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Some additive results for the generalized Drazin inverse in a Banach algebra

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Abstract: Let a, b be two generalized Drazin invertible elements in a Banach algebra. An explicit expression for the generalized Drazin inverse of the sum $a+b$ in terms of a, b, a^d, b^d is given. The generalized Drazin inverse for the sum of two elements in a Banach algebra is studied by means of the system of idempotents. It is first proved that $a+b\in A^{qnil}$ under the condition that $a, b\in A^{qnil}$, $aba=0$ and $ab^2=0$ and then the explicit expressions for the generalized Drazin inverse of the sum $a+b$ under some new conditions are given. Also, some known results are extended.

Key words: generalized Drazin inverse; Banach algebra; nilpotent element; quasi-nilpotent element

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Throughout this paper, A denotes a unital Banach algebra with 1. For $a\in A$, we use $\sigma(a)$ to denote the spectrum of a. An element $a\in A$ is called quasi-nilpotent if the spectrum $\sigma(a)=\{0\}$. Let A^{-1}, A^{nil} and A^{qnil} denote the sets of all invertible, nilpotent and quasi-nilpotent elements in A, respectively. The Drazin inverse[1] of an element $a\in A$ is the element $x\in A$, which satisfies the following three equations:

$$ax=xa,\ \ xax=x,\ \ a-a^2x\in A^{nil}$$

The Drazin inverse of $a\in A$ is denoted by a^D if it exists and it is unique. The concept of the generalized Drazin inverse in a Banach algebra was introduced by Koliha[2]. An idempotent element $p\in A$ is a spectral idempotent of $a\in A$ if $ap=pa\in A^{qnil}$ and $a+p\in A^{-1}$. The element p above is unique if it exists and it is denoted as a^{π}. If a^{π} exists, the generalized Drazin inverse of an element $a\in A$ is defined as $a^d=(a+a^{\pi})^{-1}(1-a^{\pi})=(1-a^{\pi})(a+a^{\pi})^{-1}$. Let A^d denote the set of all generalized Drazin inverse elements in A. It is obvious that $a^{\pi}=1-aa^d$. From the definition, the generalized Drazin inverse of $a\in A$ is also characterized as the unique element $x\in A$ satisfying

$$ax=xa,\ \ xax=x,\ \ a-a^2x\in A^{qnil}$$

The Drazin inverse is first studied by Drazin[1] in associative rings and semigroups. The generalized Drazin inverse is investigated for rings by Hartwig[3] and for Banach algebras by Koliha[2]. The Drazin inverse and the generalized Drazin inverses and their applications are very important in various applied mathematical fields such as singular differential equations, singular difference equations, Markov chains, iterative methods and so on[4-6].

In 1958, Drazin[1] first studied the representation for the Drazin inverse of the sum of two Drazin invertible elements in a ring and proved that $(a+b)^D=a^D+b^D$ under the condition $ab=ba=0$. Later, Koliha[2] gave the representations of $(a+b)^d$ under the same condition in a Banach algebra. In 2001, Hartwig et al.[7] gave the formula $(P+Q)^D$ under the condition $PQ=0$. Cvetković-Ilić et al.[8] generalized the result of Ref. [7] to bounded linear operators in an arbitrary complex Banach space. In 2004, González and Koliha[9] gave the formula for $(a+b)^d$ under the conditions $ab^{\pi}=a$, $a^{\pi}b=b$ and $b^{\pi}aba^{\pi}=0$ which are weaker than $ab=0$ in Banach algebras. In 2010, Deng and Wei[10] derived a result under the condition $PQ=QP$, where P, Q are bounded linear operators. In 2011, Cvetković-Ilić et al.[11] extended the result of Ref. [10] to Banach algebras. Liu et al.[12] deduced the explicit expressions for $(a+b)^D$ under the conditions $a^2b=aba$ and $b^2a=bab$, where a and b are complex matrices. Recently, Zou et al.[13] studied the corresponding results for the generalized Drazin inverse in Banach algebra. More results on the generalized Drazin inverse can be found in Refs. [14 - 15]. In this paper, we first prove that $a+b\in A^{qnil}$ under the condition that a, $b\in A^{qnil}$, $aba=0$ and $ab^2=0$. Then, we introduce some new conditions and give the explicit expressions for the generalized Drazin inverse of the sum $a+b$, where a, b are generalized Drazin invertibles in A. As corollaries, many results in Refs. [7 - 9, 14] are generalized.

Let $P=(p_1, p_2, \dots, p_n)$ be a total system of idempo-

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tents in A if $p_i p_j = \delta_{i,j} p_i$ for $i, j = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1$. Given a total system P of idempotents in A, we consider the set $M_n(A, P) \subset M_n(A)$ consisting of all matrices $M = [a_{ij}]_{i,j=1}^{n}$ with elements in A such that $a_{ij} \in p_i A p_j$ for all i, $j \in \{1, 2, \ldots, n\}$. Following Ref. [9], we know that $M_n(A, P)$ is a unital Banach algebra with unit. Also, it is proved that $\phi: A \to M_n(A, P)$ given by

$$\phi(x) = \begin{bmatrix} p_1 x p_1 & p_1 x p_2 & \cdots & p_1 x p_n \\ p_2 x p_1 & p_2 x p_2 & \cdots & p_2 x p_n \\ \vdots & \vdots & & \vdots \\ p_n x p_1 & p_n x p_2 & \cdots & p_n x p_n \end{bmatrix}$$

is an isometric Banach algebra isomorphism.

If $a \in A$ is a generalized Drazin invertible, let $p = aa^{\mathrm{d}}$ and the total system is $P = (p, 1 - p)$. Then, we have the following matrix representations:

$$a = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix},\ a^{\mathrm{d}} = \begin{bmatrix} a_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix},\ a^{\pi} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

where $a_{11} \in (pAp)^{-1}$ and $a_{22} \in ((1-p)A(1-p))^{\mathrm{qnil}}$.

1 Preliminary

Lemma 1[9] Let $p^2 = p$, $x, y \in A$ and let $x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$, $y = \begin{bmatrix} b & 0 \\ c & a \end{bmatrix}$ relative to a total system of idempotents $(p, 1-p)$ for x and $(1-p, p)$ for y.

1) If $a \in (pAp)^{\mathrm{d}}$ and $b \in ((1-p)A(1-p))^{\mathrm{d}}$, then $x, y \in A^{\mathrm{d}}$ and

$$x^{\mathrm{d}} = \begin{bmatrix} a^{\mathrm{d}} & u \\ 0 & b^{\mathrm{d}} \end{bmatrix},\ y^{\mathrm{d}} = \begin{bmatrix} b^{\mathrm{d}} & 0 \\ u & a^{\mathrm{d}} \end{bmatrix} \tag{1}$$

where $u = \sum_{n=0}^{\infty} (a^{\mathrm{d}})^{n+2} c b^n b^{\pi} + \sum_{n=0}^{\infty} a^{\pi} a^n c (b^{\mathrm{d}})^{n+2} - a^{\mathrm{d}} c b^{\mathrm{d}}$.

2) If $x \in A^{\mathrm{d}}$ [resp. $y \in A^{\mathrm{d}}$] and $a \in (pAp)^{\mathrm{d}}$, then $b \in ((1-p)A(1-p))^{\mathrm{d}}$, x^{d} [resp. y^{d}] is given by Eq. (1).

Lemma 2[9] Let $a, b \in A^{\mathrm{qnil}}$. If $ab = 0$, then $a + b \in A^{\mathrm{qnil}}$.

Lemma 3[9] Let $a, b \in A^{\mathrm{d}}$ and $ab = 0$, then $a + b \in A^{\mathrm{d}}$ and

$$(a+b)^{\mathrm{d}} = b^{\pi} \sum_{n=0}^{\infty} b^n (a^{\mathrm{d}})^{n+1} + \sum_{n=0}^{\infty} (b^{\mathrm{d}})^{n+1} a^n a^{\pi}$$

Lemma 4[13] Let $a, b \in A^{\mathrm{d}}$ such that $a^2 b = aba$ and $b^2 a = bab$, then the following conditions are equivalent:

$$a + b \in A^{\mathrm{d}},\ 1 + a^{\mathrm{d}} b \in A^{\mathrm{d}},\ c = aa^{\mathrm{d}}(a+b)bb^{\mathrm{d}} \in A^{\mathrm{d}}$$

In this case, we have

$$(a+b)^{\mathrm{d}} = a^{\mathrm{d}}(1 + a^{\mathrm{d}}b) + a^{\pi} b (a^{\mathrm{d}})^2 ((1 + a^{\mathrm{d}}b)^{\mathrm{d}})^2 + \sum_{n=0}^{\infty} (b^{\mathrm{d}})^{n+1} (-a)^n a^{\pi} + \sum_{n=0}^{\infty} (n+1) b^{\pi} a (b^{\mathrm{d}})^{n+2} (-a)^n a^{\pi}$$

2 Main Results

In this section, for $a, b \in A^{\mathrm{d}}$, we will investigate some formulae of $(a+b)^{\mathrm{d}}$ in terms of a, b, a^{d} and b^{d}. Before proving our main results, we need to prove the following result.

Lemma 5 Let $a, b \in A^{\mathrm{qnil}}$. If $aba = 0$, $ab^2 = 0$, then $a + b \in A^{\mathrm{qnil}}$.

Proof From $a, b \in A^{\mathrm{qnil}}$, it follows that $a^2, b^2 \in A^{\mathrm{qnil}}$. The condition $aba = 0$ implies that $ab, ba \in A^{\mathrm{nil}} \subset A^{\mathrm{qnil}}$. Since $ba \in A^{\mathrm{qnil}}$, $b^2 \in A^{\mathrm{qnil}}$ and $bab^2 = 0$, we have $ba + b^2 \in A^{\mathrm{qnil}}$ by Lemma 2. Similarly, we can obtain $a^2 + ab \in A^{\mathrm{qnil}}$. Also, note that $(a+b)^2 = a^2 + ab + ba + b^2 = (a^2 + ab) + (ba + b^2)$ and $(a^2 + ab)(ba + b^2) = a^2 ba + a^2 b^2 + ab^2 a + ab^3 = 0$, we deduce that $(a+b)^2 = (a^2 + ab) + (ba + b^2) \in A^{\mathrm{qnil}}$ by Lemma 2 again, which yields $a + b \in A^{\mathrm{qnil}}$.

Next we start with an important special case for our main theorem.

Theorem 1 Let $a \in A^{\mathrm{qnil}}$, $b \in A^{\mathrm{d}}$. If $ab^{\pi} = a$, $b^{\pi} aba = 0$ and $b^{\pi} ab^2 = 0$, then $a + b \in A^{\mathrm{d}}$ and

$$(a+b)^{\mathrm{d}} = b^{\mathrm{d}} + \sum_{n=0}^{\infty} (b^{\mathrm{d}})^{n+2} a (a+b)^n \tag{2}$$

Proof First, assume that $b \in A^{\mathrm{qnil}}$, then $b^{\pi} = 1$, $aba = 0$ and $ab^2 = 0$. By Lemma 5, $a + b \in A^{\mathrm{qnil}}$. Eq. (2) holds as $(a+b)^{\mathrm{d}} = 0$. If $b \notin A^{\mathrm{qnil}}$, then $p := bb^{\mathrm{d}} \neq 0$. We use a matrix representation relative to the total system $P = (p, 1-p) = (bb^{\mathrm{d}}, b^{\pi})$ of idempotents, where $p \neq 0$. We have $b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$ and $a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, where $b_1 \in (pAp)^{-1}$, $b_2 \in ((1-p)A(1-p))^{\mathrm{qnil}}$. Expressing the condition $ab^{\pi} = a$ in a matrix form, we prove that $a_{11} = 0$ and $a_{21} = 0$. For the sake of simplicity, we write $a_1 := a_{12}$ and $a_2 := a_{22}$. Then we have $a + b = \begin{bmatrix} b_1 & a_1 \\ 0 & a_2 + b_2 \end{bmatrix}$. Similarly, if we express the condition $b^{\pi} aba = 0$ and $b^{\pi} ab^2 = 0$ in a matrix form, we can prove that $a_2 b_2 a_2 = 0$ and $a_2 b_2^2 = 0$. Since $a \in A^{\mathrm{qnil}}$, then $a_2 \in ((1-p)A(1-p))^{\mathrm{qnil}}$. By Lemma 5, $a_2 + b_2 \in ((1-p)A(1-p))^{\mathrm{qnil}}$. Using Lemma 1, we can obtain that $a + b \in A^{\mathrm{d}}$ and

$$(a+b)^{\mathrm{d}} = \begin{bmatrix} b_1 & a_1 \\ 0 & a_2 + b_2 \end{bmatrix}^{\mathrm{d}} = \begin{bmatrix} b_1^{-1} & u \\ 0 & 0 \end{bmatrix}$$

where $u = \sum_{n=0}^{\infty} (b_1^{-1})^{n+2} a_1 (a_2 + b_2)^n$. Computing the right side of Eq. (2) in the matrix form, we can prove that Eq. (2) holds.

Corollary 1[9] Let $a \in A^{\mathrm{qnil}}$, $b \in A^{\mathrm{d}}$. If $ab^{\pi} = a$ and $b^{\pi} ab = 0$, then $a + b \in A^{\mathrm{d}}$ and

$$(a+b)^{\mathrm{d}} = b^{\mathrm{d}} + \sum_{n=0}^{\infty} (b^{\mathrm{d}})^{n+2} a (a+b)^n$$

Example 1 Let A be the algebra of all complex 3×3 matrices, and let $a=b=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$. Then we can check that a and b satisfy $a\in A^{\mathrm{qnil}}$, $ab^{\pi}=a$, $b^{\pi}aba=0$ and $b^{\pi}ab^2=0$, but $b^{\pi}ab\neq 0$.

Next, we present our main theorem, which is a generalization of Theorem 3.5 in Ref. [9].

Theorem 2 Let $a, b\in A^{\mathrm{d}}$. If $ab^{\pi}=a$, $a^{\pi}b=b$, $b^{\pi}abaa^{\pi}=0$ and $b^{\pi}ab^2a^{\pi}=0$, then $a+b\in A^{\mathrm{d}}$ and

$$(a+b)^{\mathrm{d}}=b^{\mathrm{d}}a^{\pi}+b^{\pi}a^{\mathrm{d}}+\sum_{n=0}^{\infty}(b^{\mathrm{d}})^{n+2}a(a+b)^na^{\pi}+\sum_{n=0}^{\infty}b^{\pi}(a+b)^nb(a^{\mathrm{d}})^{n+2}-\sum_{n=0}^{\infty}(b^{\mathrm{d}})^{n+2}a(a+b)^nba^{\mathrm{d}}-\sum_{n=0}^{\infty}b^{\mathrm{d}}a(a+b)^nb(a^{\mathrm{d}})^{n+2}-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^{\mathrm{d}})^{k+2}a(a+b)^{n+k+1}b(a^{\mathrm{d}})^{n+2} \quad (3)$$

Proof If $a\in A^{\mathrm{qnil}}$, then the conditions satisfy Theorem 1. By Theorem 1, Eq. (3) holds. If $a\in A^{-1}$, then $b=a^{\pi}b=0=b^{\mathrm{d}}$, and $b^{\pi}=1$. Obviously, Eq. (3) holds. Thus, we assume that a is neither quasinilpotent nor invertible, and use the matrix representation of elements relative to the total system $P=(p,\ 1-p)=(aa^{\mathrm{d}},\ a^{\pi})$ of idempotents, where $p\neq 0$. We have $a=\begin{bmatrix}a_1&0\\0&a_2\end{bmatrix}$ and $b=\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}$, where $a_1\in(pAp)^{-1}$, $a_2\in((1-p)A(1-p))^{\mathrm{qnil}}$. Since $a^{\pi}b=b$, we obtain that $b_{11}=0$, $b_{12}=0$. Write $b_1:=b_{21}$, $b_2:=b_{22}$. Using Lemma 1, we have

$$b=\begin{bmatrix}0&0\\b_1&b_2\end{bmatrix},\ b^{\mathrm{d}}=\begin{bmatrix}0&0\\(b_2^{\mathrm{d}})^2b_1&b_2^{\mathrm{d}}\end{bmatrix},\ b^{\pi}=\begin{bmatrix}p&0\\-b_2^{\mathrm{d}}b_1&b_2^{\pi}-p\end{bmatrix}$$

Since $ab^{\pi}=a$, we have $a_2b_2^{\mathrm{d}}b_1=0$ and $a_2b_2^{\pi}=a_2$. The conditions $b^{\pi}abaa^{\pi}=0$ and $b^{\pi}ab^2a^{\pi}=0$ give that $b_2^{\pi}a_2b_2a_2=0$ and $b_2^{\pi}a_2b_2^2=0$. Since $a_2\in((1-p)A(1-p))^{\mathrm{qnil}}$, $a_2b_2^{\pi}=a_2$, $b_2^{\pi}a_2b_2a_2=0$ and $b_2^{\pi}a_2b_2^2=0$, applying Theorem 1 to the elements a_2, b_2, we conclude that $a_2+b_2\in((1-p)A(1-p))^{\mathrm{d}}$ and

$$(a_2+b_2)^{\mathrm{d}}=b_2^{\mathrm{d}}+\sum_{n=0}^{\infty}(b_2^{\mathrm{d}})^{n+2}a_2(a_2+b_2)^n$$

By Lemma 1, we have

$$(a+b)^{\mathrm{d}}=\begin{bmatrix}a_1&0\\b_1&a_2+b_2\end{bmatrix}^{\mathrm{d}}=\begin{bmatrix}a_1^{-1}&0\\w&(a_2+b_2)^{\mathrm{d}}\end{bmatrix}$$

where $w=\sum_{n=0}^{\infty}(a_2+b_2)^{\pi}(a_2+b_2)^nb_1(a_1^{-1})^{n+2}-(a_2+b_2)^{\mathrm{d}}b_1a_1^{-1}$. Since $a_2b_2^{\mathrm{d}}=0$, we have

$$(a_2+b_2)^{\pi}=1-(a_2+b_2)(a_2+b_2)^{\mathrm{d}}=1-(a_2+b_2)\cdot\left(b_2^{\mathrm{d}}+\sum_{n=0}^{\infty}(b_2^{\mathrm{d}})^{n+2}a_2(a_2+b_2)^n\right)=1-b_2b_2^{\mathrm{d}}-\sum_{n=0}^{\infty}(b_2^{\mathrm{d}})^{n+1}a_2(a_2+b_2)^n=b_2^{\pi}-\sum_{n=0}^{\infty}(b_2^{\mathrm{d}})^{n+1}a_2(a_2+b_2)^n$$

And then

$$w=\sum_{n=0}^{\infty}b_2^{\pi}(a_2+b_2)^nb_1(a_1^{-1})^{n+2}-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b_2^{\mathrm{d}})^{k+1}a_2(a_2+b_2)^{n+k}b_1(a_1^{-1})^{n+2}-\left(b_2^{\mathrm{d}}+\sum_{n=0}^{\infty}(b_2^{\mathrm{d}})^{n+2}a_2(a_2+b_2)^n\right)b_1a_1^{-1}$$

Computing the right side of Eq. (3) in the matrix form, we can prove that Eq. (3) holds.

Corollary 2[9] Let $a, b\in A^{\mathrm{d}}$. If $ab^{\pi}=a$, $a^{\pi}b=b$, $b^{\pi}abaa^{\pi}=0$, then $a+b\in A^{\mathrm{d}}$ and

$$(a+b)^{\mathrm{d}}=b^{\mathrm{d}}a^{\pi}+b^{\pi}a^{\mathrm{d}}+\sum_{n=0}^{\infty}(b^{\mathrm{d}})^{n+2}a(a+b)^na^{\pi}+\sum_{n=0}^{\infty}b^{\pi}(a+b)^nb(a^{\mathrm{d}})^{n+2}-\sum_{n=0}^{\infty}(b^{\mathrm{d}})^{n+2}a(a+b)^nba^{\mathrm{d}}-\sum_{n=0}^{\infty}b^{\mathrm{d}}a(a+b)^nb(a^{\mathrm{d}})^{n+2}-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^{\mathrm{d}})^{k+2}a(a+b)^{n+k+1}b(a^{\mathrm{d}})^{n+2}$$

Proof It is clear that $b^{\pi}abaa^{\pi}=0$ and $b^{\pi}ab^2=0$ by $a^{\pi}b=b$ and $b^{\pi}abaa^{\pi}=0$. Then, we can complete the proof by Theorem 2.

Now, we will give another main result which generalizes Theorem 2.1 in Ref. [14].

Theorem 3 Let $a, b\in A^{\mathrm{d}}$. If $ba^{\pi}=b$, $a^{\pi}b^{\pi}aba=0$, $a^{\pi}b^{\pi}ab^2=0$ and $a^{\pi}bb^{\mathrm{d}}abb^{\mathrm{d}}=0$, then $a+b\in A^{\mathrm{d}}$ and

$$(a+b)^{\mathrm{d}}=a^{\mathrm{d}}+\sum_{n=0}^{\infty}(a^{\mathrm{d}})^{n+2}b(a+b)^n(a+b)^{\pi}+(a^{\pi}-a^{\mathrm{d}}b)\left(\left(b^{\mathrm{d}}+\sum_{n=0}^{\infty}(b^{\mathrm{d}})^{n+2}a(a+b)^n\right)\right) \quad (4)$$

Proof First, assume that $a\in A^{\mathrm{qnil}}$, then $a^{\pi}=1$, and the condition gives $b^{\pi}aba=0$, $b^{\pi}ab^2=0$ and $bb^{\mathrm{d}}abb^{\mathrm{d}}=0$. Since $b^{\pi}ab^2=0$, then $ab^2=bb^{\mathrm{d}}ab^2$. So, we have $abb^{\mathrm{d}}=bb^{\mathrm{d}}abb^{\mathrm{d}}$. Then, $ab^{\pi}=a$. By Theorem 1, $(a+b)^{\mathrm{d}}=b^{\mathrm{d}}+\sum_{n=0}^{\infty}(b^{\mathrm{d}})^{n+2}a(a+b)^n$. Then, Eq. (4) holds. If $a\in A^{-1}$, then $a^{\pi}=0$ and $b=ba^{\pi}=0$. Clearly, Eq. (4) holds. Thus, we assume that a is neither quasinilpotent nor invertible. We use a matrix representation relative to the total system $P=(p,\ 1-p)=(aa^{\mathrm{d}},\ a^{\pi})$ of idempotents, where $p\neq 0$. We have

$$a=\begin{bmatrix}a_1&0\\0&a_2\end{bmatrix},\ b=\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}$$

where $a_1\in(pAp)^{-1}$, $a_2\in((1-p)A(1-p))^{\mathrm{qnil}}$. Expressing the condition $ba^{\pi}=b$ in a matrix form, we prove that $b_{11}=0$ and $b_{21}=0$. For the sake of simplicity, we write $b_1:=b_{12}$ and $b_2:=b_{22}$. Then, we have

$$b = \begin{bmatrix} 0 & b_1 \\ 0 & b_2 \end{bmatrix},\ a + b = \begin{bmatrix} a_1 & b_1 \\ 0 & a_2 + b_2 \end{bmatrix}$$

By Lemma 1, $b^{\mathrm{d}} = \begin{bmatrix} 0 & b_1(b_2^{\mathrm{d}})^2 \\ 0 & b_2^{\mathrm{d}} \end{bmatrix}$ and $b^{\pi} = \begin{bmatrix} p & -b_1(b_2^{\mathrm{d}})^2 \\ 0 & b_2^{\pi} - p \end{bmatrix}$. The conditions $a^{\pi}b^{\pi}aba = 0$ and $a^{\pi}b^{\pi}ab^2 = 0$ give that $b_2^{\pi}\,a_2b_2a_2 = 0$ and $b_2^{\pi}\,a_2b_2^2 = 0$. Expressing the condition $a^{\pi}bb^{\mathrm{d}}abb^{\mathrm{d}} = 0$ in a matrix form, we obtain that $b_2b_2^{\mathrm{d}}\,a_2b_2b_2^{\mathrm{d}} = 0$. Since $b_2^{\pi}\,a_2b_2^2 = 0$, then $b_2^{\pi}\,a_2b_2^{\mathrm{d}} = 0$. So we have $a_2b_2^{\mathrm{d}} = b_2b_2^{\mathrm{d}}\,a_2b_2^{\mathrm{d}}$. So, $b_2b_2^{\mathrm{d}}\,a_2b_2b_2^{\mathrm{d}} = 0$ implies that $a_2b_2^{\mathrm{d}} = 0$. Thus, the following conditions $b_2 \in A^{\mathrm{d}}$, $a_2 \in A^{\mathrm{qnil}}$, $a_2b_2^{\mathrm{d}} = 0$, $b_2^{\pi}\,a_2b_2a_2 = 0$, $b_2^{\pi}\,a_2b_2^2 = 0$ are satisfied. Hence, we can apply Theorem 1 to obtain an expression of $(a_2 + b_2)^{\mathrm{d}}$.

$$(a_2 + b_2)^{\mathrm{d}} = b_2^{\mathrm{d}} + \sum_{n=0}^{\infty} (b_2^{\mathrm{d}})^{n+2} a_2 (a_2 + b_2)^n$$

By Lemma 1, we prove that

$$(a + b)^{\mathrm{d}} = \begin{bmatrix} a_1^{-1} & v \\ 0 & (a_2 + b_2)^{\mathrm{d}} \end{bmatrix}$$

where $v = \sum_{n=0}^{\infty} (a_1^{-1})^{n+2} b_1 (a_2 + b_2)^n (a_2 + b_2)^{\pi} - a_1^{-1}\,b_1(a_2 + b_2)^{\mathrm{d}}$.

Computing the right side of Eq. (4) in the matrix form, we can prove that Eq. (4) holds.

Example 2 Let A be the algebra of all complex 3×3 matrices, and take $a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Clearly, $ba^{\pi} = b$, $a^{\pi}b^{\pi}aba = 0$, $a^{\pi}b^{\pi}ab^2 = 0$ and $a^{\pi}bb^{\mathrm{d}}abb^{\mathrm{d}} = 0$, but $a^{\pi}b^{\pi}ab \neq 0$.

Remark 1 We define sum $S_n(a,b) = \sum_{i=0}^{n} b^i a^{n-i} + \sum_{i=1}^{n-1} b^{n-i-1} a^i b$ for $n \geqslant 0$. If the lower limit of a sum is greater than its upper limit, we always set the sum to be 0. When $aba = 0$ and $ab^2 = 0$, we can prove that $S_n(a, b) = (a + b)^n$. It can be proved that the $(a + b)^n$ in Theorem 1 to Theorem 3 can be replaced by $S_n(a,b)$.

Finally, we will give another main result.

Theorem 4 Let $a, b \in A^{\mathrm{d}}$. If $ab^{\pi} = a$, $b^{\pi}a^2b = b^{\pi}aba$ and $b^{\pi}b^2a = b^{\pi}bab$, then $a + b \in A^{\mathrm{d}}$ and

$$\begin{aligned}(a + b)^{\mathrm{d}} = {} & b^{\mathrm{d}} + b^{\pi}\sum_{n=0}^{\infty}(-1)^n (a^{\mathrm{d}})^{n+1} b^n + \\ & b^{\pi}\sum_{n=0}^{\infty}(-1)^n (n+1) a^{\pi} b (a^{\mathrm{d}})^{n+2} b^n + \\ & \sum_{n=0}^{\infty} (b^{\mathrm{d}})^{n+2} a (a + b)^n (a + b)^{\pi} - \\ & b^{\mathrm{d}} a \sum_{n=0}^{\infty} (-1)^n (a^{\mathrm{d}})^{n+1} b^n - \\ & b^{\mathrm{d}} a \sum_{n=0}^{\infty} (-1)^n (n+1) a^{\pi} b (a^{\mathrm{d}})^{n+2} b^n \end{aligned} \quad (5)$$

Proof First, assume that $b \in A^{\mathrm{qnil}}$, then $b^{\pi} = 1$, and the conditions $b^{\pi}a^2b = b^{\pi}aba$ and $b^{\pi}b^2a = b^{\pi}bab$ give that $a^2b = aba$ and $b^2a = bab$. By Lemma 4, $(a + b)^{\mathrm{d}} = \sum_{n=0}^{\infty}(-1)^n (a^{\mathrm{d}})^{n+1} b^n + \sum_{n=0}^{\infty}(-1)^n (n+1) a^{\pi} b (a^{\mathrm{d}})^{n+2} b^n$. Eq. (5) holds. If $b \notin A^{\mathrm{qnil}}$, we use a matrix representation relative to the total system $P = (p,\ 1 - p) = (bb^{\mathrm{d}},\ b^{\pi})$ of idempotents, where $p \neq 0$. We have

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix},\ a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where $b_1 \in (pAp)^{-1}$, $b_2 \in ((1 - p)A(1 - p))^{\mathrm{qnil}}$. Similar to the proof of Theorem 1, by $ab^{\pi} = a$, we have

$$a = \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix},\ a + b = \begin{bmatrix} b_1 & a_1 \\ 0 & a_2 + b_2 \end{bmatrix}$$

Since $b^{\pi}a^2b = b^{\pi}aba$ and $b^{\pi}b^2a = b^{\pi}bab$, we prove $b_2^2\,a_2 = b_2a_2b_2$ and $a_2^2\,b_2 = a_2b_2a_2$. By Lemma 4, we have $a_2 + b_2 \in A^{\mathrm{d}}$ if and only if $1 + b_2^{\mathrm{d}}\,a_2 \in A^{\mathrm{d}}$. Noting that $b_2 \in ((1 - p)A(1 - p))^{\mathrm{qnil}}$, we have $b_2^{\mathrm{d}} = 0$. Then

$$\begin{aligned}(a_2 + b_2)^{\mathrm{d}} = {} & \sum_{n=0}^{\infty} (a_2^{\mathrm{d}})^{n+1} (-b_2)^n + \\ & \sum_{n=0}^{\infty} (n+1) a_2^{\pi}\, b_2 (a_2^{\mathrm{d}})^{n+2} (-b_2)^n\end{aligned}$$

Using Lemma 1, we can prove that $a + b \in A^{\mathrm{d}}$ and

$$(a + b)^{\mathrm{d}} = \begin{bmatrix} b_1 & a_1 \\ 0 & a_2 + b_2 \end{bmatrix}^{\mathrm{d}} = \begin{bmatrix} b_1^{-1} & u \\ 0 & (a_2 + b_2)^{\mathrm{d}} \end{bmatrix}$$

where $u = \sum_{n=0}^{\infty} (b_1^{-1})^{n+2} a_1 (a_2 + b_2)^n (a_2 + b_2)^{\pi} - b_1^{-1}\,a_1(a_2 + b_2)^{\mathrm{d}}$.

Computing the right side of Eq. (5) in the matrix form, we can prove that Eq. (5) holds.

Example 3 Let A be the algebra of all complex 2×2 matrices, and take $a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Clearly, $b^{\mathrm{d}} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ and $b^{\pi} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Then, we can check that a, b satisfy $ab^{\pi} = a$, $b^{\pi}a^2b = b^{\pi}aba$ and $b^{\pi}b^2a = b^{\pi}bab$, but $b^2a \neq bab$.

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Banach 代数中元素之和的广义 Drazin 逆的一些结果

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摘要: 令 a,b 为 Banach 代数中的 2 个广义 Drazin 可逆的元素. 用 a,b,a^{d},b^{d} 给出元素 $a+b$ 和的广义 Drazin 逆的明确表达式. 利用 Banach 代数中的幂等系统研究了 2 个元素之和的广义 Drazin 逆. 对于 Banach 代数中元素 a,b, 首先证明了如果 $a,b\in A^{\mathrm{qnil}}$, $aba=0$ 且 $ab^{2}=0$, 则 $a+b\in A^{\mathrm{qnil}}$. 并在一些新的条件下给出了 $a+b$ 和的广义 Drazin 逆的表达式, 推广了近期的一些结果.

关键词: 广义 Drazin 逆; Banach 代数; 幂零元; 拟幂零元

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