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Construction of new bornological quantum groups based on Galois objects

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Abstract: Let A be a bornological quantum group and R a bornological algebra. If R is an essential A-module, then there is a unique extension to $M(A)$-module with $1x = x$. There is a one-to-one corresponding relationship between the actions of A and the coactions of $\hat{A}$. If R is a Galois object for A, then there exists a faithful δ-invariant functional on R. Moreover, the Galois objects also have modular properties such as algebraic quantum groups. By constructing the comultiplication Δ, counit ε, antipode S and invariant functional φ on $\hat{R}\hat{\otimes}\hat{R}$, $\hat{R}\hat{\otimes}\hat{R}$ can be considered as a bornological quantum group.

Key words: bornological quantum groups; actions and coactions; Galois theory; Galois objects

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In 1994, van Daele first introduced the concept of multiplier Hopf algebra[1] and studied algebraic quantum groups[2]. An algebraic quantum group is a multiplier Hopf algebra with invertible antipode equipped with a Haar integral. The basic example of multiplier Hopf algebra is the algebra of complex functions with finite support for a group. In order to include more examples such as smooth convolution algebras of Lie groups, Voigt[3] introduced the concept of a bornological quantum group. Moreover, van Daele and Wang[4] generalized it to the bornological quantum hypergroups case. Note that bornological quantum groups are considered over the bornological vector spaces. The bornological vector space is very important when studying various problems in noncommutative geometry and cyclic homology[5-6].

Galois objects play an important role in the operator algebra framework and they provide equivalences of certain categories. Motivated by the theory, de Commer[7] developed the theory of the Galois objects for algebraic quantum groups. So, it is natural to consider the Galois objects for bornological quantum groups.

As a generalization of the theory in Ref. [7 – 8]. We study the (co) action on bornological quantum groups, and construct the bornological quantum groups through the Galois objects. The algebras in this paper are over the field C of the complex numbers and the Sweelder notion is used for the coproduct. For two completed bornological vector spaces V and W, the tensor product is denoted by $V\hat{\otimes} W$.

1 Actions and Coactions of Bornological Quantum Groups

Definition 1 A bornological quantum group is an essential bornological algebra A satisfying the approximation property together with a comultiplication Δ: $A \rightarrow M(A\hat{\otimes}A)$ such that all Galois maps associated to Δ are isomorphisms and a faithful left invariant functional φ: $A \rightarrow C$.

A morphism between bornological quantum groups A and B is an essential algebra homomorphism f: $A \rightarrow B$ such that $(f\hat{\otimes}f)\Delta = \Delta f$.

Definition 2 Let A be a bornological quantum group. An essential A-module is an A-module X such that the module action induces a bornological isomorphism $A\hat{\otimes}_A X \cong X$.

Dually, we have the concept of an essential comodule.

Let A be a bornological quantum group. Assume that R is a bornological algebra over C probably without a unit but with non-degenerate product.

Proposition 1 Let R be an essential A-module. If $x \in R$ and $ax = 0$ for all $a \in A$, then $x = 0$.

Proposition 2 Let R be an essential A-module, then there is a unique extension to a left $M(A)$-module and $1x = x$ where $1 \in M(A)$.

Proof It is very natural to define $m(ax) = (ma)x$ for all $x \in R$, $a \in R$ and $m \in M(A)$. Since R is essential, we have $1x = x$ for all x. The action is well-defined. Assume that $\sum a_i x_i = 0, x_i \in R, a_i \in R$. Choose $e \in A$ such that $ea_i = a_i$ for all i. For any $m \in M(A)$, we have

$$\sum m(a_i x_i) = \sum (ma_i)x_i = \sum (me)(a_i x_i) = (me)\sum a_i x_i = 0$$

Therefore, we can define the action of $M(A)$ by $m(ax) = (ma)x$.

Proposition 3 Let A be a bornological quantum group. If we denote M as the category of essential left A-modules and morphisms, then M is a monoidal category with unit.

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The unit is C, and the module structure over C is $ac = \varepsilon(a)c$ for $a \in A$ and $c \in C$.

Definition 3 Let R be an essential A-module. We say that R is a left A-module algebra if $a(xy) = \sum(a_{(1)}x)(a_{(2)}y)$ for all $a \in A$ and $x, y \in R$.

Proposition 4 Let R be a left A-module algebra. We define a multiplication on $R\hat{\otimes}A$ by $(x \otimes a)(y \otimes b) = \sum x(a_{(1)}y) \otimes a_{(2)}b$ for all $x, y \in R$ and $a, b \in A$. Then $R\hat{\otimes}A$ is an essential bornological algebra.

Definition 4 Let A be a bornological quantum group and R is a bornological algebra. Γ is called the coaction of A on R if there is an essential injective homomorphism Γ: $R \to M(R\hat{\otimes}A)$ satisfying

1) $\Gamma(R)(1\otimes A) \subseteq R\hat{\otimes}A$ and $(1\otimes A)\Gamma(R) \subseteq R\hat{\otimes}A$;

2) $(\Gamma\hat{\otimes}\mathrm{id})\Gamma = (\mathrm{id}\hat{\otimes}\Delta)\Gamma$.

Γ is called reduced if $(R\otimes 1)\Gamma(R) \subseteq R\hat{\otimes}A$. If Γ is reduced, we also have $\Gamma(R)(R\otimes 1) \subseteq R\hat{\otimes}A$. In this case, R is called an A-comodule algebra.

Proposition 5 Let (A, Δ) be a bornological quantum group. If R is an A-comodule algebra, then R is an $\hat{A}$-module algebra.

Proof The action of $\hat{A}$ on R is defined by $b \cdot x = (\mathrm{id}\hat{\otimes}\varphi)((1\otimes a)\Gamma(x))$ for all $x \in R$, $b = \varphi(a\,\cdot)$, where $a \in A$. Then we need to check $(ab)\cdot x = a\cdot(b\cdot x)$ and $a\cdot(xy) = \sum(a_{(1)}\cdot x)(a_{(2)}\cdot y)$, where $a, b \in \hat{A}$. The rest proof is standard and the essentialness is easy to check.

Proposition 6 Let (A, Δ) be a bornological quantum group. If R is an A-module algebra, then R is an $\hat{A}$-comodule algebra.

Proof The coaction here is defined as

$$\Gamma(r)(1 \otimes b) = \sum S^{-1}(a_{(1)}) \cdot r \otimes \varphi(\cdot\, a_{(2)})$$

$$(1 \otimes b')\Gamma(r) = \sum S^{-1}(a'_{(2)}) \cdot r \otimes \psi(\cdot\, a'_{(1)})$$

where $b = \varphi(\cdot\, a)$ and $b' = \psi(\cdot\, a')$ for $a, a' \in A$. With this coaction, R is an $\hat{A}$-comodule algebra.

Theorem 1 Let A be a bornological quantum group and R a bornological algebra. R is A-module algebra if and only if R is an $\hat{A}$-comodule algebra.

2 Galois Objects and Main Constructions

Definition 5 Let A be a bornological quantum group and Γ is a coaction of A on a bornological algebra R. An element $f \in M(R)$ is coinvariant if $\Gamma(f) = f\otimes 1$. Let $R^{\mathrm{co}A}$ be the set of all coinvariants in $M(R)$, and $R^{\mathrm{co}A}$ is a unital subalgebra.

Definition 6 Let A be a bornological quantum group, and R is a bornological algebra. Then, the coaction Γ defined above is called Galois coaction if Γ is reduced and the map

$$V: R\hat{\otimes}_{R^{\mathrm{co}A}}R \to R\hat{\otimes}A: x\otimes y \to (x\otimes 1)\Gamma(y)$$

is bijective.

Definition 7 Let Γ be a right Galois coaction of a bornological quantum group (A, Δ) on R. Then (R, Γ) is called a right Galois object for A if $R^{\mathrm{co}A}$ is the scalar field.

Theorem 2 Let R be a right A-Galois object. There exists a faithful δ-invariant functional φ_R on R such that $(\mathrm{id}\hat{\otimes}\varphi)(\Gamma(r)) = \varphi_R(r)1$ for all $r \in R$. Moreover, there exists a non-zero invariant functional ψ_R on R.

Proof For all $r, s \in R$ and $a \in A$, let $x = (\mathrm{id}\hat{\otimes}\varphi)\cdot(\Gamma(r))$. We compute

$$\begin{aligned}\Gamma(x)(\Gamma(s)(1\otimes a)) &= \Gamma(xs)(1\otimes a) = \\ &(\mathrm{id}\hat{\otimes}\mathrm{id}\hat{\otimes}\varphi)((\Gamma\hat{\otimes}\mathrm{id})(\Gamma(r)(s\otimes 1))(1\otimes a\otimes 1)) = \\ &(\mathrm{id}\hat{\otimes}\mathrm{id}\hat{\otimes}\varphi)((\mathrm{id}\hat{\otimes}\Delta)(\Gamma(r))(s_{(0)}\otimes s_{(1)}a\otimes 1)) = \\ &r_{(0)}s_{(0)}\otimes\varphi(r_{(1)})s_{(1)}a = (x\otimes 1)(\Gamma(s)(1\otimes a))\end{aligned}$$

Since R is a Galois object, we have $\Gamma(x) = x\otimes 1 = \varphi_R(r)$ for some scalar $\varphi_R(r)$. So, we have defined a bounded linear functional φ_R on R. It is easy to obtain δ-invariance and faithfulness. Setting $\psi_R^{r'}(r) = \varphi_R(r_{(0)}r')\psi(r_{(1)})$, here r' is the element such that $\psi_R^{r'}$ is non-zero.

Proposition 7 Let A be a bornological quantum group and R is an A-Galois object. φ_R and ψ_R are defined in Theorem 2. Then for all $r, s \in R$, we have

1) There exists a unique invertible element $\delta_R \in M(A)$ such that $\varphi_R(r\delta_R) = \psi_R(r)$;

2) There exists a unique bounded algebra automorphism σ of R such that $\varphi_R(r\sigma(s)) = \varphi_R(sr)$. We call σ the modular automorphism.

Now, we construct a new bornological quantum group denoted by (X, Δ_X) which is spanned by $[\rho, \rho^1]_X$ for $\rho, \rho^1 \in \hat{R}$. Note that $\hat{R} = \{\varphi_R(\cdot\, r) \mid r \in R\}$ and $[\rho, \rho^1]_{\hat{A}}(a) = (\rho\hat{\otimes}\rho^1)(\beta(a))$, $\beta(a)$ is the element in $M(R\hat{\otimes}R)$ satisfying $(r\otimes 1)\beta(a) = V^{-1}(r\otimes a)\beta(a)(1\otimes r)$. Since there is a natural bijection between $\hat{R}\hat{\otimes}_{\hat{A}}\hat{R}$ and X, we mainly consider space X.

Define the multiplication on X as $x\cdot[\rho, \rho^1]_X = [x\cdot\rho, \rho^1]_X$ for $\rho, \rho^1 \in \hat{R}$ and $x \in X$. The associativity is straightforward.

The comultiplication Δ_X: $X \to M(X\hat{\otimes}X)$ is defined as

$$\Delta_X([\rho, \rho^1]_X) = [\rho^{(1)}, \rho^{1(2)}]\otimes[\rho^{(2)}, \rho^{1(1)}]$$

for $\rho, \rho^1 \in \hat{R}$.

The essential algebra homomorphism ε_X: $X \to C$ is defined as $\varepsilon_X([\rho, \rho^1]_X) = \rho^{(1)}\rho^1(1)$.

The antipode S_X is defined as S_X: $X \to X$: $[\rho, \rho^1]_X \to [\varphi_{\hat{R}}(\rho^1), \rho]_X$ for $\rho, \rho^1 \in \hat{R}$. Remember that $\varphi_{\hat{R}}(\rho) = \rho\circ\varphi_R$, $\delta_{\hat{A}}$ is the modular element in the dual $\hat{A}$ and σ_R is the modular automorphism of R.

Finally, given a map ν: $\hat{R} \to R$: $\psi(\cdot\, r) \to r$, the functional φ_X: $X \to C$: $[\rho, \rho^1]_X \to \rho(\nu(\rho^1))$ is a left invariant functional.

Theorem 3 Together with the maps $\Delta_X, \varepsilon_X, S_X, \varphi_X$, X is a bornological quantum group.

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基于 Galois 对象的新 bornological 量子群的构造

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摘要：设 A 为 bornological 量子群，R 为 bornological 代数．如果 R 为 essential A-模，那么 R 可以扩张为 $M(A)$-模并且满足 $1x=x$．A 上的作用与 $\hat{A}$ 上的余作用之间有一个一一对应的关系．若 R 是 A 上的 Galois 对象，则 R 上存在一个忠实的 δ-不变泛函，且拥有类似于代数量子群的 modular 性质．最后，通过构造 $\hat{R}\otimes\hat{R}$ 上的余乘 Δ、余单位 ε、对极 S 和不变泛函 φ，使之成为 bornological 量子群．

关键词：bornological 量子群；作用和余作用；Galois 理论；Galois 对象
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