{p}$, $\bar{\boldsymbol{p}}$, the nonlinear term $G = -\dfrac{1}{4}\int_0^{2\pi} u^4(x)\,\mathrm{d}x$ becomes $G = G^{40} + G^{22} + G^{04}$, where

$$G^{40}=-\frac{1}{4}\sum_{i,j,k,l\geqslant 1}G^{40}_{ijkl}(q_i+q_{-i})(q_j+q_{-j})(q_k+q_{-k})(q_l+q_{-l})$$

$$G^{22}=-\frac{1}{4}\sum_{i,j,k,l\geqslant 1}G^{22}_{ijkl}(q_i+q_{-i})(q_j+q_{-j})(p_k+p_{-k})(p_l+p_{-l})$$

$$G^{04}=-\frac{1}{4}\sum_{i,j,k,l\geqslant 1}G^{04}_{ijkl}(p_i+p_{-i})(p_j+p_{-j})(p_k+p_{-k})(p_l+p_{-l})$$

where

$$G^{40}_{ijkl}=\gamma_i\gamma_j\gamma_k\gamma_l\int_0^{2\pi}\sin ix\sin jx\sin kx\sin lx\,\mathrm{d}x \tag{8}$$

$$G^{22}_{ijkl}=\gamma_i\gamma_j\gamma_k\gamma_l\int_0^{2\pi}\cos ix\cos jx\sin kx\sin lx\,\mathrm{d}x \tag{9}$$

$$G^{04}_{ijkl}=\gamma_i\gamma_j\gamma_k\gamma_l\int_0^{2\pi}\cos ix\cos jx\cos kx\cos lx\,\mathrm{d}x \tag{10}$$

It is easy to see that the mode pair (q_j,q_{-j}), (p_j,p_{-j}) share the same frequencies, which brings much difficulty for the infinite dimensional KAM theory. Since the previous method is invalid for our problem, we are interested in special solutions. More precisely, under some parity condition of the nonlinearity, we find

$$E=\{(q_j,q_{-j},p_j,p_{-j}):\ p_j=p_{-j}=0,\ j=1,2,\cdots\}$$

$$F=\{(q_j,q_{-j},p_j,p_{-j}):\ q_j=q_{-j}=0,\ j=1,2,\cdots\}$$

are invariant subspaces for the system. Obviously, on E or F, the system is reduced to a Hamiltonian system without multiple frequencies.

Denote $E=\mathrm{span}\{\boldsymbol{\phi}_{\pm j}\}_{j\geqslant 1}$ and $F=\mathrm{span}\{\boldsymbol{\psi}_{\pm j}\}_{j\geqslant 1}$, then we have $\Sigma=E\oplus F$.

Lemma 2 The spaces E and F are invariant under the flow of the Hamiltonian system (5).

Proof Let $z(t)=\sum_{j\neq 0}r_jq_j(t)\boldsymbol{\phi}_j(x)+r_jp_j(t)\boldsymbol{\psi}_j(x)$ be a solution of system (5) with $z(0)\in E$, then $p_{\pm j}(0)=0$, $j=1,2,\cdots$. In the following, we prove that for all $t\geqslant 0$, $z(t)\in E$. It is equivalent to the following fact:

If $(\boldsymbol{q}(t),\boldsymbol{p}(t))$ is the solution of (6) with $\boldsymbol{p}(0)=0$, where $\boldsymbol{q}=(q_j)_{j\neq 0}$, $\boldsymbol{p}=(p_j)_{j\neq 0}$, then $\boldsymbol{p}(t)=0$. Obviously, the equations for $\boldsymbol{p}$ are as follows:

$$\dot{p}_j=-\mathrm{i}\mu_jp_j+\sum_{i,k,l}G^{22}_{k,l,i,j}q_kq_lp_i+\sum_{i,k,l}G^{04}_{i,j,k,l}p_ip_kp_l$$

$$\dot{p}_{-j}=\mathrm{i}\mu_jp_{-j}+\sum_{i,k,l}G^{22}_{k,l,i,-j}q_kq_lp_i+\sum_{i,k,l}G^{04}_{i,-j,k,l}p_ip_kp_l$$

where the coefficients are defined in Eqs. (8) and (9). Obviously $\boldsymbol{p}=0$ is an equilibrium point. By the elementary existence and unique theorem for the differential equation, it follows $\boldsymbol{p}(t)=0$. Thus, we have $z(t)=\sum_{j\neq 0}r_jq_j(t)\boldsymbol{\phi}_j(x)$ and so $z(t)\in E$. Similarly, we can prove that F is also invariant for the system.

Now we consider the restriction of the Hamiltonian system (6) to E, with a little abusing of notations, we still rewrite the Hamiltonian as

$$H(\boldsymbol{q})=\Lambda+G=\sum_{j\geqslant 1}\mu_jq_jq_{-j}+G^{40}(\boldsymbol{q})$$

with $G=G^{40}$ and the corresponding symplectic structure $\omega=\mathrm{i}\sum_{j\geqslant 1}\mathrm{d}\bar{q}_j\wedge\mathrm{d}q_j$. Its equations of motion are written as

$$\dot{q}_j=-\mathrm{i}\frac{\partial H}{\partial q_{-j}},\ \dot{q}_{-j}=\mathrm{i}\frac{\partial H}{\partial q_j}\qquad\forall j\geqslant 1 \tag{11}$$

Combining Lemma 1 and Lemma 2, we have

Lemma 3 Let a, $s>0$. If $I\to l^{a,s}$, $t\mapsto(\boldsymbol{q}(t),\bar{\boldsymbol{q}}(t))$ is a solution of Eq. (11) where

$$\boldsymbol{q}(t)=(q_1(t),q_2(t),\cdots,q_n(t),\cdots)$$

$$\bar{\boldsymbol{q}}(t)=(q_{-1}(t),q_{-2}(t),\cdots,q_{-n}(t),\cdots)$$

then $u(t,x)=\sum_{j\geqslant 1}\gamma_j(q_j+q_{-j})\sin jx$ is a solution of Eq. (2), where $\gamma_j=\frac{1}{\sqrt{2\pi}}\sqrt[4]{\frac{j^2}{1+j^2}}$.

Below we shall verify that the system (11) satisfies all assumptions of an infinite KAM theorem. The linear part of Eq. (11) is

$$\dot{q}_j=-\mathrm{i}\mu_jq_j,\ \dot{q}_{-j}=\mathrm{i}\mu_jq_{-j}\qquad\forall j\geqslant 1 \tag{12}$$

where $\mu_j=j\sqrt{1+j^2}$. It is easy to see that $q_j(t)=\mathrm{e}^{-\mathrm{i}\mu_jt}q_j^0$, where q_j^0 is the initial value. Depending on the number of excited modes, the combined motion is periodic, quasi-periodic or almost periodic. For any finite choice $J=\{1<n_1<n_2<\cdots<n_b\}\subset N$, there is an invariant $2b$-dimensional linear subspace E_J under the flow of (12) foliated into invariant tori $T_J(\boldsymbol{I})$ with frequencies $\mu_{n_1},\mu_{n_2},\cdots,\mu_{n_b}$. $T_J(\boldsymbol{I})$ is defined as

$$T_J(\boldsymbol{I})=\{\boldsymbol{q}\mid q_j=0,\ j\notin J,\ |q_{n_k}|=I_k,\ k=1,2,\cdots,b\}$$

where $\boldsymbol{I}=(I_1,\cdots,I_b)\in P^b$ and $P^b=\{I\in\mathbf{R}^b\mid I_j>0 \text{ for } 1\leqslant j\leqslant b\}$. Moreover, all the solutions have vanishing Lyapunov exponents.

Corresponding to the invariant torus, the linear equation $u_{tt}-u_{xx}+u_{xxxx}=0$ has an invariant manifold in $H^1(0,2\pi)$, on which the linear quasi-periodic flows correspond to quasi-periodic solutions. Thus, for the linear part, the Hamiltonian system (11) has a family of invariant tori $\{T_J(\boldsymbol{I}):\boldsymbol{I}\in P^b\}$. We aim to show that with the nonlinearity G, and a Cantorian familiy of invariant tori will be preserved. Then, equivalently, Eq. (2) has a Cantorian family of quasi-periodic solutions.

Theorem 1 For any $b\in\mathbf{N}$ $(b\geqslant 2)$ and the index set $J=\{n_1<n_2<\cdots<n_b\}\subset\mathbf{N}$, the Hamiltonian system (11) has a Cantorian manifold $\hat{E}\subset l^{a,s}$ $(s>1/2)$ of real analytic and elliptic diophantine b-tori given by a Lipschitz continuous embedding $\boldsymbol{\Phi}$: $T_J[C]\to\hat{E}$, where C is a Cantorian set of P^b and has full density at the origin, and

$\boldsymbol{\Phi}$ is close to the inclusion map $\boldsymbol{\Phi}_0$:

$$\|\boldsymbol{\Phi}-\boldsymbol{\Phi}_0\|_{a,s,B_r\cap T[C]}=O(r^{\sigma}) \qquad r\to 0$$

where B_r is a ball in $l^{a,s}$ centered at the origin with radius $r>0$, $\sigma>0$. Moreover, on these invariant tori, the Hamiltonian system (11) admits a family of linear quasi-periodic flows. Hence, Eq. (2) has a family of quasi-periodic solutions.

2 Regularity and Birkhoff Normal Form

First, we prove the regularity of the nonlinear Hamiltonian vector-field X_G corresponding to $G=G^{40}$ with $u(x)=\sum_{j\geq 1}\gamma_j(q_j+q_{-j})\sin jx$ and prove that it is bounded on $l^{a,s}$. Let $a\geq 0$ and $s\geq 0$; the subspace $l_b^{a,s}$ consists of all bi-infinite sequences $\boldsymbol{p}=\{\cdots,p_{-2},p_{-1},p_0,p_1,p_2,\cdots\}$ with finite norm $\|\boldsymbol{p}\|_{a,s}^2=|p_0|^2+\sum_j |p_j|^2|j|^{2s}e^{2|j|a}$.

We fix $a>0$ and $s>\frac{1}{2}$ later. Let l_b^2 be the Hilbert space of all bi-infinite, square summable sequences with complex coefficients. In the same way as Ref. [9], we define a mapping F by

$$F:\ l_b^2\to L^2,\ \boldsymbol{p}\ \mapsto F\boldsymbol{p}=\frac{1}{\sqrt{2\pi}}\sum_k p_k e^{ikx}$$

Obviously, F is an isometry between l_b^2 and L^2.

For $\boldsymbol{p},\boldsymbol{q}\in l_b^2$, we define the convolution of $\boldsymbol{p}$, $\boldsymbol{q}$ by $(\boldsymbol{p}*\boldsymbol{q})_k=\sum_{j\in Z}p_{k-j}q_j$.

Lemma 4[9] For $a\geq 0$, $s>\frac{1}{2}$, the space $l_b^{a,s}$ is a Hilbert algebra with respect to convolution of sequences and $\|\boldsymbol{p}*\boldsymbol{q}\|_{a,s}\leq c\|\boldsymbol{p}\|_{a,s}\|\boldsymbol{q}\|_{a,s}$ with a constant c depending only on s.

Lemma 5 For $a\geq 0$, $s>\frac{1}{2}$, the Hamiltonian vector field X_G is real analytic as a map from some neighborhood of the origin in $l^{a,s}$ into $l^{a,s}$, with $\|X_G\|_{a,s}=O(\|\boldsymbol{q}\|_{a,s}^3)$. Thus, X_G is bounded on $l^{a,s}$.

For simplicity, we extend the subscripts of γ, μ and G to all nonzero integers by

$$\mu_{-j}=-\mu_j,\ \gamma_j=\gamma_{|j|},\ G_{ijkl}=G_{|i||j||k||l|}$$
$$\forall\, i,j,k,l=\pm 1,\pm 2,\cdots$$

Thus, we have $G=-\frac{1}{4}\sum_{i,j,k,l\neq 0}G_{ijkl}q_iq_jq_kq_l$. By basic computation, it is not difficult to verify that $G_{ijkl}=0$ unless $i\pm j\pm k\pm l=0$. In particular, one has $G_{iijj}=\frac{\pi}{4}[2+\delta_{ij}]\gamma_i^2\gamma_j^2$.

Lemma 6 If i,j,k,l are nonzero integers, such that $i\pm j\pm k\pm l=0$ and each permutation of (i,j,k,l) is not of the form $(r,-r,m,-m)$, then

$$|\mu_i+\mu_j+\mu_k+\mu_l|\geq\frac{1}{2}$$

Proof Note that $\mu_j=\sqrt{j^2+j^4}=j^2+\frac{1}{2}+g(j)$, and

$$g(j)=\sqrt{j^2+j^4}-\left(j^2+\frac{1}{2}\right)=-\frac{1}{4}\frac{1}{\sqrt{j^2+j^4}+\left(j^2+\frac{1}{2}\right)}$$

then $|g(j)|\leq\frac{1}{4}\frac{1}{2j^2}=\frac{1}{8}$. Similar to Ref. [9], we can obtain the conclusion.

Lemma 7 There exists a real analytic, symplectic coordinate transformation Ψ defined in some neighborhood of the origin in $l^{a,s}$, which transforms Hamiltonian $H=\Lambda+G$ into its Birkhoff normal form up to the fourth order:

$$H\circ\Psi=\Lambda+\bar{G}+\hat{G}+K$$

where $\bar{G}=-\frac{1}{2}\sum\bar{G}_{ij}q_iq_{-i}q_jq_{-j}$, $|i|$ or $|j|\in\{n_1,n_2,\cdots,n_b\}$ with uniquely determined coefficients

$$\bar{G}_{ij}=\begin{cases}3\pi\gamma_i^2\gamma_j^2 & i\neq j\\ \bar{G}_{jj}=\frac{9\pi}{4}\gamma_j^4 & i=j\end{cases}$$

$|\hat{G}|=O(\|\hat{\boldsymbol{q}}\|_{a,s}^4)$, $\hat{\boldsymbol{q}}=\boldsymbol{q}\backslash\{q_{n_1},q_{n_2},\cdots,q_{n_b}\}$ and $|K|=O(\|\boldsymbol{q}\|_{a,s}^6)$. Moreover, $X_{\bar{G}},X_{\hat{G}}$, X_K are real analytic vector fields in a neighborhood of the origin in $l^{a,s}$. Obviously, the Hamiltonian $\Lambda+\bar{G}$ is integrable, and the fourth term $\hat{G}$ only depends on $\hat{\boldsymbol{q}}$.

Proof Let $\Psi=X_F^1=X_F^t|_{t=1}$ be the time-1-map of flow generated by $F=\sum_{i,j,k,l}F_{ijkl}q_iq_jq_kq_l$ with $F_{ijkl}=-\frac{i}{4}\cdot\frac{G_{ijkl}}{\mu_i+\mu_j+\mu_k+\mu_l}$ for the indicies (i,j,k,l), which have at least one of elements in $\{\pm n_1,\pm n_2,\cdots,\pm n_b\}$ and are not the permutation of $(r,-r,m,-m)$; $F_{ijkl}=0$ otherwise. Since in a small neighborhood of the origin in $l^{a,s}$, X_F is small, thus the flow X_F^t is well defined for $0\leq t\leq 1$. Using Taylor's formula at $t=0$, we have

$$H\circ\Psi=\Lambda+\{\Lambda,F\}+G+K$$

where $K=\int_0^1(1-t)\{\{\Lambda,F\},F\}\circ X_F^t\mathrm{d}t+\int_0^1\{G,F\}\circ X_F^t\mathrm{d}t$.

3 Cantor Manifold Theorem

By the application of the Cantor manifold theorem, which is proved in Ref. [9], we can prove Theorem 1. For the readers' convenience, we state the Cantorian manifold theorem.

Let $z=(z_1,z_2\cdots,z_n,\cdots)\in l^{a,s}$. Let Hamiltonian $H(z)$ be a real analytic in the real part and imaginary part of z in a neighborhood of the origin of $l^{a,s}$ and the form $H=\Lambda$

$+Q+R$, where $\Lambda+Q$ is the integrable normal form while R is some higher order perturbation. Let $z=(\tilde{z},\hat{z})=(z_1,z_2,\cdots,z_n,z_{n+1},z_{n+2},\cdots)\in l^{a,s}$ with $\tilde{z}=(z_1,z_2,\cdots,z_n)$ and $\hat{z}=(z_{n+1},z_{n+2},\cdots)$ and set

$$\boldsymbol{I}=(|z_1|^2,|z_2|^2,\cdots,|z_n|^2)$$

$$\boldsymbol{Z}=(|z_{n+1}|^2,|z_{n+2}|^2,\cdots)$$

We assume that the normal form consists of the following terms

$$\Lambda=\langle\boldsymbol{\alpha},\boldsymbol{I}\rangle+\langle\boldsymbol{\beta},\boldsymbol{Z}\rangle,\ Q=\frac{1}{2}\langle\boldsymbol{AI},\boldsymbol{I}\rangle+\langle\boldsymbol{BI},\boldsymbol{Z}\rangle$$

with constant vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and constant matrices $\boldsymbol{A}$, $\boldsymbol{B}$.

For the Hamiltonian $\Lambda+Q$, the equations of motion are

$$\dot{\tilde{z}}_j=\mathrm{i}(\boldsymbol{\alpha}+\boldsymbol{AI}+\boldsymbol{B}^{\mathrm{T}}\hat{\boldsymbol{z}})_j\tilde{z}_j,\ \dot{\hat{z}}_j=\mathrm{i}(\boldsymbol{\beta}+\boldsymbol{BI})_j\hat{z}_j$$

Thus, there is a complex n-dimensional invariant manifold $E=\{z=(\tilde{z},\hat{z})\in l^{a,s}\mid\hat{z}=0\}$, and up to the origin, E is completely filled with the invariant tori

$$T(\boldsymbol{I})=\{(\tilde{z},\boldsymbol{0})\mid|\tilde{z}_j|^2=I_j,1\leqslant j\leqslant n\}$$
$$\boldsymbol{I}=(I_1,\cdots,I_n)\in\bar{P}^n$$

on which the flow is given by

$$\dot{\tilde{z}}_j=\mathrm{i}\omega_j(\boldsymbol{I})\tilde{z}_j,\ \omega(\boldsymbol{I})=\boldsymbol{\alpha}+\boldsymbol{AI}$$

Moreover, on the normal space of $T(\boldsymbol{I})$, the flow is given by

$$\dot{\hat{z}}_j=\mathrm{i}\Omega_j(\boldsymbol{I})\hat{z}_j,\ \Omega(\boldsymbol{I})=\boldsymbol{\beta}+\boldsymbol{BI}$$

As the presence of higher order terms R, in general, the manifold E does not persist entirely due to resonances among oscillations. However, for the Hamiltonian $H=\Lambda+Q+R$, a large portion of E can persist near the origin and they form an invariant Cantor manifold $\hat{E}$.

To state the result, we should first make some assumptions:

1) Non-degeneracy condition. The normal form $\Lambda+Q$ is non-degenerate in the sense that for all $(\boldsymbol{k},\boldsymbol{l})\in Z^n\times Z^\infty$ with $1\leqslant|\boldsymbol{l}|\leqslant 2$: ① $\det\boldsymbol{A}\neq 0$; ② $\langle\boldsymbol{l},\boldsymbol{\beta}\rangle\neq 0$; ③ $\langle\boldsymbol{k},\boldsymbol{\omega}(\boldsymbol{I})\rangle+\langle\boldsymbol{l},\boldsymbol{\Omega}(\boldsymbol{I})\rangle\not\equiv 0$.

2) Spectral asymptotical property. There exist $d\geqslant 1$ and $\delta<d-1$ such that

$$\beta_j=j^d+\cdots+O(j^\delta)$$

where the dots stand for terms of order less than d in j.

3) Regularity. For the Hamiltonian vector-fields of Q and R, suppose that X_Q, $X_R\in A(l^{a,s},l^{a,\bar{s}})$, where $\bar{s}$ is defined by $\bar{s}\geqslant s$ for $d>1$, $\bar{s}>s$ for $d=1$.

Remark 1 By the regularity condition, the elements of $\boldsymbol{B}=(B_{ij})_{1\leqslant j\leqslant n<i}$ satisfy the estimate $B_{ij}=O(i^{s-\bar{s}})$ uniformly in $1\leqslant j\leqslant n$. Consequently, for $d=1$ there exists a maximal positive constant κ such that

$$\frac{\Omega_i-\Omega_j}{i-j}=1+O(j^{-\kappa})\qquad i>j$$

uniformly for bounded I. While for $d>1$, let $\kappa=\infty$.

Suppose that Hamiltonian $H=\Lambda+Q+R$ satisfies assumptions 1), 2), and 3), and

$$|R|=O(\|\hat{z}\|^4_{a,s})+O(\|z\|^g_{a,s})$$

with

$$g>4+\frac{4-\Delta}{\kappa},\ \Delta=\min(\bar{s}-s,1)$$

Then there exists a Cantor manifold $\hat{E}$ of real analytic, elliptic diophantine n-tori given by a Lipschitz continuous embedding $\Phi:T[C]\to\hat{E}$, where C is a Cantor set of P^n and it has full density at the origin, and Φ is close to the inclusion map Φ_0:

$$\|\Phi-\Phi_0\|_{a,\bar{s},B_r\cap T[C]}=O(r^\sigma),\ \sigma=\frac{g}{2}-\frac{\kappa+1-\Delta/4}{\kappa}>1$$

Consequently, $\hat{E}$ is tangent to E at the original point.

4 Proof of Theorem 1

For $b>1$, let $J=\{n_1<n_2<\cdots<n_b\}$, $(z_1,z_2,\cdots,z_b)=(q_{n_1},q_{n_2},\cdots,q_{n_b})$ and $(z_{b+1},z_{b+2},\cdots)$ consist of the rest components of $\boldsymbol{q}=(q_1,q_2,\cdots)$ by deleting $\{q_{n_1},q_{n_2},\cdots q_{n_b}\}$. In the same way, we define $\boldsymbol{\alpha}=(\mu_{n_1},\mu_{n_2},\cdots,\mu_{n_b})$, $\boldsymbol{\beta}=(\mu_k)_{k\neq n_1,n_2,\cdots,n_b}$. Let $\tilde{\gamma}_i=\gamma^2_{n_i}$, $i=1,2,\cdots,b$ and $(\tilde{\gamma}_{b+1},\tilde{\gamma}_{b+2},\cdots)$ consist of the rest terms of $\{\gamma_j^2\}$. From the above discussion, we have

$$\Lambda=\langle\boldsymbol{\alpha},\boldsymbol{I}\rangle+\langle\boldsymbol{\beta},\boldsymbol{Z}\rangle,\ \bar{G}=\frac{1}{2}\langle\boldsymbol{AI},\boldsymbol{I}\rangle+\langle\boldsymbol{BI},\boldsymbol{Z}\rangle$$

where $\boldsymbol{I}=(z_1\bar{z}_1,\cdots,z_b\bar{z}_b)$, $\boldsymbol{Z}=(z_{b+1}\bar{z}_{b+1},z_{b+2}\bar{z}_{b+2},\cdots)$, $\boldsymbol{A}=-\frac{3\pi}{4}\boldsymbol{DCD}$ with $\boldsymbol{D}=\mathrm{diag}(\tilde{\gamma}_1,\cdots,\tilde{\gamma}_b)$,

$$\boldsymbol{C}=\begin{bmatrix}3&4&\cdots&4\\4&3&\cdots&4\\\vdots&\vdots&&\vdots\\4&4&\cdots&3\end{bmatrix}$$

and $\boldsymbol{B}=-\frac{3\pi}{2}(\tilde{\gamma}_{b+1},\tilde{\gamma}_{b+2},\cdots)^{\mathrm{T}}(\tilde{\gamma}_1,\cdots,\tilde{\gamma}_b)$. For convenience, we write $H\circ\Psi$ as $H\circ\Psi=\Lambda+Q+R$ with $Q=\bar{G}$ and $R=\hat{G}+K$. Let $\boldsymbol{\omega}(\boldsymbol{I})=\boldsymbol{\alpha}+\boldsymbol{AI}$ and $\boldsymbol{\Omega}(\boldsymbol{I})=\boldsymbol{\beta}+\boldsymbol{BI}$.

It is obvious that the asymptotical increasing property of the normal frequencies and regularity of the vector field satisfy the assumptions of the Cantor manifold theorem. Moreover, it is not difficult to verify $\det(\boldsymbol{A})\neq 0$ and $\langle\boldsymbol{k},\boldsymbol{\omega}(\boldsymbol{I})\rangle+\langle\boldsymbol{l},\boldsymbol{\Omega}(\boldsymbol{I})\rangle\neq 0$; thus we obtain the non-degeneracy condition and non-resonance condition. By the Cantor manifold theorem, we prove Theorem 1.

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广义 Boussinesq 方程的 KAM 环面

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摘要:考虑周期边界下具有非线性项 $f(u)=u^3$ 的一维广义 Boussinesq 方程 $u_{tt}-u_{xx}+(f(u)+u_{xx})_{xx}=0$. 首先,将上述方程转化为一个哈密顿系统,并将该系统在线性算子的特征基上展开得到坐标形式下的哈密顿系统. 鉴于切频与法频之间复杂的共振关系,考虑一类具有特殊结构的拟周期解. 其次,验证了哈密顿向量场的正则性,并对四次项进行规范化,从规范形中可以得到无穷维 KAM 定理所要求的非退化和非共振条件. 利用一个 KAM 定理证明与方程等价的无穷维哈密顿系统存在许多有限维不变环面,故原方程有许多小振幅的拟周期解.

关键词:广义 Boussinesq 方程;拟周期解;哈密顿系统;不变环面

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