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Monoidal Hom-Hopf algebra on Hom-twisted product

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Abstract: Let (H, α) be a monoidal Hom-bialgebra and (B, β) be a left (H, α)-Hom-comodule coalgebra. The new monoidal Hom-algebra $B^{\#}_{\times} H$ is constructed with a Hom-twisted product $B_{\sigma}[H]$ and a $B \times H$ Hom-smash coproduct. Moreover, a sufficient and necessary condition for $B^{\#}_{\times} H$ to be a monoidal Hom-bialgebra is given. In addition, let (H, α) be a Hom-σ-Hopf algebra with Hom-σ-antipode S_H, and a sufficient condition for this new monoidal Hom-bialgebra $B^{\#}_{\times} H$ with the antipode S defined by $S(b \times h) = (1_B \times S_H(\alpha^{-1}(b_{(-1)}))) \cdot (S_B(b_{(0)}) \times 1_H)$ to be a monoidal Hom-Hopf algebra is derived.

Key words: monoidal Hom-Hopf algebra; Hom-twisted product; Hom-smash coproduct

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Caenepeel et al.[1] described Hom-structures from the point of view of monoidal categories and introduced monoidal Hom-algebras, monoidal Hom-coalgebras, monoidal Hom-Hopf algebras, etc. In 2013, Chen et al.[2] introduced integrals for monoidal Hom-Hopf algebras and Hom-smash products. Then, Liu et al.[3] constructed a Hom-smash coproduct, which generalized the classical smash coproduct and illustrated the category of Hom-Yetter-Drinfeld modules and proved that the category is a braided monoidal category. In Ref. [4], the authors constructed a Hom-crossed product, which generalizes the classical crossed product introduced in Refs. [5 − 7]. Can we give a new monoidal Hom-algebra construction of the special Hom-crossed product and the Hom-smash coproduct?

In this paper, we give a positive answer to the question above. We recall the basic definitions concerning monoidal Hom-Hopf algebras, and we find a sufficient and necessary condition for $B^{\#}_{\times} H$, with the Hom-twisted product $B_{\sigma}[H]$ and the Hom-smash coproduct $B \times H$ for a comodule coalgebra (B, β) over (H, α), to form a monoidal Hom-bialgebra and derive a sufficient condition for this new bialgebra to be a monoidal Hom-Hopf algebra.

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1 Preliminaries

Throughout this paper, we refer the readers to the books of Sweedler[8] for the relevant concepts on the general theory of Hopf algebras.

Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ denote the usual monoidal category of k-vector space and linear maps between them. Recall from Ref. [1] that there is the Hom-category $\widetilde{H}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), (k, \mathrm{id}), \tilde{a}, \tilde{l}, \tilde{r})$, a new monoidal category, associated with $\mathcal{M}_k$ as follows:

1) The objects of $\widetilde{H}(\mathcal{M}_k)$ are couples (M, μ), where $M \in \mathcal{M}_k$ and $\mu \in \mathrm{Aut}_k(M)$, the set of all k-linear automorphisms of M;

2) The morphism $f: (M, \mu) \to (N, \nu)$ is the k-linear map $f: M \to N$ in $\mathcal{M}_k$ satisfying $\nu \circ f = f \circ \mu$ for any two objects $(M, \mu), (N, \nu) \in \widetilde{H}(\mathcal{M}_k)$;

3) The tensor product is given by $(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$, for any objects $(M, \mu), (N, \nu) \in \widetilde{H}(\mathcal{M}_k)$;

4) The tensor unit is given by (k, id);

5) $\tilde{a}$ is given by the formula: $\tilde{a}_{M,N,L} = a_{M,N,L} \circ ((\mu \otimes \mathrm{id}) \otimes \zeta^{-1}) = (\mu \otimes (\mathrm{id} \otimes \zeta^{-1})) \circ a_{M,N,L,}$ for any objects $(M, \mu), (N, \nu), (L, \zeta) \in \widetilde{H}(\mathcal{M}_k)$;

6) The left and right unit constraint $\tilde{l}$ and $\tilde{r}$ are given by $\tilde{l}_M = \mu \circ l_M = l_M \circ (\mathrm{id} \otimes \mu)$, $\tilde{r}_M = \mu \circ r_M = r_M \circ (\mu \otimes \mathrm{id})$, for all $(M, \mu) \in \widetilde{H}(\mathcal{M}_k)$.

A unital monoidal Hom-assciative algebra is a vector space A together with an element $1_A \in A$ and linear maps $m: A \otimes A \to A$; $a \otimes b \mapsto ab$, $\alpha \in \mathrm{Aut}_k(A)$ such that $\alpha(1_A) = 1_A$, $a1_A = 1_A a = \alpha(a)$, $\alpha(a)(bc) = (ab)\alpha(c)$, $\alpha(ab) = \alpha(a)\alpha(b)$, for all $a, b, c \in A$.

A counital monoidal Hom-coassociative coalgebra (C, γ) is a vector space C together with linear maps $\Delta: C \to C \otimes C$, $\Delta(c) = c_1 \otimes c_2$ and $\varepsilon: C \to k$ such that $\Delta(\gamma(c)) = \gamma(c_1) \otimes \gamma(c_2)$, $\gamma^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \gamma^{-1}(c_2)$, $c_1 \varepsilon(c_2) = \gamma^{-1}(c) = c_2 \varepsilon(c_1)$, $\varepsilon(\gamma(c)) = \varepsilon(c)$, for all $c \in C$.

A monoidal Hom-bialgebra $H = (H, \alpha, m, \eta, \Delta, \varepsilon)$ in Ref. [1] is a bialgebra in the monoidal category $\widetilde{H}(\mathcal{M}_k)$. This means that (H, α, m, η) is a monoidal Hom-algebra and $(H, \alpha, \Delta, \varepsilon)$ is a monoidal Hom-coalgebra such that Δ and ε are morphisms of algebras; i.e., for any $h, g \in H$, $\Delta(hg) = \Delta(h)\Delta(g)$, $\Delta(1_H) = 1_H \otimes 1_H$, $\varepsilon(hg) = \varepsilon(h)\varepsilon(g)$, $\varepsilon(1_H) = 1$.

A monoidal Hom-bialgebra (H, α) is called a monoidal Hom-Hopf algebra if there exists a morphism S: $H \to H$ in $\widetilde{H}(\mathcal{M}_k)$ such that $S * \mathrm{id} = \eta \circ \varepsilon = \mathrm{id} * S$.

Note that a monoidal Hom-Hopf algebra is a Hopf algebra in $\widetilde{H}(\mathcal{M}_k)$. Furthermore, the antipode of monoidal Hom-Hopf algebras has almost all of the properties of antipode of Hopf algebras such as $S(hg) = S(g)S(h)$, $S(1_H) = 1_H$, $\Delta(S(h)) = S(h_2) \otimes S(h_1)$, $\varepsilon \circ S = \varepsilon$.

S is a monoidal Hom-anti-(co) algebra homomorphism. Since α is bijective and commutes with antipode S, we can also have the inverse α^{-1} commuting with S, that is, $S \circ \alpha^{-1} = \alpha^{-1} \circ S$. In the following, we recall the actions and coactions on monoidal Hom-coalgebras.

Now let (C, γ) be a monoidal Hom-coalgebra. A left (C, γ)-Hom-comodule is an object (M, μ) in $\widetilde{H}(\mathcal{M}_k)$ together with a k-linear map ρ_M: $M \to M \otimes C$, $\rho_M(m) = m_{(-1)} \otimes m_{(0)}$ such that $\varepsilon(m_{(-1)}) m_{(0)} = \mu^{-1}(m)$, $\Delta_C(m_{(1)}) \otimes \mu^{-1}(m_{(0)}) = \gamma^{-1}(m_{(-1)}) \otimes (m_{(0)(-1)} \otimes m_{(0)(0)})$, $\rho_M(\mu(m)) = \gamma(m_{(-1)}) \otimes \mu(m_{(0)})$, for all $m \in M$.

In Ref. [3], a left (H, α)-Hom-comodule coalgebra (B, β) is a monoidal Hom-coalgebra, with a monoidal Hom-bialgebra (H, α) and a left (H, α)-Hom-comodule (B, β), obeying the following axioms: for all $b \in B$, $b_{(-1)} \otimes \Delta_B(b_{(0)}) = b_{1(-1)} b_{2(-1)} \otimes b_{1(0)} \otimes b_{2(0)}$, $b_{(-1)} \varepsilon(b_{(0)}) = \varepsilon(b) 1_H$.

Recall from Ref. [3] that there is the Hom-smash coproduct $(B \times H, \beta \times \alpha)$, with a left (H, α)-Hom-comodule (B, β) and the Hom-comultiplication given by $\Delta(b \times h) = (b_1 \times b_{2(-1)} \alpha^{-1}(h_1)) \otimes (\beta(b_{2(0)}) \times h_2)$, for all $b \in B$, $h \in H$.

Recalled from Ref. [4] that the Hom-crossed product $B \#_\sigma H$ of B with H is the vector space $B \otimes H$, with a monoidal Hom-Hopf algebra (H, α) and a monoidal Hom-algebra (B, β). H acts weakly on B and σ: $H \otimes H \to B$ is convolution invertible. The multiplication is given by $(a\#h)(b\#k) = a[(h_{11}\beta^{-2}(b))\sigma(h_{12}, \alpha^{-1}(k_1))] \# \alpha(h_2 k_2)$, for any h, $k \in H$ and a, $b \in B$.

2 Monoidal Hom-Bialgebra $B^{\#}_{\times} H$

If $B \#_\sigma H$ is a Hom-associative algebra with $1 \otimes 1$ as an identity element, then we call $B \#_\sigma H$ a Hom-crossed product. A necessary and sufficient condition for $B \#_\sigma H$ to be a Hom-crossed product is that σ satisfies the following conditions:

$$(\alpha(h_1)\sigma(1_1, \alpha(k_1)))\sigma(\alpha(h_2), l_2 k_2) = \sigma(h_1, l_1)\sigma(\alpha^{-1}(h_2 l_2), \alpha^{-1}(k)) \tag{1}$$

$$\sigma(h_1, l_1)(h_2 l_2 a) = (\alpha(h_1)(l_1 \beta^{-1}(a)))\sigma(\alpha(h_2), \alpha(l_2)) \tag{2}$$

$$\sigma(1, h) = \sigma(h, 1) = \varepsilon(h) 1_B \tag{3}$$

for all h, k, $l \in H$ and $a \in B$.

If the action " · " is trivial, that is, $ha = \varepsilon_H(h) 1_B a$, for every $h \in H$, $a \in B$, then we write $B \#_\sigma H = B_\sigma[H]$. In the following, we call the Hom-crossed product $B_\sigma[H]$ a Hom-twisted product, with multiplication:

$$(a\#h)(b\#k) = \beta^{-1}(ab)\sigma(h_1, k_1)\#\alpha(h_2 k_2)$$

If the action " · " is trivial, then Eqs. (1) and (2) are, respectively, substituted into

$$\sigma(\alpha(1_1), \alpha^2(k_1))\sigma(h, l_2 k_2) = \sigma(h_1, l_1)\sigma(\alpha^{-1}(h_2 l_2), \alpha^{-1}(k)) \tag{4}$$

$$\sigma(\alpha^{-1}(h), \alpha^{-1}(l))\beta(a) = \beta(a)\sigma(h, l) \tag{5}$$

for all h, k, $l \in H$ and $a \in B$.

Lemma 1 Let (B, β) be a monoidal Hom-algebra and (H, α) be a monoidal Hom-bialgebra, with a Hom-twisted product $B_\sigma[H]$ given by

$$(a\#h)(b\#k) = \beta^{-1}(ab)\sigma(h_1, k_1)\#\alpha(h_2 k_2)$$

Then we have the following conclusions:

1) The associativity of $B_\sigma[H]$ is satisfied if and only if Eqs. (4) and (5) hold.

2) $1_B \# 1_H$ is the unit of $B_\sigma[H]$ if and only if Eq. (3) holds.

3) $B_\sigma[H]$ is an associative algebra if and only if Eqs. (3) to (5) hold.

Let (H, α) be a monoidal Hom-bialgebra. Let $(B, \beta, \Delta_B, \varepsilon_B)$ be a left (H, α)-comodule coalgebra and (B, β, m_B, η_B) be a monoidal Hom-algebra. Let σ: $H \otimes H \to B$ be a linear map. In this section, we derive necessary and sufficient conditions for $B \otimes H$ to be a monoidal Hom-bialgebra.

If $(B \otimes H, \mu_{B\sigma[H]}, m_{B\sigma[H]}, \varepsilon_{B \times H}, \Delta_{B \times H})$ is a monoidal Hom-bialgebra, we say that the triple (H, B, σ) is admissible and denote this bialgebra by $B^{\#}_{\times} H$.

First, we have the following lemmas:

Lemma 2 Let $B_\sigma[H]$ be a Hom-twisted product and $B \times H$ be a Hom-smash coproduct. The following are equivalent:

$$\beta(b_1)\varepsilon(h) \otimes \beta(b_2) = b_1 \sigma(\alpha^{-2}(h), b_{2(-1)}) \otimes \beta^2(b_{2(0)}) \tag{6}$$

$$\alpha^{-1}(h) b_{(-1)} \otimes \beta(b_{(0)}) = b_{(-1)} \alpha^{-1}(h) \otimes \beta(b_{(0)}) \tag{7}$$

for all $b \in B$ and $h \in H$.

Lemma 3 Let $B_\sigma[H]$ be a Hom-twisted product and $B \times H$ be a Hom-smash coproduct. If the identity $\rho(1) = 1 \otimes 1$ holds, then the following are equivalent:

$$\sigma(h, l)_1 \otimes \sigma(h, l)_2 = \sigma(h_1, l_1) \otimes \sigma(h_2, l_2) \tag{8}$$

$$\sigma(h_1, l_1)_{(-1)} \alpha^{-1}(h_2 l_2) \otimes \beta(\sigma(h_1, l_1)_{(0)}) = h_1 l_1 \otimes \sigma(h_2, l_2) \tag{9}$$

Lemma 4 Let $B_\sigma[H]$ be a Hom-twisted product and

$B \times H$ be a Hom-smash coproduct. If $(\beta(a_1) \otimes \alpha(a_{2(-1)})\alpha^{-1}(h_1)) \otimes \beta^2(a_{2(0)}) = (a_1\sigma(\alpha(a_{2(-1)1}), \alpha^{-1}(h_1)) \otimes \alpha^2(a_{2(-1)2})h_2) \otimes \beta^2(a_{2(0)})$ holds, then we have

$$\beta(a_1)\varepsilon(h) \otimes \beta(a_2) = a_1\sigma(a_{2(-1)}, \alpha^{-2}(h)) \otimes \beta^2(a_{2(0)})$$

for all $b \in B$ and $h \in H$.

All of the above lemmas can be directly checked, so they are left to the readers.

Theorem 1 Let (H,α) be a monoidal Hom-bialgebra over a field k, and suppose that (B,β) is a left (H,α)-Hom-comodule coalgebra and a monoidal Hom-algebra with trivial action " · ". Suppose that $(B_\sigma[H], \beta\otimes\alpha)$ is a Hom-twisted product with a convolution invertible morphism σ and $(B \times H, \beta \times \alpha)$ is a Hom-smash coproduct. Then the following conditions are equivalent:

1) $B^{\#}_{\times}H$ is a Hom-bialgebra.

2) The conditions hold for all a, $b \in B$, h, $l \in H$.

① σ is a coalgebra map;

② ε_B is an algebra map;

③ $\Delta(ab) = \beta^{-1}(a_1b_1)\sigma(a_{2(-1)}, b_{2(-1)}) \otimes \beta(a_{2(0)}, b_{2(0)})$;

④ $\beta(b_1)\varepsilon(h) \otimes \beta(b_2) = b_1\sigma(\alpha^{-2}(h), b_{2(-1)}) \otimes \beta^2(b_{2(0)})$;

⑤ $\alpha^{-1}(h)b_{(-1)} \otimes \beta(b_{(0)}) = b_{(-1)}\alpha^{-1}(h) \otimes \beta(b_{(0)})$;

⑥ $\sigma(h_1, l_1)_{(-1)}\alpha^{-1}(h_2l_2) \otimes \beta(\sigma(h_1, l_1)_{(0)}) = h_1l_1 \otimes \sigma(h_2, l_2)$;

⑦ $\Delta_B(1_B) = 1_B \otimes 1_B$;

⑧ (B,β) is a left (H,α)-comodule algebra.

Proof 1)⇒2) follows from the Lemmas 1 to 4, so it remains to show that 2)⇒1). It is easy to check $\varepsilon((a \times h)(b \times k)) = \varepsilon(a \times h)\varepsilon(b \times k)$, $\varepsilon(1_B \times 1_H) = \varepsilon(1_k)$ and $\Delta(1_B \times 1_H) = (1_B \times 1_H) \otimes (1_B \times 1_H)$. In order to prove that $\Delta((a \times h)(b \times k)) = \Delta(a \times h) \cdot \Delta(b \times k)$, it is enough to show that for every a, $b \in B$, h, $g \in H$.

$$\Delta((a \times 1_H)(1_B \times h)) = \Delta(a \times 1_H)\Delta(1_B \times h) \qquad (10)$$

$$\Delta((a \times 1_H)(b \times 1_H)) = \Delta(a \times 1_H)\Delta(b \times 1_H) \qquad (11)$$

$$\Delta((1_B \times h)(b \times 1_H)) = \Delta(1_B \times h)\Delta(b \times 1_H) \qquad (12)$$

Indeed, we use (10) and (11) to compute:

$$\begin{aligned}\Delta((a \times 1_H)(b \times g)) &= \Delta((a \times 1_H)((\beta^{-1}(b) \times 1_H) \cdot (1_B \times \alpha^{-1}(g)))) = \\ &\Delta(\beta^{-1}(ab) \times 1_H)\Delta(1_B \times g) = \\ &\Delta(a \times 1_H)(\Delta(\beta^{-1}(b) \times 1_H)\Delta(1_B \times \alpha^{-1}(g))) = \\ &\Delta(a \times 1_H)\Delta(b \times g)\end{aligned}$$

This shows that

$$\Delta((a \times 1_H)(b \times g)) = \Delta(a \times 1_H)\Delta(b \times g) \quad (13)$$

Similarly, we can obtain

$$\Delta((a \times h)(1_B \times g)) = \Delta(a \times h)\Delta(1_B \times g) \quad (14)$$

Now, we use Eqs. (12), (13) and (14):

$$\begin{aligned}&\Delta((a \times h)(b \times g)) = \Delta(((\beta^{-1}(a) \times 1_H) \cdot \\ &(1_B \times \alpha^{-1}(h)))(\beta^{-1}(b) \times 1_H)(1_B \times \alpha^{-1}(g)))) = \\ &\Delta(a \times 1_H)\Delta((1_B \times \alpha^{-1}(h)) \cdot \\ &((\beta^{-2}(b) \times 1_H)(1_B \times \alpha^{-2}(g))) = \\ &\Delta(a \times 1_H)(\Delta((1_B \times \alpha^{-2}(h)) \cdot \\ &(\beta^{-2}(b) \times 1_H))\Delta(1_B \times \alpha^{-1}(g))) = \\ &\Delta(a \times 1_H)(\Delta(1_B \times \alpha^{-1}(h)) \cdot \\ &(\Delta(\beta^{-2}(b) \times 1_H)\Delta(1_B \times \alpha^{-2}(g)))) = \\ &\Delta((\beta^{-1}(a) \times 1_H)(1_B \times \alpha^{-1}(h))) \cdot \\ &\Delta((\beta^{-1}(b) \times 1_H)(1_B \times \alpha^{-1}(g))) = \\ &\Delta(a \times h)\Delta(b \times g)\end{aligned}$$

Following this, we will show that Eqs. (10) to (12) are true.

By Lemma 4, we see that

$$\begin{aligned}\Delta(a \times 1_H)\Delta(1_B \times h) &= [(a_1 \times a_{2(-1)}\alpha^{-1}(1_H)) \otimes \\ &(\beta(a_{2(0)}) \times 1_H)][(1_B \times 1_H\alpha^{-1}(h_1)) \otimes (\beta(1_B) \times h_2)] = \\ &(a_1\sigma(\alpha(a_{2(-1)}), h_{11}) \times \alpha(\alpha(a_{2(-1)})_2h_{12})) \otimes \\ &(\beta(a_{2(0)})\sigma(1_H, h_{21}) \times \alpha^2(h_{22})) = \\ &(\beta(a_1) \times \alpha(a_{2(-1)})h_1) \otimes (\beta^2(a_{2(0)}) \times \alpha(h_2)) = \\ &\Delta(\beta(a) \otimes \alpha(h)) = \Delta((a \times 1_H)(1_B \times h))\end{aligned}$$

Also, Eq. (10) is proved. We use ③ and ⑧:

$$\begin{aligned}\Delta(a \times 1_H)\Delta(b \times 1_H) &= [(a_1 \times \alpha(a_{2(-1)})) \cdot \\ &(b_1 \times \alpha(b_{2(-1)}))] \otimes [(\beta(a_{2(0)}) \times 1_H)(\beta(b_{2(0)}) \times 1_H)] = \\ &((ab)_1 \times \alpha((ab)_{2(-1)})) \otimes (\beta((ab)_{2(0)}) \times 1_H) = \\ &\Delta(ab \times 1_H) = \Delta((a \times 1_H)(b \times 1_H))\end{aligned}$$

and (11) is proved. We use ④ and ⑤:

$$\begin{aligned}\Delta(1_B \times h)\Delta(b \times 1_H) &= [(1_B \times h_1) \otimes (1_B \times h_2)] \cdot \\ &[(b_1 \times b_{2(-1)}1_H) \otimes (\beta(b_{2(0)}) \times 1_H)] = \\ &(\beta(b_1) \times \alpha(b_{2(-1)})h_1) \otimes (\beta^2(b_{2(0)}) \times \alpha(h_2)) = \\ &\Delta(\beta(b) \otimes \alpha(h)) = \Delta((1_B \times h)(b \times 1_H))\end{aligned}$$

and (12) is proved.

Thus we have $\Delta((a \times h)(b \times g)) = \Delta(a \times h)\Delta(b \times g)$, and $(B^{\#}_{\times}H, \beta\otimes\alpha)$ is a monoidal Hom-bialgebra. The proof is completed.

Definition 1 Let (H,α) be a monoidal Hom-bialgebra, (B,β) an monoidal Hom-algebra, σ: $H\otimes H \to B$ a linear map and S_H: $H \to H$ a linear map. S_H is called a Hom-σ-antipode of (H,α) if

$$(1\otimes\alpha)(\sigma\otimes m_H)\Delta_{H\otimes H}(1\otimes S_H)\Delta(h) = \varepsilon(h)(1_B \times 1_H)$$

and

$$(1\otimes\alpha)(\sigma\otimes m_H)\Delta_{H\otimes H}(S_H\otimes 1)\Delta(h) = \varepsilon(h)(1_B \times 1_H)$$

hold for every $h \in H$. In this case, we say that (H,α) is a Hom-σ-Hopf algebra.

Proposition 1 Let $(B^{\#}_{\times}H, \beta\otimes\alpha)$ be a monoidal Hom-bialgebra. If (H,α) is a Hom-σ-Hopf algebra with Hom-

σ-antipode S_H and S_B: $B \to B$ in $\widetilde{H}(\mathcal{M}_k)$ is a convolution invertible element of $I_B \in \mathrm{Hom}(B, B)$, then $(B^{\#}_{\times} H, \beta \otimes \alpha)$ is a monidal Hom-Hopf algebra with the antipode S defined by

$$S(b \times h) = (1_B \times S_H(\alpha^{-1}(b_{(-1)})))(S_B(b_{(0)}) \times 1_H)$$

for all $b \in B$ and $h \in H$.

Proof We need to prove $S * \mathrm{id} = \eta \circ \varepsilon = \mathrm{id} * S$ holds. For all $b \times h \in B^{\#}_{\times} H$, we compute

$$\begin{aligned}
&(\mathrm{id} * S)(b \times h) = m(\mathrm{id} \otimes S)\Delta(b \times h) = \\
&m(\mathrm{id} \otimes S)((b_1 \times b_{2(-1)}\alpha^{-1}(h_1)) \otimes (\beta(b_{2(0)}) \times h_2)) = \\
&(b_1 \times b_{2(-1)}\alpha^{-1}(h_1))[(1_B \times S_H(b_{2(0)(-1)}\alpha^{-2}(h_2)) \cdot \\
&(S_B(\beta(b_{2(0)})_{(0)}) \times 1_H)] = \\
&(\beta^{-1}(\beta^{-1}(b_1 1_B)\sigma((\alpha^{-1}(b_{2(-1)}\alpha^{-1}(h_1)))_1, \\
&S_H(b_{2(0)(-1)}\alpha^{-2}(h_2))_1)) \times \\
&\alpha((\alpha^{-1}(b_{2(-1)}\alpha^{-1}(h_1)))_2) \cdot \\
&S_H(b_{2(0)(-1)}\alpha^{-2}(h_2))_2)) \cdot \\
&(S_B(\beta^2(b_{2(0)(0)})) \times 1_H) = \\
&(\beta^{-1}(b_1)\sigma((b_{2(-1)1}\alpha^{-1}(h_1))_1, \\
&(S_H(b_{2(-1)2}\alpha^{-2}(h_2)))_1) \times \alpha((b_{2(-1)1}\alpha^{-1}(h_1))_2) \cdot \\
&S_H(b_{2(-1)2}\alpha^{-2}(h_2))_2))(S_B(\beta(b_{2(0)})) \times 1_H) = \\
&(b_1 \times 1_H)(S_B(b_{2(0)}) \times 1_H)\varepsilon(h) = \\
&(1_B \times 1_H)\varepsilon(b)\varepsilon(h) = (1_B \times 1_H)\varepsilon(b \times h)
\end{aligned}$$

A similar proof shows that $S * \mathrm{id} = \eta \circ \varepsilon$. This concludes our proof.

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Hom-扭曲积上的 monoidal Hom-Hopf 代数

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摘要:设(H,α)是 monoidal Hom-Hopf 代数,(B,β)是左(H,α)-Hom-余模余代数.构造了由 Hom-扭曲积 $B_\sigma[H]$和 Hom-冲余积 $B \times H$ 构成的新 monoidal Hom-代数 $B^{\#}_{\times}H$.并给出了 $B^{\#}_{\times}H$ 成为 monoidal Hom-双代数的充分必要条件 $B^{\#}_{\times}H$.此外,设(H,α)是带有 Hom-σ-反对极 S_H 的 Hom-σ-Hopf 代数,并找到此 monoidal Hom-双代数 $B^{\#}_{\times}H$ 带有定义为 $S(b \times h) = (1_B \times S_H(\alpha^{-1}(b_{(-1)})))(S_B(b_{(0)}) \times 1_H)$的反对极 S 成为 monoidal Hom-Hopf 代数的充分条件.

关键词:monoidal Hom-Hopf 代数; Hom-扭曲积; Hom-冲余积

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