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Hypersurfaces with constant mean curvature in unit sphere

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Abstract: The pinching of n-dimensional closed hypersurface M with constant mean curvature H in unit sphere $S^{n+1}(1)$ is considered. Let $\tilde{A}=\sum_{i,j,k}h_{ijk}^2(\lambda_i+nH)^2$, $\tilde{B}=\sum_{i,j,k}h_{ijk}^2(\lambda_i+nH)\cdot(\lambda_j+nH)$, $\tilde{S}=\sum_i(\lambda_i+nH)^2$, where $h_{ij}=\lambda_i\delta_{ij}$. Utilizing Lagrange's method, a sharper pointwise estimation of $3(\tilde{A}-2\tilde{B})$ in terms of $\tilde{S}$ and $|\nabla h|^2$ is obtained, here $|\nabla h|^2=\sum_{i,j,k}h_{ijk}^2$. Then, with the help of this, it is proved that M is isometric to the Clifford hypersurface if the square norm of the second fundamental form of M satisfies certain conditions. Hence, the pinching result of the minimal hypersurface is extended to the hypersurface with constant mean curvature case.

Key words: hypersurface with constant mean curvature; unit sphere; pinching

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Let M^n be a closed hypersurface immersed in a unit sphere $S^{n+1}(1)$. The symbols S and H denote the square norm of the second fundamental form and the mean curvature of M, respectively. Assuming $H=0$, i.e., the minimal hypersurface case, Simons[1] obtained the first pinching result. More precisely, he showed that if $0\leqslant S\leqslant n$, then $S\equiv 0$ and M is totally umbilical or $S\equiv n$ and M is isometric to a Clifford torus. Later, Yang and Cheng[2] studied the scalar curvature pinching theorem. Recently, Wang and He[3] proved that if $n\leqslant S\leqslant n+\alpha(n)$, then $S\equiv n$ and M is a Clifford torus $S^k(\sqrt{k/n})\times S^{n-k}(\sqrt{(n-k)/n})(k=1,2,\dots,n-1)$, where $\alpha(n)$ is a positive constant depending only on n. Furthermore, many researchers began to consider the pinching problem for constant mean curvature hypersurfaces[4-8]. In 2013, Xu et al.[8] obtained the second pinching result for a closed hypersurface with sufficiently small mean curvature. However, they did not give the concrete gap. In this paper, we give a concrete expression to the pinching constant by making use of another method inspired by Wang and He[3]. More precisely, we prove the following theorem which is a generalization of corollary 1.1 in Ref. [3].

Theorem 1 Let M be an n-dimensional closed hypersurface with constant mean curvature H satisfying $|H|\leqslant\varepsilon(n)$ in $S^{n+1}(1)$. If $S_0\leqslant S\leqslant S_0+\alpha(n,H)$,

$$S_0=n+\frac{n^3H^2}{2(n-1)}+\frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2} \tag{1}$$

$$\alpha(n,H)=\frac{2}{3}-\left[\frac{4n^3}{9(n-1)}+\frac{2}{3}n^2+\frac{4}{9}n^2(n+2)\right]H^2-\frac{4n(n-2)}{9(n-1)}\sqrt{n^2H^4+4(n-1)H^2} \tag{2}$$

Then $S\equiv S_0$. Furthermore, M is isometric to a Clifford torus $S^k(\sqrt{k/n})\times S^{n-k}(\sqrt{(n-k)/n})$ $(k=1,2,\dots,n-1)$ if $H=0$, and M is isometric to a Clifford hypersurface $C_{1,n-1}$ if $H\neq 0$, where $\varepsilon(n)$ is a sufficiently small constant depending only on n, and $C_{1,n-1}$ is defined as

$$C_{1,n-1}=S^1\left(\frac{1}{\sqrt{1+\lambda^2}}\right)\times S^{n-1}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)$$

$$\lambda=\frac{nH+\sqrt{n^2H^2+4(n-1)}}{2}$$

Remark 1 When M is minimal, i.e., $H=0$, we obtain from (2) that $\alpha(n,0)=2/3$. Then Theorem 1 reduces to Corollary 1.1 in Ref. [3].

1 Preliminaries

We use the similar notations as in Ref. [5]. Choose a local orthonormal frame $e_1,e_2,\dots,e_{n+1}$ in $S^{n+1}(1)$ such that $e_1,e_2,\dots,e_n$ are tangent to M. Let $\omega_1,\omega_2,\dots,\omega_n$; ω_{n+1} be the corresponding dual coframe. The symmetric 2-form $h=\sum_{i,j}h_{ij}\omega_i\omega_j$ is called the second fundamental form of M. Since h_{ij} is symmetric, we can choose a local orthonormal frame $e_1,e_2,\dots,e_n$ such that $h_{ij}=\lambda_i\delta_{ij}$. Then we have

$$nH=\sum_i\lambda_i,\ S=\sum_i\lambda_i^2$$

Set

$$f_3:=\sum_{i,j,k}h_{ij}h_{jk}h_{ki}=\sum_i\lambda_i^3$$

$$f_4:=\sum_{i,j,k,l}h_{ij}h_{jk}h_{kl}h_{li}=\sum_i\lambda_i^4$$

$$A=\sum_{i,j,k}h_{ijk}^2\lambda_i^2,\ B=\sum_{i,j,k}h_{ijk}^2\lambda_i\lambda_j,\ \mu_i=\lambda_i+nH$$

$$\tilde{A}=\sum_{i,j,k}h_{ijk}^2\mu_i^2,\ \tilde{B}=\sum_{i,j,k}h_{ijk}^2\mu_i\mu_j,\ \tilde{S}=\sum_i\mu_i^2$$

Direct calculation gives

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$$\tilde{S} = S + n^2(n+2)H^2 \tag{3}$$

Lemma 1[5] Let M be an n-dimensional closed hypersurface with constant mean curvature H in $S^{n+1}(1)$, then

$$\frac{1}{2}\Delta S = S(n-S) - n^2H^2 + nHf_3 + |\nabla h|^3 \tag{4}$$

$$\frac{1}{2}\Delta(|\nabla h|^2) = (2n+3-S)|\nabla h|^2 - \frac{3}{2}|\nabla S|^2 - \frac{3}{2}(A-2B) - \frac{3}{2}(\tilde{A}-2\tilde{B}) - \frac{3}{2}n^2H^2|\nabla h|^2 + |\nabla^2 h|^2 \tag{5}$$

$$\int_M (A-2B) = \int_M \left[Sf_4 - f_3^2 - S^2 + nHf_3 - \frac{1}{4}|\nabla S|^2\right] \tag{6}$$

$$|\nabla^2 h|^2 \geqslant \frac{3}{2}[Sf_4 - f_3^2 - S^2 - S(S-n) - n^2H^2 + 2nHf_3] + \frac{3[S(S-n) + n^2H^2 - nHf_3]^2}{2(n+4)(S-nH^2)} \tag{7}$$

Lemma 2 Let M be an n-dimensional closed hypersurface with constant mean curvature H in $S^{n+1}(1)$, then

$$3(\tilde{A}-2\tilde{B}) \leqslant 2\tilde{S}|\nabla h|^2 \tag{8}$$

Proof First, we use Lagrange's method to calculate the maximum of the following function:

$$\psi = \frac{3\sum_{i\neq j} h_{iij}^2(\mu_j^2 - 4\mu_i\mu_j) + \sum_{i\neq j\neq k\neq i} h_{ijk}^2[2(\mu_i^2+\mu_j^2+\mu_k^2) - (\mu_i+\mu_j+\mu_k)^2]}{\tilde{S}|\nabla h|^2} \tag{9}$$

under the constraints

$$\sum_i \mu_i^2 = \tilde{S},\quad \sum_i \mu_i = n(n+1)H,\quad |\nabla h|^2 = a$$

Let

$$\varphi = \psi + m_1\Big(\sum_i \mu_i^2 - \tilde{S}\Big) + m_2\Big[\sum_i \mu_i - n(n+1)H\Big] + m_2(|\nabla h|^2 - a)$$

Since φ is continuous on closed hypersurface M, φ can reach its maximum $\varphi(\bar{q})$ at $\bar{q} = (\bar{\mu}_1, \bar{\mu}_2, \dots, \bar{\mu}_n, \bar{h}_{ijk})$. Direct calculation gives that

$$\sum_{i\neq j\neq k\neq i} \bar{h}_{ijk}^2[2(\bar{\mu}_i^2+\bar{\mu}_j^2+\bar{\mu}_k^2) - (\bar{\mu}_i+\bar{\mu}_j+\bar{\mu}_k)^2] \leqslant \sum_{i\neq j\neq k\neq i} 2\bar{h}_{ijk}^2(\bar{\mu}_i^2+\bar{\mu}_j^2+\bar{\mu}_k^2) \leqslant 2\tilde{S}\sum_{i\neq j\neq k\neq i}\bar{h}_{ijk}^2 \tag{10}$$

Using the similar approach as in Ref. [3], we have the following estimate:

$$\sum_{i\neq j} \bar{h}_{iij}^2(\bar{\mu}_j^2 - 4\bar{\mu}_j\bar{\mu}_i) \leqslant 2\tilde{S}\tilde{S}_j \qquad j = 1, 2, \dots, n \tag{11}$$

Adding (10) and (11), we obtain

$$3\sum_{i\neq j} \bar{h}_{iij}^2(\bar{\mu}_j^2 - 4\bar{\mu}_i\bar{\mu}_j) + \sum_{i\neq j\neq k\neq i} \bar{h}_{ijk}^2[2(\bar{\mu}_i^2+\bar{\mu}_j^2+\bar{\mu}_k^2) - (\bar{\mu}_i+\bar{\mu}_j+\bar{\mu}_k)^2] \leqslant 6\tilde{S}\tilde{S}_j + 2\tilde{S}\sum_{i\neq j\neq k\neq i}\bar{h}_{ijk}^2 = 2\tilde{S}\sum_{i\neq j}3\bar{h}_{iij}^2 + 2\tilde{S}\sum_{i\neq j\neq k\neq i}\bar{h}_{ijk}^2 \leqslant 2a\tilde{S} \tag{12}$$

Then

$$\psi(\bar{q}) \leqslant \frac{2a\tilde{S}}{a\tilde{S}} = 2 \tag{13}$$

Since $\psi(\bar{q})$ is the maximum of ψ, we have

$$3\sum_{i\neq j} h_{iij}^2(\mu_j^2 - 4\mu_i\mu_j) + \sum_{i\neq j\neq k\neq i} h_{ijk}^2[2(\mu_i^2+\mu_j^2+\mu_k^2) - (\mu_i+\mu_j+\mu_k)^2] \leqslant 2\tilde{S}|\nabla h|^2 \tag{14}$$

Hence, it is straightforward to compute that

$$3(\tilde{A}-2\tilde{B}) = 3\sum_{i,j,k} h_{ijk}^2(\mu_j^2 - 2\mu_j\mu_i) = -3\sum_i \mu_i^2 h_{iii}^2 + 3\sum_{i\neq j} h_{iij}^2(\mu_j^2 - 4\mu_j\mu_i) + \sum_{i\neq j\neq k\neq i} h_{ijk}^2[2(\mu_i^2+\mu_j^2+\mu_k^2) - (\mu_i+\mu_j+\mu_k)^2] \leqslant 3\sum_{i\neq j} h_{iij}^2(\mu_j^2 - 4\mu_j\mu_i) + \sum_{i\neq j\neq k\neq i} h_{ijk}^2[2(\mu_i^2+\mu_j^2+\mu_k^2) - (\mu_i+\mu_j+\mu_k)^2] \leqslant 2\tilde{S}|\nabla h|^2$$

2 Proof of Theorem 1

Proof By using (4), (5) and direct computations, we have

$$\int_M S(n-S) = \int_M (n^2H^2 - nHf_3 - |\nabla h|^2) \tag{15}$$

$$\int_M \frac{1}{2}|\nabla S|^2 = \int_M [S^2(S-n) + n^2H^2S - nHSf_3 - S|\nabla h|^2] \tag{16}$$

$$\int_M |\nabla^2 h|^2 = \int_M \left[\left(S - 2n - 3 + \frac{3}{2}n^2H^2\right)|\nabla h|^2 + \frac{3}{2}(A-2B) + \frac{3}{2}(\tilde{A}-2\tilde{B}) + \frac{3}{2}|\nabla S|^2\right] \tag{17}$$

It is proved directly that $S \geqslant S_0$ is equivalent to

$$\sqrt{n + \frac{n^3H^2}{4(n-1)}} - \sqrt{S - nH^2} + \frac{n(n-2)|H|}{2\sqrt{n(n-1)}} \leqslant 0$$

Then

$$S(S-n) + n^2H^2 - nHf_3 = -(S-nH^2)\left[n + nH^2 - (S-nH^2) - nH\sum_i(\lambda_i - H)^3\right] \geqslant (S-nH^2)\left[-n - nH^2 + S - nH^2 - \frac{n(n-2)|H|}{2\sqrt{n(n-1)}}\sqrt{S-nH^2}\right] = -(S-nH^2)\left[\sqrt{n+\frac{n^3H^2}{4(n-1)}} + \sqrt{S-nH^2} - \frac{n(n-2)|H|}{2\sqrt{n(n-1)}}\right] \times \left[\sqrt{n+\frac{n^3H^2}{4(n-1)}} - \sqrt{S-nH^2} + \frac{n(n-2)|H|}{2\sqrt{n(n-1)}}\right] \geqslant 0$$

Note that $L = \dfrac{3[S(S-n) + n^2H^2 - nHf_3]^2}{2(n+2)(S-nH^2)} \geqslant 0$, then

(7) gives

$$|\nabla^2 h|^2 \geqslant \frac{3}{2}[Sf_4 - f_3^2 - S^2 - S(S-n) - n^2H^2 + 2nHf_3] \tag{19}$$

We derive from (6), (15) and (19) that

$$\int_M |\nabla^2 h|^2 \geqslant \int_M \left[\frac{3}{2}(Sf_4 - f_3^2 - S^2 + nHf_3 - |\nabla h|^2)\right] \geqslant \int_M \left[\frac{3}{2}(A - 2B) + \frac{3}{8}|\nabla S|^2 - \frac{3}{2}|\nabla h|^2\right] \tag{20}$$

By the above inequality and (17), we deduce

$$\int_M \left[\left(S - 2n - \frac{3}{2} + \frac{3}{2}n^2H^2\right)|\nabla h|^2 + \frac{3}{2}(\tilde{A} - 2\tilde{B}) + \frac{9}{8}|\nabla S|^2\right] \geqslant 0 \tag{21}$$

So we obtain, from (3), (8), (16) and (21), that

$$\int_M \left[\left(-\frac{1}{4}S - 2n - \frac{3}{2}\right)|\nabla h|^2 + \left(\frac{3}{2}n^2 + n^2(n+2)\right)\cdot H^2|\nabla h|^2 + \frac{9}{4}S(S(S-n) + n^2H^2 - nHf_3)\right] \geqslant 0 \tag{22}$$

Now, we choose $\varepsilon(n)$ sufficiently small such that $|H| \leqslant \varepsilon(n)$ and $\alpha(n,H) > 0$. According to $S_0 \leqslant S \leqslant S_0 + \alpha(n,H)$, we deduce from (15) and (16) that

$$\int_M S[S(S-n) + n^2H^2 - nHf_3] \leqslant (S_0 + \alpha(n,H))\int_M [S(S-n) + n^2H^2 - nHf_3] \leqslant (S_0 + \alpha(n,H))\int_M |\nabla h|^2 \tag{23}$$

Then from (22) and (23), we obtain

$$0 \leqslant \int_M \left\{-\frac{1}{4}S - 2n - \frac{3}{2} + \frac{9}{4}[S_0 + \alpha(n,H)] + \left[\frac{3}{2}n^2 + n^2(n+2)\right]H^2\right\}|\nabla h|^2 = -\int_M \frac{1}{4}(S - S_0)|\nabla h|^2 \leqslant 0 \tag{24}$$

We finally conclude that

$$\int_M \frac{1}{4}(S - S_0)|\nabla h|^2 = 0 \tag{25}$$

Therefore, $S = S_0$ or $|\nabla h| = 0$. If $|\nabla h| = 0$, then all of the above inequalities become equalities. From (18), we also have $S = S_0$. Then $S \equiv S_0$ and M is isometric to a Clifford hypersurface. So, we complete the proof of Theorem 1.

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单位球面中具有常平均曲率的超曲面

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摘要：考虑了单位球面 $S^{n+1}(1)$ 中具有常平均曲率 H 的超曲面 M 的拼脐问题. 设 $\tilde{A} = \sum_{i,j,k} h_{ijk}^2(\lambda_i + nH)^2$, $\tilde{B} = \sum_{i,j,k} h_{ijk}^2(\lambda_i + nH)(\lambda_j + nH)$, $\tilde{S} = \sum_i (\lambda_i + nH)^2$, 其中 $h_{ij} = \lambda_i\delta_{ij}$. 利用拉格朗日方法，可以得到 $3(\tilde{A} - 2\tilde{B})$ 关于 $\tilde{S}$ 和 $|\nabla h|^2$ 的估计，其中 $|\nabla h|^2 = \sum_{i,j,k} h_{ijk}^2$. 然后，利用该估计证明了：若 M 的第二基本形式的平方范数满足一定条件，则 M 一定等距于 Clifford 超曲面. 因此，极小超曲面的拼脐结果被推广到具有常平均曲率的超曲面情形.

关键词：具有常平均曲率的超曲面；单位球面；拼脐
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