Journal of Southeast University (English Edition) Vol. 33, No. 4, pp. 517 − 520 Dec. 2017 ISSN 1003—7985

Construction of a class of H-pseudoalgebras

Liu Linlin Wang Shuanhong

(School of Mathematics, Southeast University, Nanjing 211189, China)

Abstract: Let H be a cocommutative Hopf algebra. First, a new class $\widetilde{H}$-pseudoalgebras of H-pseudoalgebras are defined by changing the regular action (i. e. left multiplication) of H on itself into an adjoint action. Secondly, a class of (H, R)-pseudoalgebras are studied by generalizing the above construction when (H, R) is a quasitriangular Hopf algebra. At the same time, the (H, R)-pseudoalgebra is constructed by both the usual algebra and the tensor product of (H, R)-pseudoalgebras. Finally, some examples of the (H, R)-pseudoalgebra are given explicitly, and the conditions for a Hopf algebra to be an (H, R)-pseudoalgebra (resp. H-pseudoalgebra) are discussed.

Key words: cocommutative Hopf algebra; H-pseudoalgebra; $\widetilde{H}$-pseudoalgebra; (H, R)-pseudoalgebra

DOI: 10.3969/j.issn.1003 − 7985.2017.04.020

The theory of conformal algebras was introduced by Kac[1] as a normal language describing the singular part of the vertex algebras, which is derived from mathematical physics. The notion of H-pseudoalgebra, or simply a pseudoalgebra over a cocommutative Hopf algebra H is a natural generalization of conformal algebra, which can be considered as multidimensional conformal algebra[2-3]. The name is motivated by the fact that this is an algebra in a pseudotensor category. Moreover, it has a close relationship to Hamiltonian formalism in the theory of nonlinear evolution equations[4-6]. The research on H-pseudoalgebra has attracted much attention recently[7-8].

In the definition of pseudoalgebra, the action of H on itself is multiplication. It is natural to ask: Can we give a new definition of pseudoalgebra by substituting adjoint action for multiplication? Moreover, due to the cocommutativity of H, can we generalize the above construction for quasitriangular Hopf algebra? And when is a Hopf algebra H an H-pseudoalgebra? The purpose of this paper is to answer the above questions.

Received 2017-03-16, **Revised** 2017-07-15.

Biographies: Liu Linlin (1990—), female, graduate; Wang Shuanhong (corresponding author), male, doctor, professor, shuanhwang@yahoo.com.

Foundation items: The National Natural Science Foundation of China (No. 11371088), the Natural Science Foundation of Jiangsu Province (No. BK20171348).

Citation: Liu Linlin, Wang Shuanhong. Construction of a class of H-pseudoalgebras[J]. Journal of Southeast University (English Edition), 2017, 33(4): 517 − 520. DOI: 10.3969/j.issn.1003 − 7985.2017.04.020.

1 Preliminaries

Unless otherwise specified, H is a cocommutative Hopf algebra over the field k. We will use Sweedler's notion $\Delta(h) = h_1 \otimes h_2$, $\forall h \in H$, and unexplained definitions and notions can refer to Ref. [9]. Now, we recall some useful definitions.

We just present the special case of the pseudotensor category[2] that will be used. $M^*(H)$ is a pseudotensor category with the identical objects as in ${}_HM$ (the category of left H-modules) but with a particular pseudotensor structure

$$\mathrm{Lin}(\{L_i\}_{i\in I}, M) = \mathrm{Hom}_{H^{\otimes I}}(\boxtimes_{i\in I} L_i, H^{\otimes I}\otimes_H M)$$

where $\boxtimes_{i\in I}$ is the tensor product functor ${}_HM^I \rightarrow {}_{H^{\otimes I}}M$.

The symmetric group S_I acts among the spaces $\mathrm{Lin}(\{L_i\}_{i\in I}, M)$ by simultaneously permuting the factors in $\boxtimes_{i\in I}$ and $H^{\otimes I}$. This is well defined only when H is cocommutative.

For example, the permutation $\sigma_{12} = (12) \in S_2$ acts on $(H\otimes H)\otimes_H M$ by $\sigma_{12}((f\otimes g)\otimes_H m) = (g\otimes f)\otimes_H m$, and this is well defined only when H is cocommutative.

An H-pseudoalgebra is an algebra A in the pseudotensor category $M^*(H)$ defined above, and it is of value to give the specific definition of H-pseudoalgebra.

Definition 1[2] An H-pseudoalgebra is an object A in ${}_HM$ together with an operation $u \in \mathrm{Lin}(\{A, A\}, A) = \mathrm{Hom}_{H\otimes H}(A\otimes A, (H\otimes H)\otimes_H A)$, called the pseudoproduct. We denote $u(a\otimes b) = a * b$, $\forall a, b \in A$. Equivalently, an H-pseudoalgebra is a left H-module A satisfying the H-bilinearity: for all $a, b \in A$, $f, g \in H$, one has

$$fa * gb = (f\otimes g)\cdot(a * b) = (ff_i\otimes gg_i)\otimes_H e_i$$

where $a * b = f_i\otimes g_i\otimes_H e_i$, in which we often omit the summation symbols for convenience.

H-pseudoalgebra is the usual algebra when $H = k$.

Let A, B be two H-pseudoalgebras. The H-linear map $f: A\rightarrow B$ is called the morphism of H-pseudoalgebra if f satisfies

$$(\mathrm{id}_{H^{\otimes 2}}\otimes_H f)u_A = u_B(f\otimes f)$$

Example 1 Let A be an H-pseudoalgebra with the pseudoproduct μ, and H' the Hopf subalgebra of H such that $H'H \subset H$. Then A is an H'-pseudoalgebra with the pseudoproduct μ.

In fact, it is clear that A is a left H'-module, and

$$\mu(f'a\otimes g'b)=(f'f_i\otimes g'g_i)\otimes_H e_i\subset(H'\otimes H')\otimes_{H'}A$$

where f', $g'\in H'$, $\mu(a\otimes b)=(f_i\otimes g_i)\otimes_H e_i$.

Proposition 1 Let M be a right H-module algebra, and A an H-pseudoalgebra. Then, $M\otimes_H A$ is a left H-module with the action

$$h\cdot(m\otimes a)=mS(h_1)\otimes h_2 a$$

for all $h\in H$, $m\in M$, $a\in A$.

Furthermore, $M\otimes_H A$ is an H-pseudoalgebra with the following pseudoproduct:

$$(m\otimes a)*(n\otimes b)=f_{i1}\otimes g_{i1}\otimes(mf_{i2})(ng_{i2})\otimes e_i$$
$$\forall\, m,n\in M,\ a,b\in A$$

where $a*b=f_i\otimes g_i\otimes_H e_i$.

Proof For all $m\in M$, $a\in A$, $h,g\in H$, we have $1\cdot(m\otimes a)=mS(1)\otimes a=m\otimes a$ and $h\cdot g\cdot(m\otimes a)=h\cdot(mS(g_1)\otimes g_2 a)=(mS(g_1))S(h_1)\otimes h_2(g_2 a)=mS(h_1g_1)\otimes(h_2g_2)a=(hg)\cdot(m\otimes a)$, then $M\otimes_H A$ is a left H-module.

For all $h,g\in H$, m, $n\in M$, $a,b\in A$, we have

$$u((f\otimes g)\cdot((m\otimes a)\otimes(n\otimes b)))=$$
$$u(mS(f_1)\otimes f_2a\otimes nS(g_1)\otimes g_2b)=$$
$$f_2f_{i1}\otimes g_2g_{i1}\otimes_H(((mS(f_1))(f_3f_{i2}))((nS(g_1))(g_3g_{i2}))\otimes e_i)=$$
$$f_3f_{i1}\otimes g_3g_{i1}\otimes_H((m(S(f_1)f_2f_{i2}))(n(S(g_1)g_2g_{i2}))\otimes e_i)=$$
$$ff_{i1}\otimes gg_{i1}\otimes_H((mf_{i2})(ng_{i2})\otimes e_i)=$$
$$(f\otimes g)\cdot(f_{i1}\otimes g_{i1}\otimes_H((mf_{i2})(ng_{i2})\otimes e_i))=$$
$$(f\otimes g)\cdot u((m\otimes a)\otimes(n\otimes b))$$

where we use the cocommutativity of H. This completes the proof.

Definition 2[10] A quasitriangular Hopf algebra is a pair (H,R), where H is a Hopf algebra, and $R\in H\otimes H$ such that

$$(\varepsilon\otimes\mathrm{id})R=(\mathrm{id}\otimes\varepsilon)R=1$$
$$(\Delta\otimes\mathrm{id})R=R_{13}R_{23}$$
$$(\mathrm{id}\otimes\Delta)R=R_{13}R_{23}$$
$$R\Delta(X)=\Delta^{\mathrm{op}}(x)R$$

where $\Delta^{\mathrm{op}}(x)=x_2\otimes x_1$; $R_{12}=R^1\otimes R^2\otimes 1$; $R_{13}=R^1\otimes 1\otimes R^2$; $R_{23}=1\otimes R^1\otimes R^2$.

Quasitriangular Hopf algebra is cocommutative Hopf algebra when $R=1\otimes 1$.

2 (H,R)-Pseudoalgebra

H is a left H-module via multiplication or adjoint action $\triangleright$, i. e. $h\triangleright g=h_1gS(h_2)$, so we modify the above definition of H-pseudoalgebra by changing the action of H. In order to distinguish the two different notions, here we denote $H=\widetilde{H}$.

Definition 3 An $\widetilde{H}$-pseudoalgebra is an object A in ${}_HM$ such that

$$fa*gb=(f\otimes g)\cdot(a*b)=f\triangleright f_i\otimes g\triangleright g_i\otimes_H e_i$$

where $a*b=f_i\otimes g_i\otimes_H e_i$.

Remark 1 $\widetilde{H}$-pseudoalgebra is the usual algebra when $H=k$.

Example 2 Let H' be the Hopf subalgebra of H such that $H'\triangleright H\subset H$, and A an $\widetilde{H}$-pseudoalgebra with the pseudoproduct μ. Then, A is an $\widetilde{H}'$-pseudoalgebra with the pseudoproduct μ.

We write the tensor product as $\hat{\otimes}$ in a braided monoidal category. Now, let us generalize the $\widetilde{H}$-pseudoalgebra as follows:

Definition 4 Let (H,R) be a quasitriangular Hopf algebra. An (H,R)-pseudoalgebra is an object A in ${}_HM$, such that

$$u((h\hat{\otimes}g)\cdot(a\hat{\otimes}b))=(h\hat{\otimes}g)\cdot u(a\hat{\otimes}b)$$

where $(h\hat{\otimes}g)\cdot(a\hat{\otimes}b)=(hR^2)\cdot a\hat{\otimes}(R^1\triangleright g)\cdot b$, $(h\hat{\otimes}g)\cdot u(a\hat{\otimes}b)=(hR^2)\triangleright f_i\otimes(R^1\triangleright g)\triangleright g_i\otimes_H e_i$, if $u(a\hat{\otimes}b)=(f_i\hat{\otimes}g_i)\hat{\otimes}_H e_i$.

Remark 2 1) ${}_HM$ is a braided monoidal category if H is a quasitriangular Hopf algebra.

2) An (H,R)-pseudoalgebra is an $\widetilde{H}$-pseudoalgebra when $R=1\otimes 1$.

Example 3 Let A be an (H,R)-pseudoalgebra with the pseudoproduct μ, and (H',R') the quasitriangular Hopf subalgebra of (H,R) such that $(H'H)\triangleright H\subset H$, $(H\triangleright H')\triangleright H\subset H'$. Then, A is an (H',R')-pseudoalgebra with the pseudoproduct μ.

Let $H'=\{x\in H\mid hx=\varepsilon(x)h,\ \forall\, h\in H\}$. Then, it is easy to see that (H',R') is the quasitriangular Hopf subalgebra of quasitriangular Hopf algebra (H,R).

Example 4 Let A be an (H',R')-pseudoalgebra. Then, $H\otimes_{H'}A$ is an (H,R)-pseudoalgebra with the following pseudoproduct:

$$u((f\otimes a)\hat{\otimes}(g\otimes b))=(ff_i\hat{\otimes}gg_i)\hat{\otimes}_H(1\otimes_{H'}e_i)$$

for all h, $g\in H$ and $a*b=(f_i\hat{\otimes}g_i)\hat{\otimes}_{H'}e_i\in(H'\hat{\otimes}H')\hat{\otimes}_{H'}A$.

Indeed, it is easy to check that $H\otimes_{H'}A$ is a left H-module with $h\cdot(g\otimes a)=h\triangleright g\otimes a$.

If $a*b=(f_i\hat{\otimes}g_i)\hat{\otimes}_{H'}e_i$, then for all $h,g,x,y\in H$, f_i, $g_i\in H'$, $a,b\in A$, we have

$$u((x\hat{\otimes}y)\cdot((h\otimes a)\hat{\otimes}(g\otimes b)))=$$
$$u(((xR^2)\triangleright h\otimes a)\hat{\otimes}((R^1\triangleright y)\triangleright g\otimes b))=$$
$$((xR^2)\triangleright h)f_i\hat{\otimes}((R^1\triangleright y)\triangleright g)g_i\hat{\otimes}_H(1\otimes e_i)=$$
$$((xR^2)\triangleright h)\varepsilon(f_i)\hat{\otimes}((R^1\triangleright y)\triangleright g)\varepsilon(g_i)\hat{\otimes}_H(1\otimes e_i)$$

and

$$(x\hat{\otimes}y)\cdot u((h\otimes a)\hat{\otimes}(g\otimes b))=$$
$$(x\hat{\otimes}y)\cdot(hf_i\hat{\otimes}gg_i\hat{\otimes}_H(1\otimes e_i))=$$

$$(xR^2)\rhd(hf_i)\hat{\otimes}(R^1\rhd y)\rhd(gg_i)\hat{\otimes}_H(1\otimes e_i)=$$
$$(xR^2)\rhd h\varepsilon(f_i)\hat{\otimes}(R^1\rhd y)\rhd g\varepsilon(g_i)\hat{\otimes}_H(1\otimes e_i)$$

Proposition 2 Let A be the usual algebra, and H a Hopf algebra. Then $H\otimes A$ is an (H,R)-pseudoalgebra under the pseudoproduct

$$u((h\otimes a)\hat{\otimes}(g\otimes b))=(h\otimes g)\hat{\otimes}_H(1\otimes ab)$$
$$\forall a,b\in A,\ h,g\in H$$

Proof $H\otimes A$ is a left H-module with the following structure:

$$h\cdot(g\otimes a)=h\rhd g\otimes a\qquad\forall\ h,g\in H,\ a\in A$$

For all $x,y,h,g\in H$, $a,b\in A$, we obtain

$$u((x\hat{\otimes}y)\cdot((h\otimes a)\hat{\otimes}(g\otimes b)))=$$
$$u(x\cdot R^2\cdot(h\otimes a)\hat{\otimes}(R^1\rhd y)\cdot(g\otimes b))=$$
$$u(((xR^2)\rhd h\otimes a)\hat{\otimes}((R^1\rhd y)\rhd g\otimes b))=$$
$$((xR^2)\rhd h\hat{\otimes}(R^1\rhd y)\rhd g)\hat{\otimes}_H(1\otimes ab)$$

and

$$(x\hat{\otimes}y)\cdot u((h\otimes a)\hat{\otimes}(g\otimes b))=$$
$$(x\hat{\otimes}y)\cdot((h\hat{\otimes}g)\hat{\otimes}_H(1\otimes ab))=$$
$$((xR^2)\rhd h\hat{\otimes}(R^1\rhd y)\rhd g)\hat{\otimes}_H(1\otimes ab)$$

Therefore, we have $u((x\hat{\otimes}y)\cdot((h\otimes a)\hat{\otimes}(g\otimes b)))=(x\hat{\otimes}y)\cdot u((h\otimes a)\hat{\otimes}(g\otimes b))$.

Proposition 3 Let A_1 be an (H_1,R)-pseudoalgebra, and A_2 an (H_2,r)-pseudoalgebra. Then, $A\equiv A_1\otimes A_2$ is $(H\equiv H_1\otimes H_2,\ \overline{R}\equiv R^1\otimes r^1\otimes R^2\otimes r^2)$-pseudoalgebra with the following pseudoproduct:

$$(a\otimes a')*(b\otimes b')=(f_i\otimes f'_j)\otimes(g_i\otimes g'_j)\otimes_H(e_i\otimes e'_j)$$

where $a*b=f_i\otimes g_i\otimes_{H_1}e_i\in H_1\otimes H_1\otimes_{H_1}A_1$, $a'*b'=f'_j\otimes g'_j\otimes_{H_2}e'_j\in H_2\otimes H_2\otimes_{H_2}A_2$.

Proof First, it is easy to formulate that $(H,\overline{R})$ is a quasitriangular Hopf algebra. The muitiplication and comultiplication are the usual tensor product and coproduct.

Secondly, A is a left H-module with $(h\otimes h')\cdot(a\otimes a')=h\cdot a\otimes h'\cdot a'$, where $h\in H$, $h'\in H'$, $a\in A$, $a'\in A'$.

Finally, since A_1 is an (H_1,R)-pseudoalgebra and A_2 is an (H_2,r)-pseudoalgebra, we have that for all $h,g,x\in H$, $h',g',x'\in H'$, $a,b\in A$, $a',b'\in A'$,

$$u((hR^2)\cdot a\hat{\otimes}(R^1g)\cdot b)=$$
$$((hR^2)\rhd f_i\hat{\otimes}(R^1\rhd g)\rhd g_i)\hat{\otimes}_{H_1}e_i$$
$$u((h'r^2)\cdot a'\hat{\otimes}(r^1g')\cdot b')=$$
$$((h'r^2)\rhd f'_j\hat{\otimes}(r^1\rhd g')\rhd g')_j\hat{\otimes}_{H_2}e'_j$$

Using the above two equations, we do a calculation as follows:

$$((h\otimes h')\hat{\otimes}(g\otimes g'))\cdot u((a\otimes a')\hat{\otimes}(b\otimes b'))=$$
$$((h\otimes h')\hat{\otimes}(g\otimes g'))\cdot$$
$$((f_i\otimes f'_j)\hat{\otimes}(g_i\otimes g'_j)\otimes(e_i\hat{\otimes}_He'_j))=$$
$$((hR^2)\rhd f_i\otimes(h'r^2)\rhd f'_j)\hat{\otimes}$$
$$((R^1\rhd g)\rhd g_i\otimes(r^1\rhd g')\rhd g'_j)\hat{\otimes}_H(e_i\otimes e_j)=$$
$$u(((hR^2)\cdot a\otimes(h'r^2)\cdot a')\hat{\otimes}((R^1\rhd g)\cdot$$
$$b\otimes(r^1\rhd g')\cdot b'))=$$
$$u(((h\otimes h')\hat{\otimes}(g\otimes g'))\cdot((a\otimes a')\hat{\otimes}(b\otimes b')))$$

This completes the proof.

We call $(H,\overline{R})$-pseudoalgebra $A_1\otimes A_2$ a tensor product of the (H_1,R)-pseudoalgebra A_1 and (H_2,r)-pseudoalgebra A_2.

Theorem 1 Let (H,R) be a quasitriangular Hopf algebra. Then, H is an (H,R)-pseudoalgebra (resp. H-pseudoalgebra) with the pseudoproduct

$$f*g=(f\otimes g)\otimes_H1\qquad\forall f,g\in H$$

Proof If we consider the action of H on itself is the adjoint action, then for all $h,g,a,b\in H$, we have

$$u((h\otimes g)\cdot(a\otimes b))=$$
$$((hR^2)\rhd a\otimes(R^1\rhd g)\rhd b)\otimes_H1=$$
$$(h\otimes g)\cdot((a\otimes b)\otimes_H1)=$$
$$(h\otimes g)\cdot u(a\otimes b)$$

Similarly, we can prove that H is an H-pseudoalgebra if the action of H on itself is multiplication. This completes the proof.

Remark 3 The Hopf algebra H is an $\widetilde{H}$-pseudoalgebra when $R=1\otimes1$.

References

[1] Kac V. *Vertex algebras for beginners* [M]. American Mathematical Society, 1998.

[2] Bakalov B, D'Andrea A, Kac V G. Theory of finite pseudoalgebras [J]. *Advances in Mathematics*, 2001, **162**(1): 1-140. DOI:10.1006/aima.2001.1993.

[3] Boyallian C, Liberati J Y. On pseudo-bialgebras [J]. *Journal of Algebra*, 2012, **372**: 1-34. DOI:10.1016/j.jalgebra.2012.08.009.

[4] Dorfman I. *Dirac structures and integrability of nonlinear evolution equations* [M]. Wiley & Sons, 1993.

[5] Gelfand I M, Dorfman I J. Hamiltonian operators and infinite-dimensional Lie algebra [J]. *Functional Analysis and Its Applications*, 1981, **15**(3): 173-187.

[6] Xu X P. Equivalence of conformal superalgebras to Hamiltonian superoperators [J]. *Algebra Colloquium*, 2001, **8**(1): 63-92.

[7] Sun Q. Generalization of H-pseudoalgebraic structures [J]. *Journal of Mathematical Physics*, 2012, **53**(1): 012105. DOI:10.1063/1.3665708.

[8] Wu Z. Leibniz H-pseudoalgebras [J]. *Journal of Algebra*, 2015, **437**: 133. DOI:10.1016/j.jalgebra.2015.04.019.

[9] Sweedler M E. *Hopf algebras* [M]. New York: W. A. Benjamin, 1969.

[10] Chari V, Pressley A. *A guide to quantum groups* [M]. Cambridge, UK: Cambridge University Press, 1995.

一类 H-伪代数的构造

刘琳琳　王栓宏

(东南大学数学学院, 南京 211189)

摘要: 设 H 是一个余交换的 Hopf 代数. 首先,通过把 H 在其自身上的正则作用(即左乘作用)替换成伴随作用,从而引入一类新的 H-伪代数,称为 $\widetilde{H}$-伪代数. 其次,设 (H,R) 是一个拟三角 Hopf 代数,通过把上述得到的一类新的 H-伪代数,即 $\widetilde{H}$-伪代数,推广到拟三角 Hopf 代数 (H,R) 上,构造了一类 (H,R)-伪代数. 并且,由一般代数及 (H,R)-伪代数的张量积给出了 (H,R)-伪代数的构造. 最后,给出了 (H,R)-伪代数的一些例子,以及 Hopf 代数成为 (H,R)-伪代数(或者 H-伪代数)的条件.

关键词: 余交换 Hopf 代数; H-伪代数; $\widetilde{H}$-伪代数; (H,R)-伪代数

中图分类号: O153