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Equivalence of crossed product of linear categories and generalized Maschke theorem

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Abstract: Some sufficient and necessary conditions are given for the equivalence between two crossed product actions of Hopf algebra H on the same linear category, and the Maschke theorem is generalized. Based on the result of the crossed product in the classic Hopf algebra theory, first, let A be a k-linear category and H be a Hopf algebra, and the two crossed products $A\#_\sigma H$ and $A\#'_{\sigma'} H$ are isomorphic under some conditions. Then, let $A\#_\sigma H$ be a crossed product category for a finite dimensional and semisimple Hopf algebra H. If V is a left $A\#_\sigma H$-module and $W \subseteq V$ is a submodule such that W has a complement as a left A-module, then W has a complement as a $A\#_\sigma H$-module.

Key words: linear category; inner action; crossed product; generalized Maschke theorem

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Group-graded rings and algebras have been the study focus and several fundamental results have been obtained. Cohen and Montgomery[1] obtained the duality theorem for graded algebras. In this paper, the graded algebras are treated from the point of view of Hopf algebras. A G-graded algebra A can be viewed as a kG-comodule algebra, and thus can be viewed as a k^G-module algebra when G is a finite group. In fact, the concept of graded algebras can be extended to linear categories, leading to group-graded k-linear categories. Similarly, if a k-linear category is graded by a finite group G, then it can be called a k^G-module category. More generally, it is natural to define the concept of the module category for any Hopf algebra H, as done in Refs. [2-4].

As a continuation of the work in Ref. [5], we provide some sufficient and necessary conditions for the equivalence between two crossed product actions of Hopf algebra H on the same linear category, and generalize the Maschke theorem.

Throughout this paper, we work over field k, and all the vector spaces, algebras and tensor product are over k.

1 Preliminary

In this section, we recall the concept of the crossed product of a linear category with a Hopf algebra H. For the crossed product action of a Hopf algebra H on an algebra, one can refer to Refs. [6-7]. If A is a linear category and x, y are objects of A, we denote ${}_yA_x$ the space $\mathrm{Hom}(x, y)$, and denote A_0 the class of objects.

Definition 1 The weak action of a Hopf algebra H on a linear category A is defined by the map $H\otimes {}_yA_x \to {}_yA_x$ given by $h\otimes {}_yf_x \mapsto h\cdot {}_yf_x$, for any $x, y\in A_0$ and $h\in H$ such that

$$h\cdot({}_zg_y\circ {}_yf_x) = \sum (h_1\cdot {}_zg_y)\circ(h_2\cdot {}_yf_x)$$
$$h\cdot 1_x = \varepsilon(h)1_x,\quad 1_H\cdot {}_yf_x = {}_yf_x$$

where ${}_yf_x\in {}_yA_x$, ${}_zg_y\in {}_zA_y$.

Definition 2 Let H be a Hopf algebra and A be a k-linear category. Assume the weak action of H on A and $\sigma=\{\sigma_x\}_{x\in A_0}\in \mathrm{Hom}(H\otimes H, A)$ with $\sigma_x\in\mathrm{Hom}(H\otimes H, {}_xA_x)$ convolution invertible. The crossed product $A\#_\sigma H$ of A with H is a category such that $(A\#_\sigma H)_0 = A_0$, and for any objects x, y and z, ${}_y(A\#_\sigma H)_x = {}_yA_x\otimes H$ as a vector space. Therefore, the composition of morphisms is given by

$$({}_zg_y\#h)\circ({}_yf_x\#k) = \sum {}_zg_y\circ(h_1\cdot {}_yf_x)\circ\sigma_x(h_2,k_1)\#h_3k_2$$

for any h, $k\in H$, ${}_zg_y\in {}_zA_y$, ${}_yf_x\in {}_yA_x$.

In what follows, we assume that the morphisms of all crossed product categories are associative with identity morphisms $\{1_x\#1_H\}_{x\in A_0}$.

2 Equivalence Between Two Crossed Product Categories

Definition 3 Let H be a Hopf algebra and A be a k-linear category. Consider the crossed product $A\#_\sigma H$. We call the action of H on A inner if there is a collection of maps $u=\{u_x\}_{x\in A_0}$ with $u_x\in\mathrm{Hom}(H, {}_xA_x)$ convolution invertible such that for any ${}_yf_x\in {}_yA_x$ and $h\in H$,

$$h\cdot {}_yf_x = \sum u_y(h_1)\circ {}_yf_x\circ u_x^{-1}(h_2) \tag{1}$$

The above equation has an equivalent form:

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$$\sum (h_1 \cdot {}_yf_x) \circ u_x(h_2) = \sum u_y(h) \circ {}_yf_x \quad (2)$$

Proposition 1 Let A be a k-linear category and $A\#_\sigma H$ be the crossed product such that the action of H on A is inner, via some invertible $u \in \mathrm{Hom}(H, A)$. Define $\tau = \{\tau_x\}_{x\in A_0} \in \mathrm{Hom}(H\otimes H, A)$ by

$$\tau_x(h, k) = \sum u_x^{-1}(k_1) u_x^{-1}(h_1) \sigma_x(h_2, k_2) u_x(h_3k_3) \quad (3)$$

Then τ is a cocycle and $A\#_\sigma H \cong A_\tau[H]$, a twisted product with trivial action and a category isomorphism which is also a left A-module, right H-comodule map.

Proof Define $\varphi: A\#_\sigma H \to A_\tau[H]$ by ${}_yf_x\#h \to \sum {}_yf_x u_x(h_1) \otimes h_2$. It is easy to confirm that φ is a left A-module and a right H-comodule map. Then define ψ: $A_\tau[H] \to A\#_\sigma H$ by ${}_yf_x \otimes h \to \sum {}_yf_x u_x^{-1}(h_1)\#h_2$.

It is straightforward to confirm that φ and ϕ are inverses. We show that φ is an algebra map. For any ${}_zf_y, {}_yg_x$ in A, and $h, k \in H$,

$$\varphi[({}_zf_y\#h)({}_yg_x\#k)] = \varphi\left[\sum {}_zf_y(h_1 \cdot {}_yg_x)\sigma_x(h_2,k_1)\#h_3k_2\right] =$$
$$\sum {}_zf_y(h_1 \cdot {}_yg_x)\sigma_x(h_2,k_1)u_x(h_3k_2)\otimes h_4k_3 =$$
$$\sum {}_zf_y u_y(h_1)\, {}_yg_x u_x^{-1}(h_2)\sigma_x(h_3,k_1)u_x(h_4k_2)\otimes h_5k_3 =$$
$$\sum {}_zf_y u_y(h_1)\, {}_yg_x u_x(k_1)\tau_x(h_2,k_2)\otimes h_3k_3 =$$
$$\left[\sum {}_zf_y u_y(h_1)\otimes h_2\right]\left[\sum {}_yg_x u_x(k_1)\otimes k_2\right] =$$
$$\varphi({}_zf_y\#h)\varphi({}_yg_x\#k)$$

Since $A_\tau[H] \cong A\#_\sigma H$ as categories, the composition in $A_\tau[H]$ is associative and τ is a cocycle.

Now we give some necessary and sufficient conditions for the two crossed products to be isomorphic.

Theorem 1 Let A be a k-linear category and H be a Hopf algebra with two crossed product actions $h\otimes {}_yf_x \mapsto h \cdot {}_yf_x$ and $h\otimes {}_yf_x \mapsto h \triangleright {}_yf_x$ with respect to two cocycles σ, σ': $H\otimes H \to A$, respectively. Assume that

$$\varphi: A\#_\sigma H \to A\#'_{\sigma'} H$$

is an isomorphism of linear category, which is also a left A-module and a right H-comodule map. Then there is a collection of invertible maps $u = \{u_x\}_{x\in A_0} \in \mathrm{Hom}(H, A)$ such that for any ${}_yf_x \in {}_yA_x$, $h, k \in H$,

1) $\varphi({}_yf_x\#h) = \sum {}_yf_x u_x(h_1)\#'h_2$;

2) $h \triangleright {}_yf_x = \sum u_y^{-1}(h_1)(h_2 \cdot {}_yf_x)u_x(h_3)$;

3) $\sigma'_x(h,k) = \sum u_x^{-1}(h_1)[h_2 \cdot u_x^{-1}(k_1)]\sigma_x(h_3,k_2)u_x(h_4k_3)$.

Conversely given a collection of maps $u = \{u_x\}_{x\in A_0} \in \mathrm{Hom}(H, A)$ such that 2) and 3) hold, then the map φ in 1) is an isomorphism.

Proof Define $u_x \in \mathrm{Hom}(H, {}_xA_x)$ by $u_x(h) = (\mathrm{id}\otimes\varepsilon)\varphi(1_x\otimes h)$ for any $h \in H$. Then

$$(\mathrm{id}\otimes\varepsilon)\varphi({}_yf_x\#h) = (\mathrm{id}\otimes\epsilon)\{({}_yf_x\otimes 1)[\varphi(1_x\#h)]\} =$$
$${}_yf_x \circ u_x(h)$$

as φ is a left A-module map. Since φ is a right H-comodule map, we have

$$(\mathrm{id}\otimes\Delta)\circ\varphi = (\varphi\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)$$

Apply $\mathrm{id}\otimes\varepsilon\otimes\mathrm{id}$ to both sides of the equation. The left side becomes φ, and the right side becomes $[(\mathrm{id}\otimes\varepsilon)\circ\varphi\otimes\mathrm{id}]\circ(\mathrm{id}\otimes\Delta)$, which evaluated at ${}_yf_x\#h$ is

$$\sum (\mathrm{id}\otimes\varepsilon)\circ\varphi({}_yf_x\#h_1)\otimes h_2 = \sum {}_yf_x u_x(h_1)\otimes h_2$$

This proves 1).

Similarly, if $\varphi^{-1}: A\#'_\sigma H \to A\#_{\sigma'} H$ is an isomorphism satisfying the same hypotheses of φ, we may set $v_x(h) = (\mathrm{id}\otimes\varepsilon)\varphi^{-1}(1_x\otimes h)$ and conclude as above that $\varphi^{-1}({}_yf_x\#'h) = \sum {}_yf_x v_x(h_1)\#h_2$. We claim that $v = u^{-1}$.

$$1_x\#h = \varphi^{-1}\varphi(1_x\#h) = \varphi^{-1}\left[\sum u_x(h_1)\#'h_2\right] =$$
$$\sum u_x(h_1)v(h_2)\#h_3$$

Applying $\mathrm{id}\otimes\varepsilon$ to both sides, we obtain $\sum u_x(h_1)v_x(h_2) = \varepsilon(h)1_x$. Similarly we obtain $\sum v_x(h_1)u_x(h_2) = \varepsilon(h)1_x$, and thus $v = u^{-1}$.

Now the equation $\varphi^{-1}[({}_zg_y\#'h)({}_yf_x\#'k)] = \varphi^{-1}\cdot({}_zg_y\#'h)\varphi^{-1}({}_yf_x\#'k)$ becomes

$$\sum {}_zg_y(h_1 \triangleright {}_yf_x)\sigma'_x(h_2, k_1)v_x(h_3k_2)\#h_4k_3 =$$
$$\sum {}_zg_y v_y[(h_1)(h_2 \cdot {}_yf_x v_x(k_1))]\sigma_x(h_3,k_2)\#h_4k_3$$

Set $x = y = z$ and $f = g = 1_x$, and apply $\mathrm{id}\otimes\varepsilon$ to both sides, we obtain

$$\sum \sigma'_x(h_1, k_1)v_x(h_2k_2) = \sum v_x\{(h_1)[h_2 \cdot v_x(k_1)]\}\sigma_x(h_3, k_2)$$

This proves 3) after inverting $v_x(hk)$ and using $v = u^{-1}$.

Again using the above equation with $y = z$, $g = 1_z$ and $k = 1$, and applying $\mathrm{id}\otimes\varepsilon$ to both sides, we have

$$\sum (h_1 \triangleright {}_yf_x)v_x(h_2) = \sum v_y(h_1)(h_2 \cdot {}_yf_x)$$

Inverting v, we obtain 2).

The converse follows as in the proof of Proposition 1.

3 Generalized Maschke Theorem

Recall from Ref. [5] that for a crossed product category $A\#_\sigma H$, the family of maps γ_x: $H \to {}_xA_x\#_{\sigma_x}H$ given by $h \mapsto 1_x\#h$ is invertible in $\mathrm{Hom}(H, {}_xA_x\#_{\sigma_x}H)$. Then by the equation

$$(1_y\#h)({}_yf_x\#1) = \sum (h_1 \cdot f)\#h_2$$

denoting $f = f\#1$, we have

$$h \cdot f = \sum \gamma_y(h_1) f \gamma_x^{-1}(h_2) \quad (4)$$

for any $h \in H$ and $f \in {}_yA_x$.

Recall Ref. [8], let A be a k-linear category. A left module over A is a family of linear spaces $M=\{{}_xM\}_{x\in A_0}$ together with the structure maps $\cdot : {}_yA_x \otimes {}_xM \to {}_yM$ satisfying

$$ {}_zf_y \cdot ({}_yg_x \cdot m) = fg \cdot m \qquad 1_x \cdot m = m $$

A morphism of the left A-module is a family of linear maps $u=\{{}_xu : {}_xM \to {}_xN\}_{x\in A_0}$ satisfying

$$ {}_yu(f\cdot m) = f\cdot {}_xu(m) $$

for an ${}_yf \in {}_yA_x$ and $m \in {}_xM$.

Proposition 2 Let $A\#_\sigma H$ be a crossed product category for a finite dimensional, semisimple Hopf algebra H. If $V \in {}_{A\#_\sigma H}M$ and $W \subseteq V$ is a submodule such that W has a complement in ${}_AM$, then W has a complement in ${}_{A\#_\sigma H}M$.

Proof Let $\pi : V \to W$ be an A-projection with $\pi_x : {}_xV \to {}_xW$. Choose $t \in \int_H^r$ with $\varepsilon(t)=1$. Define $\tilde{\pi} : V \to W$ by

$$ \tilde{\pi}_x(v) = \sum \gamma_x^{-1}(t_1)\pi_x[\gamma_x(t_2)\cdot v] $$

for any $x \in A_0$ and $v \in {}_xV$.

First, for any $f \in {}_yA_x$ and $v \in {}_xV$

$$ \begin{aligned} \pi_y(f\cdot v) &= \sum \gamma_y^{-1}(t_1)\pi_y[\gamma_y(t_2)\cdot(f\cdot v)] = \\ &\sum \gamma_y^{-1}(t_1)\pi_y[(t_2\cdot f)\gamma_x(t_3)\cdot v] = \\ &\sum \gamma_y^{-1}(t_1)(t_2\cdot f)\pi_x[\gamma_x(t_3)\cdot v] = \\ &\sum f\gamma_y^{-1}(t_1)\pi_x[\gamma_x(t_2)\cdot v] = f\,\tilde{\pi}(v) \end{aligned} $$

where the second identity uses Eq. (4), and the third identity is obtained by π being A-linear.

To show that $\tilde{\pi}$ is also an H-map, first we have

$$ h\otimes\Delta(t) = \sum h_1 \otimes \Delta[t\varepsilon(h_2)] = \sum h_1 \otimes t_1h_2 \otimes t_2h_3 $$

Then for any $h \in H$ and $v \in {}_xV$.

$$ \begin{aligned} \tilde{\pi}[\gamma_x(h)\cdot v] &= \sum \gamma_x^{-1}(t_1)\pi_x[\gamma_x(t_2)\gamma_x(h)\cdot v] = \\ &\sum \gamma_x^{-1}(t_1)\pi_x[\sigma_x(t_2,h_1)\gamma_x(t_3h_2)\cdot v] = \\ &\sum \gamma_x^{-1}(t_1)\sigma_x(t_2,h_1)\pi_x[\gamma_x(t_3h_2)\cdot v] = \\ &\sum \gamma_x(h_1)\gamma_x^{-1}(t_1h_2)\pi_x[\gamma_x(t_2h_3)\cdot v] = \\ &\sum \gamma_x(h)\gamma_x^{-1}(t_1)\pi_x[\gamma_x(t_2)\cdot v] = \\ &\gamma_x(h)\tilde{\pi}_x(v) \end{aligned} $$

Thus $\tilde{\pi}$ is a left $A\#_\sigma H$-map. Finally, if $W \in {}_xW$, then it is easy to confirm that $\tilde{\pi}(w)=w$. The proof is completed.

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线性范畴交叉积等价及广义 Maschke 定理

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摘要:给出了 Hopf 代数与线性范畴 2 个不同交叉积之间等价的充要条件,并推广了 Maschke 定理. 基于经典 Hopf 代数的方法,首先设 A 为 k-线性范畴且 H 为 Hopf 代数,则 2 个交叉积 $A\#_\sigma H$ 与 $A\#'_{\sigma'}H$ 在某些条件下是同构的. 其次设 $A\#_\sigma H$ 为有限维半单 Hopf 代数 H 的交叉积范畴. 若 V 为左 $A\#_\sigma H$-模且 $W\subseteq V$ 为 V 的子模,W 作为左 A-模在 V 中有补,则 W 作为左 $A\#_\sigma H$-模在 V 中有补.

关键词:线性范畴;内作用;交叉积;广义 Maschke 定理

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