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Representations of the Drazin inverse involving idempotents in a ring

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Abstract: An element a of a ring R is called Drazin invertible if there exists $b \in R$ such that $ab = ba$, $bab = b$, and $a - a^2 b$ is nilpotent. The element b above is unique if it exists and is denoted as a^{D}. The equivalent conditions of the Drazin inverse involving idempotents in R are established. As applications, some formulae for the Drazin inverse of the difference and the product of idempotents in a ring are given. Hence, a number of results of bounded linear operators in Banach spaces are extended to the ring case.

Key words: idempotent; Drazin inverse; spectral idempotent
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Let R be an associative ring with unity 1. The symbols R^{-1}, R^{D} and R^{nil} denote the sets of invertible, Drazin invertible and nilpotent elements of R, respectively. The commutant of an element $a \in R$ is defined as $\mathrm{comm}(a) = \{x \in R: xa = ax\}$. Recall that an element $a \in R$ is said to have a Drazin inverse[1] if there is $b \in R$ such that $b \in \mathrm{comm}(a)$, $bab = b$, $a - a^2 b \in R^{\mathrm{nil}}$. The element $b \in R$ above is unique if it exists and is denoted by a^{D}. In this case, we call $a^{\pi} = 1 - aa^{\mathrm{D}}$ the spectral idempotent of a. The nilpotency index of $a - a^2 b$ is called the Drazin index of a, denoted by $\mathrm{ind}(a)$. By R^{D} we mean the set of all Drazin invertible elements in R. It is well known that $a \in R^{\mathrm{D}}$ implies that $a^2 \in R^{\mathrm{D}}$ and $(a^2)^{\mathrm{D}} = (a^{\mathrm{D}})^2$.

Groβ and Trenkler[2] considered the invertibility of $p - q$ for general matrix projectors p, q. Koliha and Rakocevic[3] studied the invertibility of the sum $p + q$ and described the relationship between the invertibility of $p - q$ and $p + q$ for idempotents p and q in a ring. Later, Koliha and Rakocevic[4] obtained the equivalent conditions for the invertibility of $p - q$ in a ring.

Many authors considered Drazin invertibility in different sets. For example, Deng[5] considered the Drazin inverse of the difference and the product of projections in Hilbert spaces. Deng and Wei[6] presented the formulae for the Drazin inverse involving idempotent bounded linear operators in Banach spaces. More results on the Drazin inverse of the difference and the product of idempotents can be found in Refs. [7 - 9].

In this paper, we present the formulae for the Drazin inverse of the difference and the product of idempotents in a ring. Moreover, the equivalent relationships of Drazin inverse involving idempotents are established. Hence, the results in Refs. [5 - 6] are extended to a general ring case. Note that dimensional analysis and spectral decompositions cannot be used in a ring case. The results in this paper are proved by a purely algebraic method.

1 Some Lemmas

In what follows, p, q always mean any two idempotents in a ring R. We first state several known results in the form of lemmas.

Lemma 1[10] Let $S = \{p - q, 1 - pq, p - pq, p - qp, p - pqp, 1 - qp, q - pq, q - qp, p + q - pq\}$. If one of the elements in the set S is Drazin invertible, then all elements in S are Drazin invertible.

Lemma 2[10] The following statements are equivalent:

1) $pq \in R^{\mathrm{D}}$;

2) $1 - p - q \in R^{\mathrm{D}}$;

3) $(1 - p)(1 - q) \in R^{\mathrm{D}}$.

Lemma 3[11] (Cline's formula) Let $a, b \in R^{\mathrm{D}}$. Then $(ba)^{\mathrm{D}} = b((ab)^{\mathrm{D}})^2 a$.

Lemma 4[1] Let $a, b \in R^{\mathrm{D}}$ with $ab = ba$. Then $(ab)^{\mathrm{D}} = b^{\mathrm{D}} a^{\mathrm{D}} = a^{\mathrm{D}} b^{\mathrm{D}}$.

Lemma 5[12] Let $a, b \in R$. If $1 - ab \in R^{\mathrm{D}}$ with $\mathrm{ind}(1 - ab) = k$, then $1 - ba \in R^{\mathrm{D}}$ with $\mathrm{ind}(1 - ba) = k$ and

$$(1 - ba)^{\mathrm{D}} = 1 + b[(1 - ab)^{\mathrm{D}} - (1 - ab)^{\pi} r] a$$

where $r = \sum_{i=0}^{k-1} (1 - ab)^i$.

2 Main Results

In this section, we present some formulae on the Drazin inverse of the difference and the product of idempotents of ring R.

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Definition 1 Let $p-q\in R^{\mathrm{D}}$. Define F, G and H as

$F=p(p-q)^{\mathrm{D}}$, $G=(p-q)^{\mathrm{D}}p$, and $H=(p-q)^{\mathrm{D}}(p-q)$.

Theorem 1 Let $p-q\in R^{\mathrm{D}}$. Then F, G and H above are idempotents and

1) $F=(p-q)^{\mathrm{D}}(1-q)$;
2) $G=(1-q)(p-q)^{\mathrm{D}}$.

Proof Since p, q are idempotents, we obtain $p(p-q)^2=(p-q)^2p=p-pqp$. Note that $a\in R^{\mathrm{D}}$ and $ab=ba$ imply $a^{\mathrm{D}}b=ba^{\mathrm{D}}$ by Theorem 1 of Ref.[1]. It follows that $p\in\mathrm{comm}((p-q)^{\mathrm{D}})^2$. Hence, we have

$$\begin{aligned}F=p(p-q)^{\mathrm{D}}=p[(p-q)^{\mathrm{D}}]^2(p-q)=\\ [(p-q)^{\mathrm{D}}]^2p(p-q)=\\ [(p-q)^{\mathrm{D}}]^2(p-q)(1-q)\\ =(p-q)^{\mathrm{D}}(1-q)\end{aligned}$$

Next, we prove that F is idempotent. From $p(p-q)^{\mathrm{D}}=(p-q)^{\mathrm{D}}(1-q)$, we have

$$\begin{aligned}F^2=(p-q)^{\mathrm{D}}(1-q)p(p-q)^{\mathrm{D}}=\\ (p-q)^{\mathrm{D}}(1-q)(p-q)(p-q)^{\mathrm{D}}=\\ p(p-q)^{\mathrm{D}}(p-q)(p-q)^{\mathrm{D}}\\ =p(p-q)^{\mathrm{D}}=F\end{aligned}$$

Similarly, $G^2=G=(1-q)(p-q)^{\mathrm{D}}$. It is clear that H is idempotent and

$$H=(p-q)(p-q)^{\mathrm{D}}=(p-q)^{\mathrm{D}}(p-q)$$

Similarly, we obtain more relationships among F, G and H.

Corollary 1 Let $p-q\in R^{\mathrm{D}}$. Then

1) $q(p-q)^{\mathrm{D}}=(p-q)^{\mathrm{D}}(1-p)$;
2) $(p-q)^{\mathrm{D}}q=(1-p)(p-q)^{\mathrm{D}}$;
3) $qH=Hq$;
4) $G(1-q)=(1-q)F$.

Proof 1) and 2) can be obtained by a similar way of Theorem 1.

3) Since $H=(p-q)^{\mathrm{D}}(p-q)$, we have

$$\begin{aligned}qH=q(p-q)^{\mathrm{D}}(p-q)=(p-q)^{\mathrm{D}}(1-p)(p-q)=\\ (p-q)^{\mathrm{D}}(p-q)q=Hq\end{aligned}$$

4) By Theorem 1, we have

$$\begin{aligned}G(1-q)=(p-q)^{\mathrm{D}}p(p-q)=(1-q)(p-q)^{\mathrm{D}}(p-q)=\\ (1-q)(1-q-1+p)(p-q)^{\mathrm{D}}=\\ (1-q)p(p-q)^{\mathrm{D}}=(1-q)F\end{aligned}$$

Proposition 1 Let $p-q\in R^{\mathrm{D}}$. Then

1) $Fp=pG=pH=Hp$;
2) $qHq=qH=Hq=HqH$.

Proof 1) It is clear that $Fp=pG$, we only need to show $pG=pH$ and $pH=Hp$.

$$\begin{aligned}pG=p(p-q)^{\mathrm{D}}p=(p-q)^{\mathrm{D}}(1-q)p=\\ (p-q)^{\mathrm{D}}(p-q)p=Hp\end{aligned}$$

According to Theorem 1, we obtain

$$\begin{aligned}pH=p(p-q)^{\mathrm{D}}(p-q)=(p-q)^{\mathrm{D}}(1-q)(p-q)=\\ (p-q)^{\mathrm{D}}(p-q)p=Hp\end{aligned}$$

Hence, 1) holds.

2) Note that $qH=Hq$ in 3) of Corollary 1. We obtain that $qHq=(Hq)q=Hq$. Since H is idempotent, $HqH=H^2q=Hq$.

Thus, $qHq=qH=Hq=HqH$.

The following theorems, the main results of this paper, give the formulae of the Drazin inverses of the difference and the product of idempotents in a ring R.

Theorem 2 Let $p-q\in R^{\mathrm{D}}$. Then

1) $(1-pqp)^{\mathrm{D}}=[(p-q)^{\mathrm{D}}]^2p+1-p$;
2) $(p-pqp)^{\mathrm{D}}=[(p-q)^{\mathrm{D}}]^2p=p[(p-q)^{\mathrm{D}}]^2$;
3) $(p-pq)^{\mathrm{D}}=p[(p-q)^{\mathrm{D}}]^3$;
4) $(p-qp)^{\mathrm{D}}=[(p-q)^{\mathrm{D}}]^3p$;
5) If $\mathrm{ind}\,(p-q)=k$, then

$$(1-pq)^{\mathrm{D}}=1-p+[(p-q)^{\mathrm{D}}]^2[p+pq(1-p)]+\left[\sum_{i=0}^{k-1}(p-q)^{\pi}(p-q)^{2i}\right]pq(p-1)$$

Proof 1) As $1-pqp=(p-q)^2p+1-p$, $[(p-q)^2]^{\mathrm{D}}=[(p-q)^{\mathrm{D}}]^2$ and $(p-q)^2p(1-p)=(1-p)(p-q)^2p=0$; then $(1-pqp)^{\mathrm{D}}=[(p-q)^{\mathrm{D}}]^2p+1-p$ by Corollary 1 of Ref. [1].

2) Observing that $p-pqp=p(p-q)^2=(p-q)^2p$, we obtain $(p-pqp)^{\mathrm{D}}=[(p-q)^{\mathrm{D}}]^2p=p[(p-q)^{\mathrm{D}}]^2$ from Lemma 4.

3) Let $x=p[(p-q)^{\mathrm{D}}]^3$. We prove that x is the Drazin inverse of $p-pq$ by showing the following conditions hold.

① From $p(p-q)^2=(p-q)^2p=(p-pq)p$, it follows that

$$\begin{aligned}(p-pq)x=(p-pq))p[(p-q)^{\mathrm{D}}]^3=\\ p(p-q)^2[(p-q)^{\mathrm{D}}]^3=p(p-q)^{\mathrm{D}}\end{aligned}$$

and

$$\begin{aligned}x(p-pq)=p[(p-q)^{\mathrm{D}}]^3(p-pq)=\\ p(p-q)^{\mathrm{D}}[(p-q)^{\mathrm{D}}]^2p(p-q)=\\ p(p-q)^{\mathrm{D}}p(p-q)^{\mathrm{D}}=\\ p(p-q)^{\mathrm{D}}=(p-pq)x\end{aligned}$$

② Note that $(p-pq)x=p(p-q)^{\mathrm{D}}$. We have

$$\begin{aligned}x(p-pq)x=p[(p-q)^{\mathrm{D}}]^3p(p-q)^{\mathrm{D}}=\\ [(p-q)^{\mathrm{D}}]^2p(p-q)^{\mathrm{D}}p(p-q)^{\mathrm{D}}=\\ [(p-q)^{\mathrm{D}}]^2p(p-q)^{\mathrm{D}}=\\ p[(p-q)^{\mathrm{D}}]^3=x\end{aligned}$$

③ Since $(p-pq)x=p(p-q)^{\mathrm{D}}$, we obtain that

$$\begin{aligned}(p-pq)-(p-pq)^2x=(p-pq)-(p-pq)p(p-q)^{\mathrm{D}}=\\ p(p-q)-p(p-q)^2(p-q)^{\mathrm{D}}=\\ p(p-q)(p-q)^{\pi}\end{aligned}$$

According to $pH = Hp$ and $qH = Hq$, it follows that $p(p-q)(p-q)^{\pi} = (p-q)^{\pi}p(p-q)$. By induction, one can obtain $[p(p-q)]^m = p(p-q)^{2m-1}$. Take $m \geqslant \mathrm{ind}(p-q)$, then $[p(p-q)(p-q)^{\pi}]^m = p(p-q)^{2m-1}(p-q)^{\pi} = 0$. This implies that $(p-pq)-(p-pq)^2x$ is nilpotent.

Therefore, $(p-pq)^{\mathrm{D}} = p[(p-q)^{\mathrm{D}}]^3$.

4) Use a similar proof of 3).

5) It follows from Lemma 1 that $1-pq \in R^{\mathrm{D}}$. Lemma 5 guarantees that

$$(1-pq)^{\mathrm{D}} = 1 + p[(1-pqp)^{\mathrm{D}} - (1-pqp)^{\pi}r]pq \quad (1)$$

where $r = \sum_{i=0}^{k-1}(1-pqp)^i$.

Note that

$$\begin{aligned}(1-pqp)^{\mathrm{D}} &= [(p-q)^{\mathrm{D}}]^2p + 1 - p\\ (1-pqp)^{\pi} &= p - p(p-q)(p-q)^{\mathrm{D}}\end{aligned} \quad (2)$$

Substituting Eq.(2) into Eq.(1), we have

$$(1-pq)^{\mathrm{D}} = 1 - p + [(p-q)^{\mathrm{D}}]^2[p+pq(1-p)] + \Big[\sum_{i=0}^{k-1}(p-q)^{\pi}(p-q)^{2i}\Big]pq(p-1)$$

Theorem 3 Let $1-p-q \in R^{\mathrm{D}}$. Then

1) $(pqp)^{\mathrm{D}} = [(1-p-q)^{\mathrm{D}}]^2p = p[(1-p-q)^{\mathrm{D}}]^2$;

2) $(pq)^{\mathrm{D}} = [(1-p-q)^{\mathrm{D}}]^4pq$.

Proof 1) By $pqp = p(1-p-q)^2 = (1-p-q)^2p$ and Lemma 4, it follows that $(pqp)^{\mathrm{D}} = [(1-p-q)^{\mathrm{D}}]^2p = p[(1-p-q)^{\mathrm{D}}]^2$.

2) From $pq = ppq$ and Lemma 3, we have $(pq)^{\mathrm{D}} = p[(pqp)^{\mathrm{D}}]^2pq = [(pqp)^{\mathrm{D}}]^2pq$. According to Eq.(1), we obtain $(pq)^{\mathrm{D}} = [(pqp)^{\mathrm{D}}]^2pq = [(1-p-q)^{\mathrm{D}}]^4pq$.

Deng[5] and Li[13] considered the following result for projections in Hilbert spaces, C^*-algebras, respectively. Indeed, they still hold for idempotents in a ring.

Theorem 4 Let $pq \in R^{\mathrm{D}}$. Then

1) $(pq)^{\mathrm{D}} = (pqp)^{\mathrm{D}} - p[(1-q)(1-p)]^{\mathrm{D}}$;

2) $(pq)^{\mathrm{D}}pq = (pqp)^{\mathrm{D}}pq$.

Proof 1) By 4) of Theorem 2, we have $(p-qp)^{\mathrm{D}} = [(p-q)^{\mathrm{D}}]^3p$ and $(q-pq)^{\mathrm{D}} = [(q-p)^{\mathrm{D}}]^3q = -[(p-q)^{\mathrm{D}}]^3q$.

Hence,

$$\begin{aligned}(q-pq)^{\mathrm{D}} + (p-qp)^{\mathrm{D}} &= [(p-q)^{\mathrm{D}}]^3(-q) + [(p-q)^{\mathrm{D}}]^3p\\ &= [(p-q)^{\mathrm{D}}]^2\end{aligned} \quad (3)$$

We replace p by $1-p$ in Eq.(3) to obtain

$$(pq)^{\mathrm{D}} + [(1-q)(1-p)]^{\mathrm{D}} = [(1-p-q)^{\mathrm{D}}]^2 \quad (4)$$

Multiplying Eq.(4) by p on the left yields

$$p(pq)^{\mathrm{D}} + p[(1-q)(1-p)]^{\mathrm{D}} = p[(1-p-q)^{\mathrm{D}}]^2 \quad (5)$$

Note that $p(pq)^{\mathrm{D}} = p(pq)(pq)^{\mathrm{D}}(pq)^{\mathrm{D}} = (pq)^{\mathrm{D}}$ and Theorem 3. We have

$$(pq)^{\mathrm{D}} = (pqp)^{\mathrm{D}} - p[(1-q)(1-p)]^{\mathrm{D}}$$

2) By Lemma 3, we have

$$(pqp)^{\mathrm{D}}pq = pq[(pq)^{\mathrm{D}}]^2pq = (pq)^{\mathrm{D}}pq$$

The proof is completed.

Theorem 5 Let $1-pq \in R^{\mathrm{D}}$. Then $p-q \in R^{\mathrm{D}}$ and

$$(p-q)^{\mathrm{D}} = (1-pq)^{\mathrm{D}}(p-pq) + (p+q-pq)^{\mathrm{D}}(pq-q)$$

Proof By 5) of Theorem 2, we have

$$(1-pq)^{\mathrm{D}} = 1-p+[(p-q)^{\mathrm{D}}]^2[p+pq(1-p)] + \Big[\sum_{i=0}^{k-1}(p-q)^{\pi}(p-q)^{2i}\Big]pq(p-1) \quad (6)$$

Substituting p and q by $1-p$ and $1-q$, respectively, in Eq.(6), we obtain

$$(p+q-pq)^{\mathrm{D}} = p + [(p-q)^{\mathrm{D}}]^2[1-p+(1-p)(1-q)p] + \Big[\sum_{i=0}^{k-1}(p-q)^{\pi}(p-q)^{2i}\Big](1-p)(1-q)p \quad (7)$$

Multiplying Eq.(6) by $p-pq$ on the right yields

$$(1-pq)^{\mathrm{D}}(p-pq) = p(p-q)^{\mathrm{D}} = (p-q)^{\mathrm{D}}(1-q) \quad (8)$$

Multiplying Eq.(7) by $pq-p$ on the right yields

$$(p+q-pq)^{\mathrm{D}}(pq-p) = (p-q)^{\mathrm{D}}q \quad (9)$$

From (8) and (9), one can obtain

$$\begin{aligned}(1-pq)^{\mathrm{D}}(p-pq) + (p+q-pq)^{\mathrm{D}}(pq-q) &=\\ (p-q)^{\mathrm{D}}(1-q) + (p-q)^{\mathrm{D}}q &= (p-q)^{\mathrm{D}}\end{aligned}$$

The proof is complete.

Let p, q be two idempotents in a Banach algebra. Then, $p+q \in R^{\mathrm{D}}$ if and only if $p-q \in R^{\mathrm{D}}$. However, in general, this need not be true in a ring. For example, let $R = Z$ and $p = q = 1$. Then $p-q = 0 \in R^{\mathrm{D}}$, but $p+q = 2 \notin R^{\mathrm{D}}$. Next, we consider what conditions p and q satisfy, and $p-q \in R^{\mathrm{D}}$ implies that $p+q \in R^{\mathrm{D}}$.

The following result, proved by Deng and Wei[6] for bounded linear operators in Banach spaces, indeed holds in a ring.

Theorem 6 Let $p-q \in R^{\mathrm{D}}$. If F, G and H are given by Definition 1 and $(p+q)(p-q)^{\pi} \in R^{\mathrm{nil}}$, then

1) $(p+q)^{\mathrm{D}} = (p-q)^{\mathrm{D}}(p+q)(p-q)^{\mathrm{D}}$;

2) $(p-q)^{\mathrm{D}} = (p+q)^{\mathrm{D}}(p-q)(p+q)^{\mathrm{D}}$;

3) $(p-q)^{\pi} = (p+q)^{\pi}$;

4) $(p-q)^{\mathrm{D}} = F+G-H$;

5) $(p+q)^{\mathrm{D}} = (2G-H)(F+G-H)$.

In Theorem 6, we know that if $p-q \in R^{\mathrm{D}}$ and $(p+q)(p-q)^{\pi} \in R^{\mathrm{nil}}$, then $p+q \in R^{\mathrm{D}}$. Naturally, does the reverse statement above hold? Next, we illustrate that it is not true. Such as, take $R = Z_7$, $p = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $q = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \in M_2(R)$. Then p and q are idempotents. Moreover, $p+q=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$, $p-q=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. It is clear that $p+q$ and $p-q$ are Drazin invertible. However, $(p+q)(p-q)^{\pi}$ is not nilpotent.

Koliha et al.[4] proved that $p-q\in R^{-1}$ implies that $p+q\in R^{-1}$ for idempotents p and q in a ring R. Hence, we have the following results.

Corollary 2[14] Let $p-q\in R^{-1}$. If $F=p(p-q)^{-1}$ and $G=(p-q)^{-1}p$, then

1) $(p+q)^{-1}=(p-q)^{-1}(p+q)(p-q)^{-1}$;
2) $(p-q)^{-1}=(p+q)^{-1}(p-q)(p+q)^{-1}$;
3) $(p-q)^{-1}=F+G-1$;
4) $(p+q)^{-1}=(2G-1)(F+G-1)$.

Corollary 3 Let $p-qp\in R^{\mathrm{D}}$, and then $(p-q)^{\mathrm{D}}=(p-q)^2[(p-qp)^{\mathrm{D}}-(q-qp)^{\mathrm{D}}]$.

Proof Since $(p-qp)^{\mathrm{D}}=[(p-q)^{\mathrm{D}}]^3p$ and $(q-qp)^{\mathrm{D}}=q[(q-p)^{\mathrm{D}}]^3$, we obtain

$$\begin{aligned}&(p-q)^2[(p-qp)^{\mathrm{D}}-(q-qp)^{\mathrm{D}}]=\\&\quad(p-q)^2([(p-q)^{\mathrm{D}}]^3p-q[(q-p)^{\mathrm{D}}]^3)=\\&\quad(p-q)^{\mathrm{D}}p+q(p-q)^{\mathrm{D}}=\\&\quad(p-q)^{\mathrm{D}}p+(p-q)^{\mathrm{D}}(1-p)=(p-q)^{\mathrm{D}}\end{aligned}$$

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环中涉及幂等元的 Drazin 逆的表示

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摘要：称环 R 中的元素 a 为 Drazin 可逆的，如果存在 R 中的元素 b 使得 $ab=ba$，$bab=b$，$a-a^2b$ 是幂零的. 上述元素 b 如果存在则是唯一的，并表示为 a^{D}. 给出了一些环中涉及幂等元的 Drazin 逆的等价条件. 作为应用，给出了环中幂等元的积与差的 Drazin 逆的公式. 因此，一些关于 Banach 空间中有界线性算子的结果被推广到环上.

关键词：幂等元；Drazin 逆；谱幂等元

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