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Enveloping algebras of generalized H-Hom-Lie algebras

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Abstract: Let H be a Hopf algebra and ${}_H^H\mathrm{YD}$ the Yetter-Drinfeld category over H. First, the enveloping algebra of generalized H-Hom-Lie algebra L, i. e., Hom-Lie algebra L in the category ${}_H^H\mathrm{YD}$, is constructed. Secondly, it is obtained that $U(L) = T(L)/I$, where I is the Hom-ideal of $T(L)$ generated by $\{l\otimes l' - l_{(-1)} \cdot l'\otimes l_0 - [l,l'] \mid l,l' \in L\}$, and u: $L\to T(L)/I$ is the canonical map. Finally, as the applications of the result, the enveloping algebras of generalized H-Lie algebras, i. e., the Lie algebras in the category ${}_H^H\mathrm{YD}$ and the Hom-Lie algebras in the category of left H-comodules are presented, respectively.

Key words: enveloping algebra; generalized H-Hom-Lie algebra; Yetter-Drinfeld category

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Hom-Lie algebras were first studied by Hartwig et al. in Ref. [1], where they introduced the structure of the Hom-Lie algebras in the context of the deformations of Witt and Virasoro algebras. The ideal is that the Jacobi identity is replaced by the so-called Hom-Jacobi identity, namely,

$$[\alpha(x),[y,z]] + [\alpha(y),[z,x]] + [\alpha(z),[x,y]] = 0$$

where α is an endomorphism of Lie algebras. Hom-algebras were first studied by Makhlouf and Silvestrov in Ref. [2], in which the associativity is replaced by the Hom-associativity, namely,

$$\alpha(x)(yz) = (xy)\alpha(z)$$

Dually, Makhlouf et al.[3-4] gave the Hom-coassociativity for Hom-coalgebras. Later, Chen et al.[5] studied Hom-Lie bialgebras as a natural generalization of Lie bialgerbas. Caenepeel and Goyvaerts[6] studied Hom-Hopf algebras from a categorical view point, and Yau[7] introduced the notion of quasitriangular Hom-Hopf algebras. Also, he proved that each quasitriangular Hom-Hopf algebra produces a solution of the Hom-Yang-Baxter equation, and constructed the enveloping algebras of Hom-Lie algebras in Ref. [8].

Motivated by Wang et al.[9], we considered Hom-Lie algebras in Yetter-Drinfeld categories and proved that each H-Hom-algebra gives rise to a generalized H-Hom-Lie algebra. It is a natural question whether we can construct enveloping algebras of generalized H-Hom-Lie algebras or not. This paper will give a positive answer to this question.

Throughout this paper, all algebraic systems are supposed to be over a field k. About the Hom-algebras and Hom-Lie algebras, the readers can be referred to Caenepeel and Goyvaerts[6] as general references, about Hopf algebras to Sweedler[10] and Yetter-Drinfeld categories to Radford[11]. If C is a coalgebra, we use the Sweedler-type notation for the comultiplication: $\Delta(c) = c_1\otimes c_2$, for all $c\in C$.

1 Generalized H-Hom-Lie Coalgebras

From now on, we always assume that H is a Hopf algebra with a bijective antipode S. The Yetter-Drinfeld category ${}_H^H\mathrm{YD}$ is a braided monoidal category whose objects are both left H-modules and left H-comodules, and morphisms are both left H-linear and left H-colinear maps and satisfy the compatibility condition

$$\rho(h\cdot m) = h_1 m_{(-1)} S(h_3)\otimes h_2\cdot m_0$$

where the H-module action is denoted by $h\cdot m$ and the H-comodule structure map is denoted by $\rho: M\to H\otimes M$, $\rho(m) = m_{(-1)}\otimes m_0$, for all $h\in H$, $m\in M$. The braiding τ is given by $\tau(m\otimes n) = m_{(-1)}\cdot n\otimes m_0$, for all $m\in M$, $n\in N$, M, N are objects in ${}_H^H\mathrm{YD}$.

Letting A be an object in ${}_H^H\mathrm{YD}$, the braiding τ is called symmetric on A if the following condition holds:

$$(a_{(-1)}\cdot b)_{(-1)}\cdot a_0\otimes(a_{(-1)}\cdot b)_0 = a\otimes b \qquad a,b\in A$$

Definition 1[9] An H-Hom-algebra is a triple (A, m, α) consisting of a linear space A in ${}_H^H\mathrm{YD}$, a bilinear map $m: A\otimes A\to A$, and a homomorphism $\alpha: A\to A$ such that

$$\alpha(a)(bc) = (ab)\alpha(c),\ \alpha(ab) = \alpha(a)\alpha(b)$$
$$a1_A = 1_A a = \alpha(a),\ \alpha(1_A) = 1_A \qquad a,b,c\in A$$

Definition 2[9] A generalized H-Hom-Lie algebra is a triple $(L, [\,,], \alpha)$ consisting of a linear space L in ${}_H^H\mathrm{YD}$, a bilinear map $[\,,]: L\otimes L\to L$, and a homomorphism $\alpha: L\to L$ in ${}_H^H\mathrm{YD}$ satisfying

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1) H-anti-commutativity

$$[l, l'] = -[l_{(-1)} \cdot l', l_0] \qquad l, l' \in L$$

2) H-Hom-Jacobi identity

$$\{l \otimes l' \otimes l''\} + (\tau \otimes 1)(1 \otimes \tau)\{l \otimes l' \otimes l''\} + (1 \otimes \tau)(\tau \otimes 1)\{l \otimes l' \otimes l''\} = 0$$

for all $l, l', l'' \in L$, where $\{l \otimes l' \otimes l''\}$ denotes $[\alpha(l), [l', l'']]$.

Proposition 1[9] Let (A, α) be an H-Hom-algebra. Assume that the braiding τ is symmetric on A. Then the triple $(A, [\,,], \alpha)$ is a generalized H-Hom-Lie algebra, where the bracket product is defined by

$$[\,,]: A \otimes A \to A,\ [a, b] = ab - (a_{(-1)} \cdot b)a_0 \qquad a, b \in A$$

Dually, we can present the definitions of Hom-coalgebras and Hom-Lie coalgebras in the category ${}_H^H\mathrm{YD}$.

Definition 3 An H-Hom-coalgebra is a triple (A, Δ, α) consisting of a linear space C in ${}_H^H\mathrm{YD}$, a linear map $\Delta: C \to C \otimes C$, and a homomorphism $\alpha: C \to C$ such that

$$\alpha^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \alpha^{-1}(c_2)$$
$$\Delta(\alpha(c)) = \alpha(c_1) \otimes \alpha(c_2)$$
$$c_1 \varepsilon(c_2) = \alpha^{-1}(c) = \varepsilon(c_1)c_2,\ \varepsilon(\alpha(c)) = \varepsilon(c)$$

Definition 4 A generalized H-Hom-Lie coalgebra C is an object in ${}_H^H\mathrm{YD}$ together with a linear map $\delta: C \to C \otimes C$ (called the cobracket) and a homomorphism $\alpha: C \to C$ in ${}_H^H\mathrm{YD}$ subject to the following conditions:

1) H-anti-cocommutativity

$$\delta = -\tau\delta$$

2) H-Hom-coJacobi identity

$$(1 + (\tau \otimes 1)(1 \otimes \tau) + (1 \otimes \tau)(\tau \otimes 1))(\alpha \otimes \delta)\delta = 0$$

Proposition 2 Let (C, Δ, α) be a generalized H-Hom-coalgebra. Assume that the braiding τ is symmetric on C. Then the triple (C, δ, α) is a generalized H-Hom-Lie coalgebra, where the cobracket is defined by

$$\delta: C \to C \otimes C,\ \delta(c) = c_1 \otimes c_2 - (c_{1(-1)} \cdot c_2) \otimes c_{10} \qquad c \in C$$

Proof We first show that δ is a morphism in ${}_H^H\mathrm{YD}$. For any $c \in C$ and $h \in H$, we have

$$\begin{aligned}\delta(h \cdot c) &= (h \cdot c)_1 \otimes (h \cdot c)_2 - ((h \cdot c)_{1(-1)} \cdot (h \cdot c)_2) \otimes (h \cdot c)_{10} = h_1 \cdot c_1 \otimes h_2 \cdot c_2 - (h_{11}c_{1(-1)}S(h_{13})) \cdot (h_2 \cdot c_2) \otimes h_{12} \cdot c_{10} = h_1 \cdot c_1 \otimes h_2 \cdot c_2 - \\ &\quad h_1 c_{1(-1)} \cdot c_2 \otimes h_2 \cdot c_{10} = h \cdot (c_1 \otimes c_2) - h \cdot (c_{1(-1)} \cdot c_2 \otimes c_{10})\end{aligned}$$

So δ is left H-linear. We can also conclude that

$$(1 \otimes \delta)\rho(c) = c_{(-1)} \otimes (c_{01} \otimes c_{02} - c_{01(-1)} \cdot c_{02} \otimes c_{010}) = c_{1(-1)}c_{2(-1)} \otimes (c_{10} \otimes c_{20} - c_{10(-1)} \cdot c_{20} \otimes c_{100})$$

$$\begin{aligned}\rho\delta(c) &= c_{1(-1)}c_{2(-1)} \otimes c_{10} \otimes c_{20} - (c_{1(-1)} \cdot c_2)_{(-1)}c_{10(-1)} \otimes (c_{1(-1)} \cdot c_2)_0 \otimes c_{100} = c_{1(-1)}c_{2(-1)} \otimes c_{10} \otimes c_{20} - \\ &\quad c_{1(-1)1}c_{2(-1)}S(c_{1(-1)3})c_{10(-1)} \otimes c_{1(-1)2} \cdot c_{20} \otimes c_{100} = c_{1(-1)}c_{2(-1)} \otimes (c_{10} \otimes c_{20} - c_{10(-1)} \cdot c_{20} \otimes c_{100})\end{aligned}$$

Hence, $(1 \otimes \delta)\rho = \rho\delta$, that is, δ is left H-colinear.

Next, we verify that the cobracket δ is compatible with α. In fact, for any $c \in C$, we obtain

$$\begin{aligned}\delta\alpha(c) &= \alpha(c)_1 \otimes \alpha(c)_2 - (\alpha(c)_{1(-1)} \cdot \alpha(c)_2) \otimes \alpha(c)_{10} = \\ &\quad \alpha(c_1) \otimes \alpha(c_2) - (\alpha(c_1)_{(-1)} \cdot \alpha(c_2)) \otimes \alpha(c_1)_0 = \\ &\quad \alpha(c_1) \otimes \alpha(c_2) - c_{1(-1)} \cdot \alpha(c_2) \otimes \alpha(c_{10}) = \\ &\quad \alpha(c_1) \otimes \alpha(c_2) - \alpha(c_{1(-1)} \cdot c_2) \otimes \alpha(c_{10})\end{aligned}$$

as required. To show that (C, Δ, α) is a generalized H-Hom-Lie coalgebra in the sense of Definition 4, we verify the H-anti-cocommutativity and H-Hom-coJacobi identity. However, this is a routine work since τ is symmetric on C. This completes the proof.

2 Enveloping Algebras of Generalized *H*-Hom-Lie Algebras

In this section, we will construct the enveloping algebra $U(L)$ of a generalized H-Hom-Lie algebra L.

Definition 5 Let (L_1, α_1) and (L_2, α_2) be two generalized H-Hom-Lie algebras. A Hom-Lie homomorphism $f: L_1 \to L_2$ is an ${}_H^H\mathrm{YD}$-morphism such that

$$f\alpha_1 = \alpha_2 f,\ f([x, y]_{L_1}) = [f(x), f(y)]_{L_2}$$

for all $x, y \in L_1$.

Let A be an H-Hom-algebra and $\bar{L}$ a Yetter-Drinfeld submodule of A. Then $\bar{L}$ is called a derived Hom-Lie algebra in A and denoted by $\bar{L}^{-A}$, and the bracket product is induced by A, that is, $[\bar{l}, \bar{l}'] = \bar{l}\bar{l}' - (\bar{l}_{(-1)} \cdot \bar{l}')\bar{l}_0$, for all $\bar{l}, \bar{l}' \in \bar{L}$.

Let L be a generalized H-Hom-Lie algebra and A an H-Hom-algebra. We call a map $f: L \to A$ a Hom-Lie homomorphism if there is a derived Hom-Lie algebra $\bar{L}^{-A}$ in A such that $f: L \to \bar{L}^{-A} \subset A$ is a Hom-Lie homomorphism.

Definition 6 Let L be a generalized H-Hom-Lie algebra. Then, by an enveloping algebra of L, we mean a pair (U, u), where $U = (U, \alpha)$ is an H-Hom-algebra with 1, $u: L \to U$ is a Hom-Lie homomorphism and the following assertion holds. For any H-Hom-algebra A and any Hom-Lie homomorphism $f: L \to A$, there is a unique Hom-algebra homomorphism g, which is also a morphism in ${}_H^H\mathrm{YD}$ such that $gu = f$.

Now, for the given generalized H-Hom-Lie algebra L, $T(L)$ is the tensor Hom-algebra given by Caenepeel and Goyvaerts[6]. Note that $T(L)$ is not a Hom-algebra. It is not difficult to verify that $T(L)$ is an object in ${}_H^H\mathrm{YD}$.

Theorem 1 Let (L, α) be a generalized H-Hom-Lie algebra. Take $U(L) = T(L)/I$, where I is the Hom-ideal of $T(L)$ generated by

$$\{l \otimes l' - l_{(-1)} \cdot l' \otimes l_0 - [l, l'] \mid l, l' \in L\}$$

and let $u: L \to T(L)/I$ be the canonical map. Then $(U(L), u)$ is an enveloping algebra for (L, α).

Proof We first show that I is an object in ${}_H^H\mathrm{YD}$. For any $l, l' \in L$ and $h \in H$, we have

$$\begin{aligned}h \cdot (l \otimes l' - l_{(-1)} \cdot l' \otimes l_0 - [l, l']) &= \\ h_1 \cdot l \otimes h_2 \cdot l' - h_1 l_{(-1)} \cdot l' \otimes h_2 \cdot l_0 - [h_1 \cdot l, h_2 \cdot l'] &= \\ h_1 \cdot l \otimes h_2 \cdot l' - (h_1 \cdot l)_{(-1)} \cdot (h_2 \cdot l') \otimes (h_1 \cdot l)_0 - &\end{aligned}$$

$[h_1\cdot l,h_2\cdot l']=h_1\cdot l\otimes h_2\cdot l'-h_1l_{(-1)}S(h_3)\cdot(h_4\cdot l')\otimes h_2\cdot l_0-[h_1\cdot l,h_2\cdot l']=h_1\cdot l\otimes h_2\cdot l'-h_1\cdot(l_{(-1)}\cdot l')\otimes h_2\cdot l_0-[h_1\cdot l,h_2\cdot l']\in I$

So I is an H-module. Similarly, one can show that I is also an H-comodule, as desired. According to Ref. [8], $U(L)$ is a Hom-algebra, one can easily check that it is an object in ${}_H^H\mathrm{YD}$.

Secondly, let u be the restriction to L of the canonical homomorphism π of $T(L)$ onto $U(L)$. Then, we can claim that $u:L\rightarrow U(L)$ is a Hom-Lie homomorphism in ${}_H^H\mathrm{YD}$. Obviously, u is both H-linear and H-colinear. We show that $\bar{\alpha}u=u\alpha$. In fact,

$$\bar{\alpha}u(l)=\bar{\alpha}(l+I)=\alpha(l)+I=u\alpha(l)\qquad l\in L$$

Finally, we show that the following statement holds. For any H-Hom-algebra A in ${}_H^H\mathrm{YD}$ and any Hom-Lie homomorphism $f:L\rightarrow A$, there exists a unique ${}_H^H\mathrm{YD}$-morphism $g:U(L)\rightarrow A$ such that $gu=f$. To prove this statement, we first consider a unique homomorphism f^* which maps $T(L)$ onto A by extending the k-homomorphism f of L into A. Since f is a Lie homomorphism, for any $\bar{l}$, $\overline{l'}\in L$, it follows that

$$f([l,\ l'])=[f(l),f(l')]=f(l)f(l')-(f(l)_{(-1)}\cdot f(l'))f(l)_0$$

Hence, $f^*([l,\ l']-f(l)f(l')+(l_{(-1)}\cdot f(l'))f(l_0)=0$. This shows that $I\subset\ker f^*$, and we have a unique homomorphism g of $U(L)=T(L)/I$ into A such that $g(l+I)=f(l)$ or $gu(l)=f(l)$. Hence, $f=gu$ since L generates $T(L)$. Also, it is not difficult to check that g is an ${}_H^H\mathrm{YD}$-morphism. This completes the proof.

Example 1 It is clear that the generalized H-Lie algebras[12] are examples of generalized H-Hom-Lie algebras by setting $\alpha=id$. By theorem 1, one can obtain the enveloping algebra for any generalized H-Lie algebra L. $U(L)=T(L)/I$, where I is the ideal of $T(L)$ generated by

$$\{l\otimes l'-l_{(-1)}\cdot l'\otimes l_0-[l,\ l']\mid l,\ l'\in L\}$$

and $u:L\rightarrow T(L)/I$ is the canonical map.

Example 2 If $H=(H,\langle|\rangle,\alpha)$ is a cotriangular Hopf algebra[12], then the category ${}^H M$ is a Yetter-Drinfeld category under the left H-module action: $h\cdot m=\langle h\mid m_{(-1)}\rangle m_0$, for all $h\in H$ and $m\in M\in{}^H M$. For any Hom-Lie algebra (L,α) in ${}^H M$, the enveloping algebra for (L,α) is $(U(L),u)$, where $U(L)=T(L)/I$, I is the ideal of $T(L)$ generated by

$$\{l\otimes l'-<l_{(-1)}\mid l'_{(-1)}>l'_0\otimes l_0-[l,\ l']\mid l,\ l'\in L\}$$

and $u:L\rightarrow T(L)/I$ is the canonical map.

References

[1] Hartwig J T, Larsson D, Silvestrov S D. Deformations of Lie algebras using σ-derivations [J]. *J Algebra*, 2006, **295**(2): 314－361.

[2] Makhlouf A, Silvestrov S D. Hom-algebra structures [J]. *J Gen Lie Theory*, 2008, **3**(2): 51－64.

[3] Makhlouf A, Silvestrov S D. Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras [C]//*Generalized Lie Theory in Mathematics, Physics and Beyond*. Berlin: Springer-Verlag, 2009:189－206.

[4] Makhlouf A, Silvestrov S D. Hom-algebras and Hom-coalgebras [J]. *J Algebra Appl*, 2010, **9**(4): 553－589.

[5] Chen Y Y, Wang Z W, Zhang L Y. Quasi-triangular Hom-Lie bialgebras [J]. *J Lie Theory*, 2012, **22**(4): 1075－1089.

[6] Caenepeel S, Goyvaerts I. Monoidal Hom-Hopf algebras [J]. *Comm Algebra*, 2011, **39**(6): 2216－2240.

[7] Yau D. The Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras [J]. *J Phys A*, 2009, **42**(16): 165202-1－165202-12.

[8] Yau D. Enveloping algebra of Hom-Lie algebras [J]. *J Gen Lie Theory Appl*, 2008, **2**(2): 95－108.

[9] Wang S X, Wang S H. Hom-Lie algebras in Yetter-Drinfeld categories [J]. *Comm Algebra*, 2014, **42**(10): 4540－4561.

[10] Sweedler M E. *Hopf algebras* [M]. New York: Benjamin, 1969.

[11] Radford D E. The structure of Hopf algebra with a projection [J]. *J Algebra*, 1985, **92**(2): 322－347.

[12] Wang S H. On the generalized *H*-Lie structure of associative algebras in Yetter-Drinfeld categories [J]. *Comm Algebra*, 2002, **30**(1): 307－325.

广义 H-Hom-李代数的包络代数

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摘要：设 H 是一个 Hopf 代数，${}_H^H\mathrm{YD}$ 是 H 上的 Yetter-Drinfeld 范畴. 首先，构造了广义 H-Hom-李代数 L，即 Hom-李代数 L 是范畴${}_H^H\mathrm{YD}$ 中对象的包络代数. 其次，证明了 $U(L)=T(L)/I$，其中 I 是由$\{l\otimes l'-l_{(-1)}\cdot l'\otimes l_0-[l,l']\mid l,l'\in L\}$生成的 $T(L)$的 Hom-理想，$u:L\rightarrow T(L)/I$ 是典范同态. 最后，作为应用，分别得到了广义 H-李代数，即范畴${}_H^H\mathrm{YD}$ 中的李代数和左 H-余模范畴中广义 H-Hom-李代数的包络代数.

关键词：包络代数；广义 H-Hom-李代数；Yetter-Drinfeld 范畴

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