

Characterizations of EP, normal and Hermitian elements in rings using generalized inverses

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Abstract: The properties and some equivalent characterizations of equal projection (EP), normal and Hermitian elements in a ring are studied by the generalized inverse theory. Some equivalent conditions that an element is EP under the existence of core inverses are proposed. Let $a \in R^{\oplus}$, then a is EP if and only if $aa^{\oplus}a^{\#} = a^{\#}aa^{\oplus}$. At the same time, the equivalent characterizations of a regular element to be EP are discussed. Let $a \in R$, then there exist $b \in R$ such that $a = aba$ and a is EP if and only if $a \in R^{\oplus}$, $a^{\oplus} = a^{\oplus}ba$. Similarly, some equivalent conditions that an element is normal under the existence of core inverses are proposed. Let $a \in R^{\oplus}$, then a is normal if and only if $a^*a^{\oplus} = a^{\oplus}a^*$. Also, some equivalent conditions of normal and Hermitian elements in rings with involution involving powers of their group and Moore-Penrose inverses are presented. Let $a \in R^{\oplus} \cap R^{\#}$, $n \in N$, then a is normal if and only if $a^*a^{\dagger}(a^{\#})^n = a^{\#}a^*(a^{\dagger})^n$. The results generalize the conclusions of Mosić et al.

Key words: equal projection (EP) elements; normal elements; Hermitian elements; core inverse; Moore-Penrose inverse; group inverse

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Let R be an associative ring with unity. An involution $*$ is an anti-automorphism of degree 2 in R , i. e., $(a^*)^* = a$, $(a + b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for all $a, b \in R$. Let R be a ring with involution and $a \in R$, we say that $x \in R$ is the Moore-Penrose inverse (MP-inverse for short) of a , if the following equations hold:

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad (xa)^* = xa$$

Under this condition, x is unique if it exists, which is denoted by a^{\dagger} . The set of all MP-invertible elements of R is denoted by R^{\dagger} . If a satisfies the equation $axa = a$, then, we state that a is regular. An element $a \in R$ has a

group inverse if there exists $x \in R$ such that

$$axa = a, \quad xax = x, \quad ax = xa$$

Then x is called a group inverse of a and it is unique when it exists, denoted by $a^{\#}$. We use $R^{\#}$ to represent the set of all group invertible elements of R .

In Ref. [1], Baksalary and Trenkler introduced the core inverse of complex matrices. Later, Rakić et al. [2] generalized the core inverse of complex matrices to the case of elements in rings, and they used five equations to characterize the core inverse of elements, i. e., let $a \in R$, if there exist $x \in R$ satisfying

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad xa^2 = a, \quad ax^2 = x$$

then x is the core inverse of a . In this case, x is unique if it exists, denoted by a^{\oplus} . The set of all core invertible elements of R is denoted by R^{\oplus} . They also showed that each core invertible element is group invertible. It is clear to see that if $a \in R^{\dagger} \cap R^{\#}$, then, we have $a \in R^{\oplus}$, and the converse does not hold. Recently, Xu et al. [3] proved that $a^{\oplus} = x$ if and only if

$$(ax)^* = ax, \quad xa^2 = a, \quad ax^2 = x$$

An element $a \in R$ satisfying $a^*a = aa^*$ is called normal. If $a \in R$ such that $a^* = a$, then, a is called Hermitian (resp. symmetric). Let $n \in N$, an element $a \in R$ satisfying $a^*a^n = a^n a^*$ is called generalized normal. Any $a \in R$ satisfying $a^n = (a^*)^n$ is called generalized Hermitian. An element $a \in R$ is called EP (equal projection) if $a \in R^{\dagger} \cap R^{\#}$ and $a^{\dagger} = a^{\#}$. The set of all EP elements of R is denoted by R^{EP} .

In Refs. [4–8], the authors focused on EP matrices or EP linear operators on Banach or Hilbert spaces. Later, Mosić et al. [9] presented some characterizations, for example, elements of rings are EP by using their group inverses and MP-inverses, which extended the results in Refs. [4–7]. They showed that if $a \in R^{\dagger} \cap R^{\#}$, then, a is EP if and only if $aa^{\dagger}a^{\#} = a^{\#}aa^{\dagger}$. Motivated by them, we consider similar conditions for elements in rings in terms of their core inverses. For example, we prove that if $a \in R^{\oplus}$, then, a is EP if and only if $aa^{\oplus}a^{\#} = a^{\#}aa^{\oplus}$. Moreover, Mosić et al. [10] also considered the necessary and sufficient conditions for regular elements to be EP. They showed that a is EP with $a = aba$ if and only if $a \in$

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$R^\dagger \cap R^\#$, $a^\dagger = a^\dagger ba$. Inspired by them, we investigate similar cases for elements in rings by their core inverses. For example, we present that a is EP with $a = aba$ if and only if $a \in R^{\oplus}$, $a^{\oplus} = a^{\oplus} ba$.

For a long time, normal and Hermitian matrices, as well as normal and Hermitian linear operators on Banach or Hilbert spaces have attracted much attention^[4-7, 11-12]. In Ref. [13], Mosić et al. gave several equivalent conditions for elements in rings to be normal by using their group inverses and MP-inverses, which generalized the results in Refs. [4-7]. Mosić et al. proved that if $a \in R^\dagger$, then, a is normal if and only if $a \in R^\#$, $a^* a^\# = a^\# a^*$. Inspired by them, we consider the corresponding conditions for elements in rings by their core inverses. For example, we show that if $a \in R^{\oplus}$, then, a is normal if and only if $a^* a^{\oplus} = a^{\oplus} a^*$. Later, in Ref. [14], Mosić et al. also presented some equivalent characterizations for elements in rings to be generalized normal (resp. generalized Hermitian) involving powers of their group inverses and MP-inverses. Motivated by Refs. [13-14], we investigate some new characterizations for elements in rings to be normal (resp. Hermitian) by powers of their group inverses and MP-inverses. We prove that if $a \in R^\dagger \cap R^\#$, $n \in \mathbb{N}$, then, a is normal if and only if $a^* a^\dagger (a^\dagger)^n = a^\dagger a^* (a^\dagger)^n$.

Lemma 1^[15] Let $a \in R^\#$, $b \in R$ such that $ab = ba$. Then, $a^\# b = ba^\#$.

Lemma 2^[9] $a \in R^{\text{EP}}$ if and only if $a \in R^\dagger$ and $aa^\dagger = a^\dagger a$.

Lemma 3^[16] Let $a \in R^\dagger$ and $b \in R$. If $ab = ba$ and $a^* b = ba^*$, then, $a^\dagger b = ba^\dagger$.

Lemma 4^[2] Let $a \in R$. The following assertions are equivalent: 1) $a \in R^{\text{EP}}$; 2) $a \in R^{\oplus}$, $a^\# = a^{\oplus}$.

Lemma 5^[17] Let $a \in R$. Then the following statements are equivalent: 1) $a \in R^{\text{EP}}$; 2) $a \in R^\#$ and $Ra \subseteq Ra^*$; 3) $a \in R^\#$ and $Ra^* \subseteq Ra$.

In the following theorem, we use core inverses to characterize EP elements.

Theorem 1 Let $a \in R^{\oplus}$. Then, $a \in R^{\text{EP}}$ if and only if one of the following equivalent conditions holds:

1) $aa^{\oplus} a^\# = a^\# aa^{\oplus}$; 2) $aa^{\oplus} a^\# = a^\# a^{\oplus} a$; 3) $aa^{\oplus} a^\# = a^{\oplus} aa^\#$; 4) $a^\# a^{\oplus} = (a^\#)^2$; 5) $a^{\oplus} aa^\# = a^{\oplus}$; 6) $a^\# a^{\oplus} a = a^{\oplus}$.

Proof If $a \in R^{\text{EP}}$, by Lemma 4 it is easy to verify that conditions 1) to 6) hold. Conversely, to conclude that $a \in R^{\text{EP}}$, we show that one of the conditions of Lemma 4 or Lemma 5 holds.

1) The equality $aa^{\oplus} a^\# = a^\# aa^{\oplus}$ gives $aa^{\oplus} a = aaa^{\oplus}$ by Lemma 1. Then, $a = aa^{\oplus} a = aaa^{\oplus} = a(a^{\oplus})^* a^*$, which follows $Ra \subseteq Ra^*$. By Lemma 5, we have $a \in R^{\text{EP}}$.

2) Assume that $aa^{\oplus} a^\# = a^\# a^{\oplus} a$. Then, $Ra^* = Ra^{\oplus} = Raa^{\oplus} a^\# = Ra^\# a^{\oplus} a \subseteq Ra$, which implies $a \in R^{\text{EP}}$ by Lemma 5.

3) Since $aa^{\oplus} a^\# = a^{\oplus} aa^\#$, we have $aa^{\oplus} = (aaa^{\oplus}) a^\# = a(a^{\oplus} aa^\#) = aa^\#$. Thus, $a^{\oplus} = a^\#$. Then, $a \in R^{\text{EP}}$ by Lem-

ma 4.

4) Using the equality $a^\# a^{\oplus} = (a^\#)^2$, we obtain $a^{\oplus} = aa^\# a(a^{\oplus})^2 = aa^\# a^{\oplus} = a(a^\#)^2 = a^\#$. Then, $a \in R^{\text{EP}}$ by Lemma 4.

5) The assumption $a^{\oplus} aa^\# = a^{\oplus}$ gives $Ra^* = Ra^{\oplus} = Ra^{\oplus} aa^\# \subseteq Ra$. Thus, $a \in R^{\text{EP}}$ by Lemma 5.

6) It is similar to 5).

Now, we consider the equivalent conditions of an EP element a which satisfies $a = aba$, where $b \in R$.

Theorem 2 Let $a \in R$. Then, the following conditions are equivalent:

- 1) $a = aba$, $a \in R^{\text{EP}}$;
- 2) $a \in R^{\oplus}$, $a^* = a^* ba$;
- 3) $a \in R^{\oplus}$, $a^{\oplus} = a^{\oplus} ba$.

Proof 1) \Rightarrow 2). Assume that $a \in R^{\text{EP}}$, then, $a^{\oplus} = a^\#$ by Lemma 4. It follows that

$$\begin{aligned} a^* &= a^* aa^{\oplus} = a^* aa^\# = a^* a^\# a = a^* a^\# aba = \\ &= a^* aa^\# ba = a^* aa^{\oplus} ba = a^* ba \end{aligned}$$

2) \Rightarrow 3). Suppose that $a^* = a^* ba$, we have

$$\begin{aligned} a^{\oplus} &= a^{\oplus} aa^{\oplus} = a^{\oplus} (a^{\oplus})^* a^* = a^{\oplus} (a^{\oplus})^* a^* ba = \\ &= a^{\oplus} aa^{\oplus} ba = a^{\oplus} ba \end{aligned}$$

3) \Rightarrow 1). The assumption $a^{\oplus} = a^{\oplus} ba$ gives

$$aa^{\oplus} = aa^{\oplus} ba = aa^{\oplus} ba a^\# a = aa^{\oplus} a^\# a = aa^{\oplus} a(a^\#)^2 a = a^\# a$$

Therefore, we obtain $a^{\oplus} = a^\#$. Then, $a \in R^{\text{EP}}$ by the Lemma 4. Hence, it follows that $a = aba$.

Next, we investigate some results such that elements in rings are normal when their core inverses exist. Now, we begin with an auxiliary lemma.

Lemma 6 Let $a \in R^{\oplus}$ with $aa^* = a^* a$, then, $a \in R^{\text{EP}}$.

Proof Note that

$$\begin{aligned} aa^{\oplus} &= (a^\# aa^{\oplus} aa) a^{\oplus} = a^\# (a^{\oplus})^* a^* aaa^{\oplus} = \\ &= a^\# (a^{\oplus})^* aa^* aa^{\oplus} = a^\# (a^{\oplus})^* aa^* = \\ &= a^\# (a^{\oplus})^* a^* a = a^\# a \end{aligned}$$

Thus, we have $a^{\oplus} = a^\#$, which implies $a \in R^{\text{EP}}$ by Lemma 4.

In the following, motivated by Ref. [13], some characterizations of normal elements in rings are presented under the condition that $a \in R^{\oplus}$.

Theorem 3 Let $a \in R^{\oplus}$. Then, a is normal if and only if one of the following equivalent conditions holds:

- 1) $a^* a^{\oplus} = a^{\oplus} a^*$; 2) $a^{\oplus} a^* a = a^*$; 3) $aa^* a = a^* aa$;
- 4) $aa^* a^\# = a^*$; 5) $a^* a^{\oplus} a^\# = a^* a^* a^{\oplus}$; 6) $aa^* a^{\oplus} = a^* a^{\oplus} a$;
- 7) $a^* a^\# a^{\oplus} = a^{\oplus} a^* a^\#$.

Proof As $aa^* = a^* a$ and Lemma 6, we have $a \in R^{\text{EP}}$. Thus, it is easy to verify 1) to 7). Conversely, we show that a is normal.

1) As $a^* a^{\oplus} = a^{\oplus} a^*$, we have

$$(a^{\oplus} a)^* = a^* (a^{\oplus})^* = a^* aa^{\oplus} (a^{\oplus})^* =$$

$$\begin{aligned} a^* (a^{\oplus} a^2) a^{\oplus} (a^{\oplus})^* &= a^{\oplus} a^* a^2 a^{\oplus} (a^{\oplus})^* = \\ a^{\oplus} a a^{\oplus} a^* a^2 a^{\oplus} (a^{\oplus})^* &= a^{\oplus} a a^* a^{\oplus} a^2 a^{\oplus} (a^{\oplus})^* = \\ a^{\oplus} a a^* a a^{\oplus} (a^{\oplus})^* &= a^{\oplus} a a^* (a^{\oplus})^* = a^{\oplus} a (a^{\oplus} a)^* \end{aligned}$$

Then, we can obtain $a^{\dagger} = a^{\oplus}$. Thus, $a(a^{\dagger})^2 = a^{\dagger}$, $a^{\dagger} a^2 = a$, which implies $a^{\dagger} a = a^{\dagger} a a a^{\dagger} = a a^{\dagger}$. By Lemma 2, we have $a^{\#} = a^{\dagger} = a^{\oplus}$. Thus, $a^* a^{\oplus} = a^{\oplus} a^*$ yields $a^* a^{\#} = a^{\#} a^*$. Then, by Lemma 1 we have $a^* a = a a^*$.

2) From $a^{\oplus} a^* a = a^*$, we have $a^{\oplus} = a^{\#}$. Thus, $a^{\oplus} a^* a = a^*$, it follows that $a^* a = a a^*$.

3) The equality $a a^* a = a^* a a$ gives $a^{\#} a = a a^{\oplus}$. So, we obtain $a^{\oplus} = a^{\#}$. Thus, $a a^* a = a^* a a$ implies that $a a^* = a a^* a a^{\oplus} = a a^* a a^{\#} = a^* a a a^{\#} = a^* a$.

4) We can easily obtain that 3) holds.

5) Using $a^* a^{\oplus} a^{\#} = a^{\#} a^* a^{\oplus}$, we have $a a^{\#} a^* = a^*$. It follows that $a^{\#} a = a a^{\oplus}$. Thus, $a^{\oplus} = a^{\#}$. Hence, $a^* a^{\oplus} a^{\#} = a^{\#} a^* a^{\oplus}$ yields $a^* (a^{\#})^2 = a^{\#} a^* a^{\#}$. So, $a^* a^{\#} = a^{\#} a^*$. By Lemma 1, we have $a^* a = a a^*$.

6) By Lemma 1, we have 5) holds.

7) Since $a^* a^{\#} a^{\oplus} = a^{\oplus} a^* a^{\#}$, we have $a^{\oplus} = a^{\#}$. Then, $a^* a^{\#} a^{\oplus} = a^{\oplus} a^* a^{\#}$ implies $a a^* = a^* a$. Therefore, a is normal.

In Refs. [13–14], the authors showed that the equivalent conditions such that $a \in R$ is a normal element, are closely related to the powers of the group inverse and MP-inverse of a . Motivated by Ref. [13], in the following two theorems we present some new characterizations for elements in rings to be normal (resp. Hermitian) involving powers of their group inverses and MP-inverses.

Theorem 4 Let $a \in R^{\dagger}$, $n \in N$. Then a is normal if and only if $a \in R^{\#}$ and one of the following conditions holds:

- 1) $a^* a^{\dagger} (a^{\#})^n = a^{\#} a^* (a^{\dagger})^n$; 2) $a^* a^{\#} (a^{\dagger})^n = a^{\dagger} a^* (a^{\#})^n$;
- 3) $(a^{\dagger})^n a^* a^{\#} = (a^{\#})^n a^{\dagger} a^*$; 4) $(a^{\dagger})^n a^{\#} a^* = (a^{\#})^n a^* a^{\dagger}$;
- 5) $a^{\dagger} (a^{\#})^n a^* = (a^{\#})^n a^{\dagger} a^*$; 6) $a^{\dagger} a^* (a^{\#})^n = a^* (a^{\#})^n a^{\dagger}$.

Proof If a is a normal element, then $a^{\dagger} = a^{\#}$ by Lemma 3. We also have $a^* a^{\#} = a^{\#} a^*$ by Lemma 1. Thus, it is easy to verify that conditions 1) to 6) hold. Conversely, we prove a is normal.

1) Suppose that $a^* a^{\dagger} (a^{\#})^n = a^{\#} a^* (a^{\dagger})^n$, we have $a^{\dagger} a = a^{\#} a$. So, $a^{\dagger} = a^{\#}$. Therefore, $a^* a^{\dagger} (a^{\#})^n = a^{\#} a^* (a^{\dagger})^n$ yields $a^* (a^{\#})^{n+1} = a^{\#} a^* (a^{\#})^n$. Thus, $a a^* = a^* a$.

2) The equation $a^* a^{\#} (a^{\dagger})^n = a^{\dagger} a^* (a^{\#})^n$ implies $a a^{\dagger} = a a^{\#}$. So, we obtain $a^{\dagger} = a^{\#}$. Thus, $a^* a^{\#} (a^{\dagger})^n = a^{\dagger} a^* (a^{\#})^n$ becomes $a^* (a^{\#})^{n+1} = a^{\#} a^* (a^{\#})^n$, which follows $a a^* = a^* a$.

3) Assume that $(a^{\dagger})^n a^* a^{\#} = (a^{\#})^n a^{\dagger} a^*$, we have $a a^{\dagger} = a a^{\#}$. So, $a^{\dagger} = a^{\#}$.

Therefore, $(a^{\dagger})^n a^* a^{\#} = (a^{\#})^n a^{\dagger} a^*$ gives $(a^{\#})^n a^* a^{\#} = (a^{\#})^{n+1} a^*$. Thus, $a a^* = a^* a$.

4) Using the assumption $(a^{\dagger})^n a^{\#} a^* = (a^{\#})^n a^* a^{\dagger}$, we can obtain $a^{\dagger} a = a^{\#} a$. Then, we have $a^{\dagger} = a^{\#}$. Hence, $(a^{\dagger})^n a^{\#} a^* = (a^{\#})^n a^* a^{\dagger}$ implies $(a^{\#})^{n+1} a^* = (a^{\#})^n a^* a^{\#}$. So, $a a^* = a^* a$.

5) The equality $a^{\dagger} (a^{\#})^n a^* = (a^{\#})^n a^{\dagger} a^*$ gives $a^{\#} a =$

$a^{\dagger} a$. Then, we obtain $a^{\dagger} = a^{\#}$.

Therefore, $a^{\dagger} (a^{\#})^n a^* = (a^{\#})^n a^* a^{\dagger}$ yields $(a^{\#})^{n+1} a^* = (a^{\#})^n a^* a^{\#}$. So, $a a^* = a^* a$.

6) Since $a^{\dagger} a^* (a^{\#})^n = a^* (a^{\#})^n a^{\dagger}$, we obtain $a a^{\dagger} = a a^{\#}$. Then, we obtain $a^{\dagger} = a^{\#}$. Also, $a^{\dagger} a^* (a^{\#})^n = a^* (a^{\#})^n a^{\dagger}$ implies $a^* a^{\#} = a^{\#} a^*$. Therefore, we have $a^* a = a a^*$ by Lemma 1.

Corollary 1^[13] Let $a \in R^{\dagger}$. Then, a is normal if and only if $a \in R^{\#}$ and one of the following conditions holds:

- 1) $a^* a^{\dagger} a^{\#} = a^{\#} a^* a^{\dagger}$; 2) $a^* a^{\#} a^{\dagger} = a^{\dagger} a^* a^{\#}$; 3) $a^{\dagger} a^* a^{\#} = a^{\#} a^{\dagger} a^*$; 4) $a^{\dagger} a^{\#} a^* = a^{\#} a^{\dagger} a^*$.

Next, we consider the cases of Hermitian elements in rings.

Theorem 5 Let $a \in R^{\dagger}$ and $n \in N$. Then a is Hermitian if and only if $a \in R^{\#}$ and one of the following conditions holds:

- 1) $a (a^{\#})^n = a^* (a^{\dagger})^n$; 2) $a^* (a^{\dagger})^{n+1} = (a^{\#})^n$;
- 3) $a^* (a^{\#})^{n+1} = (a^{\#})^n$; 4) $a^* a^{\dagger} (a^{\#})^n = (a^{\#})^n$;
- 5) $(a^{\#})^n a^* a^{\#} = (a^{\dagger})^n$; 6) $a^{\#} a^* (a^{\#})^n = (a^{\dagger})^n$; 7) $(a^{\dagger})^n a = (a^{\#})^n a^*$; 8) $a^{\#} a^* a^{\dagger} = a^{\#}$; 9) $(a^*)^{n+1} + 1 a^{\#} = (a^*)^n$.

Proof Since Hermitian elements are normal, we have $a a^{\dagger} = a^{\dagger} a$ by Lemma 3. Using $a^* = a$, we can obtain $a^* a^{\dagger} = a^{\dagger} a^*$. Thus, it is easy to verify that conditions 1) to 9) hold. Conversely, we show that a is Hermitian.

1) Suppose that $a (a^{\#})^n = a^* (a^{\dagger})^n$, we have $a a^{\dagger} = a a^{\#}$. Thus, $a^{\dagger} = a^{\#}$. It follows that $a (a^{\#})^n = a^* (a^{\dagger})^n$ yields $a (a^{\#})^n = a^* (a^{\#})^n$, which implies $a = a^*$.

2) The equality $a^* (a^{\dagger})^{n+1} = (a^{\#})^n$ gives $a a^{\dagger} = a a^{\#}$. So, we obtain $a^{\dagger} = a^{\#}$. Therefore, $a^* (a^{\dagger})^{n+1} = (a^{\#})^n$ implies $a^* (a^{\#})^{n+1} = (a^{\#})^n$. Thus, we have $a^* = a$.

3) Using the assumption $a^* (a^{\#})^{n+1} = (a^{\#})^n$, we have $a^{\dagger} a = a^{\#} a$. Then, $a^{\dagger} = a^{\#}$. Therefore, by $a^* (a^{\#})^{n+1} = (a^{\#})^n$, we have $a^* = a$.

4) If $a^* a^{\dagger} (a^{\#})^n = (a^{\#})^n$, then, $a^{\dagger} a = a^{\#} a$. So, we obtain $a^{\dagger} = a^{\#}$. It follows that

$$a^* a^{\dagger} (a^{\#})^n = (a^{\#})^n \text{ yields } a^* (a^{\#})^{n+1} = (a^{\#})^n. \text{ Thus, } a^* = a.$$

5) From the equality $(a^{\#})^n a^* a^{\#} = (a^{\dagger})^n$, we have $a a^{\dagger} = a a^{\#}$. Thus, we obtain $a^{\dagger} = a^{\#}$. It follows that $(a^{\#})^n a^* a^{\#} = (a^{\dagger})^n$ turns into $(a^{\#})^n a^* a^{\#} = (a^{\#})^n$. So, $a^* = a$.

6) By $a^{\#} a^* (a^{\#})^n = (a^{\dagger})^n$, we obtain $a^{\#} a = a^{\dagger} a$. Then, $a^{\dagger} = a^{\#}$. Therefore, $a^{\#} a^* (a^{\#})^n = (a^{\dagger})^n$ gives $a^{\#} a^* (a^{\#})^n = (a^{\dagger})^n$. Thus, $a^* = a$.

7) Let $(a^{\dagger})^n a = (a^{\#})^n a^*$, we obtain $a^{\dagger} a = a^{\#} a$. Then, we deduce that $a^{\dagger} = a^{\#}$. Therefore, $(a^{\dagger})^n a = (a^{\#})^n a^*$ yields $(a^{\#})^n a = (a^{\#})^n a^*$. Thus, $a = a^n (a^{\#})^n a = a^n (a^{\#})^n a^* = a^*$.

8) The assumption $a^n a^* a^{\dagger} = a^n$ implies $a a^{\dagger} = a^{\#} a$. Then, $a^{\dagger} = a^{\#}$. Hence, $a^n a^* a^{\dagger} = a^n$ implies $a^n a^* a^{\#} = a^n$. Thus, $a^* = a$.

9) Using the equality $(a^*)^{n+1} a^{\#} = (a^*)^n$, we have $a a^{\dagger} = a a^{\#}$. Then, $a^{\dagger} = a^{\#}$. From $(a^*)^{n+1} a^{\#} = (a^*)^n$, it follows that $(a^*)^{n+1} = (a^*)^n a$. Thus, we can obtain $a =$

a^* .

Corollary 2^[13] Let $a \in R^\dagger$. Then, a is Hermitian if and only if $a \in R^\#$ and one of the following conditions holds:

- 1) $aa^\# = a^* a^\dagger$; 2) $a^* (a^\dagger)^2 = a^\#$; 3) $a^* (a^\#)^2 = a^\#$;
- 4) $a^* a^\dagger a^\# = a^\#$; 5) $a^\# a^* a^\# = a^\dagger$; 6) $a^\dagger a = a^\# a^*$;
- 7) $aa^* a^\dagger = a$; 8) $(a^*)^2 a^\# = a^*$.

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环上 EP 元、正规元和对称元的广义逆刻画

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摘要: 结合广义逆理论研究了环中平等投影 (EP) 元、正规元和对称元的性质和一些等价刻画. 给出了在核逆存在的情况下元素为 EP 元的一些等价条件. 设 $a \in R^\oplus$, 那么 a 是 EP 元当且仅当 $aa^\oplus a^\# = a^\# aa^\oplus$. 同时, 讨论了正则元是 EP 元的等价刻画. 设 $a \in R$, 那么存在 $b \in R$, 使得 $a = aba$ 且 a 是 EP 元当且仅当 $a \in R^\oplus, a^\oplus = a^\oplus ba$. 同样地, 给出了在核逆存在的情况下元素为正规元的一些等价条件. 设 $a \in R^\oplus$, 那么 a 是正规元当且仅当 $a^* a^\oplus = a^\oplus a^*$. 而且在群逆和 Moore-Penrose 逆存在的情况下给出了元素为正规元和对称元的一些涉及次数的等价条件. 设 $a \in R^\dagger \cap R^\#$, 且存在 $n \in N$, 那么 a 是正规元当且仅当 $a^* a^\dagger (a^\#)^n = a^\# a^* (a^\dagger)^n$. 结果推广了 Mosić 等人的结论.

关键词: EP 元; 正规元; 对称元; 核逆; Moore-Penrose 逆; 群逆

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