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Pricing of discrete barrier options based on an analytical method

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Abstract: The problem of analytically pricing the discrete monitored European barrier options is studied under the assumption of the Black-Scholes market. First, using variable transformation, the mean vector and covariance matrix of multi-dimensional marginal distribution are given. Secondly, the analytical pricing formulas of the discrete monitored up-knock-out European call option and the discrete monitored down-knock-out European put option are obtained by using the conditional probability and the characteristics of the multi-dimensional normal distribution. Finally, the effects of the discrete monitoring barriers on the prices of the barrier options are discussed and analyzed. The research results state that the price of the discrete monitored up-knock-out European call option increases with the increase in the up barrier, and the price of the discrete monitored down-knock-out European put option decreases with the increase in the down barrier.

Key words: discrete monitored; barrier options; pricing; analytical method

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One of the most important issues in pricing barrier options is whether or not the barrier is monitored at continuous time. Continuous monitoring of the barrier is assumed in most models. Pricing for continuous monitoring of barrier options is a key issue in financial engineering. Merton[1] gave the formula for pricing the continuous monitoring barrier option under the classical Brownian motion. Chen et al. [2] studied the pricing of a two-barrier option when the underlying price was considered a sub-transmission system. In this case, the option price is modified by the Black-Scholes formula with a time-fraction derivative of the constraint. Appolloni et al. [3] argued that the binomial lattice method based on interpolation techniques can be used to price one-step two-barrier options. Boyarchenko et al. [4] proved the advantages of the conformal deformation in the integration of the pricing formula in the context of a wide class of Lie-dimensional models and the effective conformal deformation of the Heston model and constructed contours. Chandra et al. [5] proposed a method for calculating the value of a class of Levy processes under the incomplete market. Thakoor et al. [6] proposed a fourth-order numerical method for the continuous discrete-time monitoring of barriers and demonstrated their technology's superiority over existing Black-Scholes models and constant elasticity diffusion. Fajardo[7] proposed a new pricing formula for some underlying barrier-type contracts when the underlying process is driven by a type of important Levy method, which includes the CGMY model, the generalized hyperbolic model and the Meixner model. Cassagnes et al. [8] used the path integral approach to frame the outer barrier to the Asian option pricing problem as the Wiener functional integral form. Zhang et al. [9] proposed the least squares Monte Carlo (LSMC) method to estimate the American barrier option, which modified the LSMC method. Dokuchaev[10] suggested that the applicable market-time model can be applied, where the market dynamics are described by stochastic differential equations with stochastic coefficients in the European barrier option pricing and hedging theorem. Dai et al. [11] obtained an accurate formula for assessing dividends for continuous monitoring of barrier options. Escobar et al. [12] relied on the context-dependent perturbation theory of two-dimensional asset models with stochastic correlation structures to derive derivatives from two asset paths. Jun and Ku[13] studied two barriers to which the barriers alternated with the activity of the US barrier option valuation analysis. These options are used in analytical pricing formulas where constant and exponential barriers are the best early functional exercise policies.

However, in practice, most of the bargain options traded in the market are discretely monitored. In other words, they monitor the barrier at a fixed time (usually a daily shutdown). In addition to the practical implementation issues, there are some legal and financial reasons why the option of monitoring the barrier with the discrete monitoring option is preferable to continuous monitoring. Some traders in the discussion of the "Derivatives Weekly" on May 29, 1995 announced that since the continuous monitoring barriers can exist in less liquid markets, while the rest of the western markets are restrained by excess barrier damage, this may lead to some arbitrary opportu-

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nities[14]. While discrete monitoring barrier options are popular, it is important that their pricing is more difficult than their sequential counterparts. The most important is the monitoring frequency of the barrier, that is, the frequency of the observed event. With separate monitoring barriers, they check for a fixed time (e. g. weekly or monthly) triggering. The result is a knock-out (knock-in) option with an increased number of discrete monitoring barriers becoming less (more) expensive to monitor. Aitsahlia et al. [15] is the first person to study the problem of pricing discrete monitoring barrier options. Broadie et al. [16] used the continuous barrier formula to quite accurately set a pricing barrier option with a simple continuity correction barrier. Sullivan[17] proposed a classical Brownian separation for the barrier option based on the numerical integration method. Duan et al. [18] proposed a new Markov chain technique for pricing all of these problems at any time, obstructing options. Ahmadian et al. [19] developed a numerical method for pricing the variance model with elastic-invariant and jump-diffusion (CEVJD) underlying the description of discrete barrier options. Li and Linetsky[20] developed a feature-based approach to solve the discrete-time first-pass detection problem by a rich class of Markov processes, including diffusion and jumps, of which transitions or Ferman-Katz semigroups possess eigenfunction expansions in L2-space. Farnoosh et al. [21] studied the method for pricing the discrete time-dependent parameters with dual barrier options in the Black-Scholes framework. Rostan et al. [22-23] applied the original variance reduction technique to monitor the European double barrier option pricing in discrete time. Umezawa et al. [24] proposed a discrete monitoring path-dependent derivative pricing method when the underlying asset price is time-driven by the Levy process. Using the derived feature representation, they obtained the semi-analytical pricing formula for geometric Asian, forward-start, attenuator and look back options. Even though discretely monitored barrier options are popular and important, pricing them is more difficult than that of their continuous counterparts. The existing literature on the pricing of the option with discretely monitoring barriers is focused on the numerical method. This paper studies the analytical method of pricing option with discretely monitoring barriers.

1 Market Model

Assume that the market is a Black-Scholes market[25], and the motion of free risky asset is as follows:

$$dB_t = B_t r dt \qquad B_0 = 1 \tag{1}$$

where $r > 0$ is a constant free risk interest rate. The price of risk asset (stock) is depicted by a stochastic difference equation (geometric Brownian motion) under the risk-neutral measure

$$dS_t = S_t(r dt + \sigma dW_t) \qquad S_0 > 0 \tag{2}$$

where $\sigma > 0$ is the constant volatility of the risk asset; W_t is the standard Wiener process. Let $0 < s < t$, and we have

$$E[W_s W_t] = s \tag{3}$$

According to the ITO formula, the price of risk asset at the time point t is obtained by

$$S_t = S_0 e^{(r - \sigma^2/2)t + \sigma W_t} \tag{4}$$

where $W_t \pm N(0, \sigma^2 t)$; $\phi(x; \boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\Psi(x; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ are the probability density function (PDF) and the cumulative distribution function (CDF) of a multi-dimension Gaussian distribution with mean vector $\boldsymbol{\mu}$ and covariable matrix $\boldsymbol{\Sigma}$, respectively; and $\phi(\cdot)$, $\Psi(\cdot)$ are the probability density function (PDF) and the cumulative distribution function (CDF) of a standard Gaussian distribution, respectively.

For the European barrier options, the barriers are divided into the up barriers and the down barriers on the basis of the relationship between the barriers and the initial price of risk asset or on the basis of the behaviors that the price of risk asset passes through the barriers. Since the barrier options are activated or extinguished while the price of risk asset crosses the barriers, the barrier options can be classified as the knocked-in barrier options and the knocked-out barrier options. The European barrier option can also be divided into call barrier options and put barrier options on the basis of the payment function at the mature time. With different combinations, there are eight-type barrier options with a single barrier. Since there is some relationship among those of the eight-type barrier options, we just consider the up-knocked-out European call options and the down-knocked-out European put options, in which these two type barrier options are the most important barrier options. T is the mature time of the barrier option, in the time interval $(0, T)$, $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T$ are the discrete monitoring time points, where the barriers are $B_t \geqslant 0$, $i = 1, 2, \cdots, n-1$. Without loss of generality, let $t_i = i\tau$, $i = 1, 2, \cdots, n$, where $\tau = T/n$, and for the pithiness of the symbols, denote $S_i = S_{t_i}$. With the strike price $K(>0)$, the price of the discrete monitored barrier up-knocked-out European call option can be described as

$$C = e^{-rT} \int_K^{+\infty} (S - K) \, d\text{pro}(S_1 \leqslant B_1, S_2 \leqslant B_2, \cdots, S \leqslant S_n) \tag{5}$$

The price of discrete monitored barrier down-knocked-out European put option can be described as

$$P = e^{-rT} \int_0^K (K - S) \, d\text{pro}(S_1 \geqslant B_1, S_2 \geqslant B_2, \cdots, S \leqslant S_n) \tag{6}$$

2 Pricing Discrete Monitored Barrier Options

2.1 Prices of the discrete monitored up-knock-out European call options

Let

$$\tilde{W}_i = \frac{\ln(S_i) - \ln(S_0) - \left(r - \frac{\sigma^2}{2}\right)i\tau}{\sigma\sqrt{\tau}}$$

According to the characteristic of the Wiener process, we have $\tilde{W}_i \pm N(0,i)$, and $E[\tilde{W}_i\ \tilde{W}_j] = i, i \leqslant j$. Furthermore, the distribution $\tilde{W}_1, \tilde{W}_2, \cdots, \tilde{W}_n$ is a multivariate normal distribution, and its covariance matrix is

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & \cdots & n-1 & n-1 \\ 1 & 2 & \cdots & n-1 & n \end{bmatrix} \quad (7)$$

Denote

$$\boldsymbol{\Sigma}_{11} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & n-1 \end{bmatrix}$$

and

$$\boldsymbol{\Sigma}_{21} = \boldsymbol{\Sigma}_{12}^{\mathrm{T}} = [1 \quad 2 \quad \cdots \quad n-1]$$

and then, we have

$$\overset{-1}{\boldsymbol{\Sigma}} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}_{n\times n} \quad (8)$$

$$\boldsymbol{\Sigma}_{11}^{-1} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}_{(n-1)\times(n-1)} \quad (9)$$

Let

$$\tilde{B}_i = \frac{\ln(B_i) - \ln(S_0) - \left(r - \frac{\sigma^2}{2}\right)i\tau}{\sigma\sqrt{\tau}} \qquad i = 1,2,\cdots,n-1 \quad (10)$$

$$\tilde{B}_n = \frac{\ln(S_T) - \ln(S_0) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{\tau}}$$

Accordingly, we have

$$\mathrm{pro}(S_1 \leqslant B_1, S_2 \leqslant B_2, \cdots, S \leqslant S_n) = \mathrm{pro}(\tilde{W}_1 \leqslant \tilde{B}_1, \tilde{W}_2 \leqslant \tilde{B}_2, \cdots, \tilde{W}_n \leqslant \tilde{B}_n) \quad (11)$$

where $\mathrm{pro}(\tilde{W}_1, \tilde{W}_2, \cdots, \tilde{W}_n) \pm N(0,\boldsymbol{\Sigma})$, and the $\boldsymbol{\Sigma}$ covariance matrix is presented by Eq. (7). On the basis of the multivariate normal distribution, we have $\mathrm{pro}(\tilde{W}_n \leqslant \tilde{B}_n \mid \tilde{W}_1 \leqslant \tilde{B}_1, \cdots, \tilde{W}_{n-1} \leqslant \tilde{B}_{n-1})$, and

$$\tilde{\mu} = 0 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_{n-1}) = \tilde{B}_{n-1} \quad (12)$$

$$\tilde{\sigma}^2 = n - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{21}^{T} = n-1 \quad (13)$$

Since $\mathrm{pro}(\tilde{W}_1 \leqslant \tilde{B}_1, \cdots, \tilde{W}_{n-1} \leqslant \tilde{B}_{n-1}) \pm N(0,\boldsymbol{\Sigma}_{11})$, we have

$$\begin{aligned} &\mathrm{pro}(\tilde{W}_1 \leqslant \tilde{B}_1, \tilde{W}_2 \leqslant \tilde{B}_2, \cdots \tilde{W}_n \leqslant \tilde{B}_n) = \\ &\quad \mathrm{pro}(\tilde{W}_n \leqslant \tilde{B}_n \mid \tilde{W}_1 \leqslant \tilde{B}_1, \tilde{W}_2 \leqslant \tilde{B}_2, \cdots \tilde{W}_{n-1} \leqslant \tilde{B}_{n-1}) \times \\ &\quad \mathrm{pro}(\tilde{W}_1 \leqslant \tilde{B}_1, \tilde{W}_2 \leqslant \tilde{B}_2, \cdots \tilde{W}_{n-1} \leqslant \tilde{B}_{n-1}) = \\ &\quad \Psi(\tilde{B}_n; \tilde{\mu}, \tilde{\sigma}^2)\Psi((\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_{n-1}); 0, \boldsymbol{\Sigma}_{11}) \end{aligned} \quad (14)$$

Let

$$\tilde{K} = \frac{\ln(K) - \ln(S_0) - (r - \sigma^2/2)T}{\sigma\sqrt{\tau}}$$

On the basis of Eqs. (5) and (14), we have

$$\begin{aligned} C = {} & \Psi((\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_{n-1}); 0, \boldsymbol{\Sigma}_{11}) \\ & \mathrm{e}^{-rT}\int_{\tilde{K}}^{+\infty} (S_0 \mathrm{e}^{(r-\sigma^2/2)T + \sigma\sqrt{\tau}B} - K)\mathrm{d}\Psi(B; \tilde{\mu}, \tilde{\sigma}^2) \end{aligned} \quad (15)$$

Clearly, we have

$$\int_{\tilde{K}}^{+\infty} K\mathrm{d}\Psi(B; \tilde{\mu}, \tilde{\sigma}^2) = K\Psi(\tilde{B}_{n-1} - \tilde{K}) \quad (16)$$

And then, we have

$$\begin{aligned} &\int_{\tilde{K}}^{+\infty} S_0 \mathrm{e}^{(r-\sigma^2/2)T + \sigma\sqrt{\tau}B}\mathrm{d}\psi(B:\tilde{\mu}, \tilde{\sigma}^2) = \\ &\qquad S_0 \mathrm{e}^{(r-\sigma^2/2)T + \tilde{B}_{n-1}\sigma\sqrt{\tau} + \sigma^2\tau/2}\psi(\tilde{B}_{n-1} + \sigma\sqrt{\tau} - \tilde{K}) \end{aligned} \quad (17)$$

According to Eqs. (16) and (17), we can obtain the price of the discrete monitored up-knock-out European call option

$$\begin{aligned} C = {} & \Psi((\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_{n-1}); 0, \boldsymbol{\Sigma}_{11}) \\ & \mathrm{e}^{-rT}[S_0 \mathrm{e}^{(r-\sigma^2/2)T + \tilde{B}_{n-1}\sigma\sqrt{\tau} + \sigma^2\tau/2}\Psi(\tilde{B}_{n-1} + \sigma\sqrt{\tau} - \tilde{K}) - \\ & K\Psi(\tilde{B}_{n-1} - \tilde{K})] \end{aligned} \quad (18)$$

2.2 Prices of the discrete monitored down-knock-out European put options

For the down-knocked-out barrier options, on the basis of the symmetries of the Wiener process and the normal distribution, we have

$$\text{pro}(S_1 \geqslant B_1, S_2 \geqslant B_2, \cdots, S_{n-1} \geqslant B_{n-1}, S_n \leqslant B_n) =$$
$$\text{pro}(\tilde{W}_1 \leqslant -\tilde{B}_1, \tilde{W}_2 \leqslant -\tilde{B}_2, \cdots, \tilde{W}_{n-1} \leqslant -\tilde{B}_{n-1}, \tilde{W}_n \leqslant \tilde{B}_n) =$$
$$\text{pro}(\tilde{W}_n \leqslant \tilde{B}_n \mid \tilde{W}_1 \leqslant -\tilde{B}_1, \tilde{W}_2 \leqslant -\tilde{B}_2, \cdots, \tilde{W}_{n-1} \leqslant -\tilde{B}_{n-1}) \cdot$$
$$\text{pro}(\tilde{W}_1 \leqslant -\tilde{B}_1, \tilde{W}_2 \leqslant -\tilde{B}_2, \cdots, \tilde{W}_{n-1} \leqslant -\tilde{B}_{n-1}) \tag{19}$$

$\tilde{W}_1, \tilde{W}_2, \cdots, \tilde{W}_n$ are depicted by a multivariate normal distribution, and its covariance matrix is still presented by Eq. (7), so we have

$$\text{pro}(\tilde{W}_n \leqslant \tilde{B}_n \mid \tilde{W}_1 \leqslant -\tilde{B}_1, \tilde{W}_2 \leqslant -\tilde{B}_2, \cdots, \tilde{W}_{n-1} \leqslant -\tilde{B}_{n-1}) \pm N(\bar{\mu}, \bar{\sigma}^2)$$

where

$$\bar{\mu} = 0 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(-\tilde{B}_1, -\tilde{B}_2, \cdots, -\tilde{B}_{n-1}) = -\tilde{B}_{n-1} \tag{20}$$

$$\bar{\sigma}^2 = n - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{21}^{\mathrm{T}} = 1 \tag{21}$$

The price of the discrete monitored down-knock-out European put option can be presented by

$$P = \Psi((-\tilde{B}_1, -\tilde{B}_2, \cdots, -\tilde{B}_{n-1}); 0, \boldsymbol{\Sigma}_{11}) \cdot \mathrm{e}^{-rT}\int_{-\infty}^{\tilde{K}} (K - S_0 \mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{\tau}B}) \,\mathrm{d}\Psi(B; \bar{\mu}, \bar{\sigma}^2) \tag{22}$$

Since

$$\int_{-\infty}^{\tilde{K}} K \mathrm{d}\Psi(B; \bar{\mu}, \bar{\sigma}^2) = K\Psi(\tilde{K}; -\tilde{B}_{n-1}, 1) = K\Psi(\tilde{K} + \tilde{B}_{n-1})$$

and

$$\int_{-\infty}^{\tilde{K}} S_0 \mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{\tau}B} \mathrm{d}\Psi(B; -\tilde{B}_{n-1}, 1) =$$
$$S_0 \mathrm{e}^{(r-\sigma^2/2)T} \int_{-\infty}^{\tilde{K}} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\sigma\sqrt{\tau}B - (B+\tilde{B}_{n-1})^2/2} \mathrm{d}B =$$
$$S_0 \mathrm{e}^{(r-\sigma^2/2)T+\sigma^2\tau/2-\tilde{B}_{n-1}\sigma\sqrt{\tau}} \int_{-\infty}^{\tilde{K}} \frac{2}{\sqrt{2\pi}} \mathrm{e}^{-(B+\tilde{B}_{n-1}-\sigma\sqrt{\tau})^2/2} \mathrm{d}B =$$
$$S_0 \mathrm{e}^{(r-\sigma^2/2)T+\sigma^2\tau/2-\tilde{B}_{n-1}\sigma\sqrt{\tau}} \Psi(\tilde{K} + \tilde{B}_{n-1} - \sigma\sqrt{\tau})$$

The analytical formula for pricing the discrete monitored down-knock-out European put option can be obtained as

$$P = \Psi((-\tilde{B}_1, -\tilde{B}_2, \cdots, -\tilde{B}_{n-1}); 0, \boldsymbol{\Sigma}_{11}) \mathrm{e}^{-rT}(K\Psi(\tilde{K} + \tilde{B}_{n-1}) - S_0 \mathrm{e}^{(r-\frac{\sigma^2}{2})T+\frac{\sigma^2\tau}{2}-\tilde{B}_{n-1}\sigma\sqrt{\tau}} \Psi(\tilde{K} + \tilde{B}_{n-1} - \sigma\sqrt{\tau})) \tag{23}$$

3 Sensitive Analysis of Barriers

According to Eq. (10), for $\forall i = 1, 2, \cdots, n$, we have

$$\left.\begin{array}{ll} \tilde{B}_i \to +\infty & \text{if } B_i \to +\infty \\ \tilde{B}_i \to -\infty & \text{if } B_i \to 0 \end{array}\right\} \tag{24}$$

For the discrete monitored up-knock-out European call option, if all the barriers at the discrete monitoring time points are tend to be $+\infty$, which are equal to the fact that the up barriers disappear, the discrete monitored up-knock-out European call option degenerates into an ordinary European call option. In Eq. (18),

$$\Psi((+\infty, \cdots, +\infty); 0, \boldsymbol{\Sigma}_{11}) = 1$$

and while $\tilde{B}_{n-1} \to +\infty$,

$$\mathrm{e}^{-rT}[S_0 \mathrm{e}^{(r-\sigma^2/2)T+\tilde{B}_{n-1}\sigma\sqrt{\tau}+\sigma^2\tau/2}\Psi(\tilde{B}_{n-1} + \sigma\sqrt{\tau} - \tilde{K}) - K\Psi(\tilde{B}_{n-1} - \tilde{K})] =$$
$$S_0\Psi\left(\frac{\ln\left(\frac{S_0}{K}\right) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{\mathrm{T}}}\right) - \mathrm{e}^{-rT}\Psi\left(\frac{\ln\left(\frac{S_0}{K}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

The pricing formula (18) degenerates to the Black-Scholes formula for pricing European call options. According to the nature of normal distribution, the price of the discrete monitored up-knock-out European call option increases with the increase in the up barrier $B_i, i = 1, 2, \cdots, n-1$. The price of the discrete monitored up-knock-out European call option decreases with the decrease in the up barrier $B_i, i = 1, 2, \cdots, n-1$.

For the discrete monitored down-knock-out European put option, if all the barriers at the discrete monitoring time points tend to be zeros, which are equal to the fact that the down barriers disappear, the discrete monitored down-knock-out European put option degenerates into an ordinary European put option. In fact, in Eq. (23), we have

$$\Psi((-\tilde{B}_1, \cdots, -\tilde{B}_1); 0, \boldsymbol{\Sigma}_{11}) = \Psi((+\infty, \cdots, +\infty); 0, \boldsymbol{\Sigma}_{11}) = 1$$

Furthermore, while $\tilde{B}_{n-1} \to -\infty$, we have

$$\mathrm{e}^{-rT}(K\Psi(\tilde{K} + \tilde{B}_{n-1}) - S_0 \mathrm{e}^{(r-\sigma^2/2)T+\sigma^2\tau/2+\tilde{B}_{n-1}\sigma\sqrt{\tau}}) \cdot$$
$$\Psi(\tilde{K} + \tilde{B}_{n-1} - \sigma\sqrt{\tau}) =$$
$$\mathrm{e}^{-rT}\Psi\left(-\frac{\ln\left(\frac{S_0}{K}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) -$$
$$S_0\Psi\left(-\frac{\ln\left(\frac{S_0}{K}\right) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{\mathrm{T}}}\right)$$

The pricing formula (23) degenerates into the Black-Scholes formula for pricing European put options. According to the nature of Gaussian distribution, the price of the discrete monitored down-knock-out European put option decreases with the increase in the down barrier $B_i, i=1,2,\cdots,n-1$. The price of the discrete monitored down-knock-out European put option increases with the decrease in the down barrier B_i, $i=1,2,\cdots,n-1$.

4 Conclusion

The problem of analytically pricing the discrete monitored European barrier options is studied under the assumption of the Black-Scholes market in this paper. By making use of variable transformation, the mean vector and covariance matrix of the multivariate normal distribution, which can be used to price the discrete monitored barrier options, are obtained according to the Wiener process. For the discrete monitored up-knock-out European call option and the discrete monitored down-knock-out European put option, we study their pricing problems, and we obtain the analytical pricing formula. The effects of the discrete monitoring barriers on the prices of the barrier options are discussed and analyzed. The results of this research state that the discrete monitored down-knock-out European put option degenerates into the European call option, and the price of the discrete monitored up-knock-out European call option increases with the increase in the up barrier, and the price of the discrete monitored up-knock-out European call option decreases with the decrease in the up barrier; the discrete monitored down-knock-out European put option degenerates to the European call option, but the relationship between the barrier and the price of barrier option contradicts this. The analytical pricing formula given in this paper can be used to quickly and real-time estimate the barrier option price on the market. Compared with the existing numerical methods, it has characteristics of simplicity and rapidity.

The method pricing the discrete monitored barrier European options can be easily expanded to the pricing any discrete monitored barrier options with the single barrier. We will consider the discrete monitored double barrier options and the discrete monitored barrier American-style options.

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基于解析方法的离散障碍期权定价

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摘要:在 Black-Scholes 市场假设下,研究了离散监督障碍期权的解析定价问题. 首先,通过变量变换,给出了多维边际分布的平均向量和协方差矩阵. 其次,通过使用条件概率和多维正态分布的特征,获得离散监督向上敲出欧式看涨期权和离散监督向下敲出欧式看跌期权的解析定价公式. 最后,讨论和分析了离散监督障碍对障碍期权价格的影响. 研究结果表明,随着上障碍的增大,离散监督向上敲出的欧式看涨期权价格变大;离散监督向下敲出欧式看跌期权价格随着下障碍的增加而变小.

关键词:离散监督;障碍期权;定价;解析方法

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