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Some new bound estimates of the Hermitian positive definite solutions of the nonlinear matrix equation $X^s + A^* X^{-t} A = Q$

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Abstract: The range and existence conditions of the Hermitian positive definite solutions of nonlinear matrix equations $\boldsymbol{X}^s + \boldsymbol{A}^* \boldsymbol{X}^{-t} \boldsymbol{A} = \boldsymbol{Q}$ are studied, where $\boldsymbol{A}$ is an $n \times n$ non-singular complex matrix and $\boldsymbol{Q}$ is an $n \times n$ Hermitian positive definite matrix and parameters $s, t > 0$. Based on the matrix geometry theory, relevant matrix inequality and linear algebra technology, according to the different value ranges of the parameters s, t, the existence intervals of the Hermitian positive definite solution and the necessary conditions for equation solvability are presented, respectively. Comparing the existing correlation results, the proposed upper and lower bounds of the Hermitian positive definite solution are more accurate and applicable.

Key words: nonlinear matrix equation; Hermitian positive definite solution; solution bound; matrix inequality

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In this paper, we consider the following nonlinear matrix equations:

$$\boldsymbol{X}^s + \boldsymbol{A}^* \boldsymbol{X}^{-t} \boldsymbol{A} = \boldsymbol{Q} \tag{1}$$

where $\boldsymbol{A}^*$ denotes the conjugate transpose of the $n \times n$ nonsingular complex matrix $\boldsymbol{A}$ and $\boldsymbol{Q}$ is a Hermitian positive definite matrix with the general case $s, t > 0$.

The nonlinear matrix equation $\boldsymbol{X}^s + \boldsymbol{A}^* \boldsymbol{X}^{-t} \boldsymbol{A} = \boldsymbol{Q}$ has many applications in control theory, dynamic programming, stochastic filtering and statistics etc., see Refs. [1 − 5] and the relevant reference therein. In recent years, the theory and computation of the Hermitian positive definite solutions of Eq. (1), such as existence conditions, perturbation analysis, and the numerical algorithm, have been widely studied. The case $s = t = 1$ has been discussed by several authors[1-2, 4-8]. Hasanov and Ivanov[9-10] discussed the Hermitian positive definite solutions with the case $s = 1$, $t \in \mathbf{N}^+$, where $\mathbf{N}^+$ denotes the set of positive integers. Liu et al.[11-13] considered the case $s, t \in \mathbf{N}^+$. The cases $s = 1$, $t \in [0, 1]$ and $s = 1$, $t \in [1, +\infty)$ were discussed by Hasanov et al.[14-15]. Cai et al.[16-17] considered the case: $0 < t \leqslant 1$, $s \geqslant 1$ or $t \geqslant 1$, $0 < s \leqslant 1$. However, the general case $s, t > 0$ is rarely considered.

In this paper, we study the bounds of the Hermitian positive definite solutions of Eq. (1) in the general case $s, t > 0$. In practice, giving precise estimates of solution bounds before actually solving the problem can help to search for the solutions and reduce computational burdens. It also can be a check of whether the solution techniques result in valid solutions. The following notations are used throughout this paper. $\boldsymbol{C}^{m \times n}$ denotes the set of $m \times n$ complex matrices. Let $\boldsymbol{B} > 0 (\boldsymbol{B} \geqslant 0)$ denote that $\boldsymbol{B}$ is a Hermitian positive definite (semidefinite) matrix. For Hermitian matrices $\boldsymbol{B}, \boldsymbol{C} \in \mathbf{C}^{n \times n}$, we write $\boldsymbol{B} > \boldsymbol{C} (\boldsymbol{B} \geqslant \boldsymbol{C})$ if $\boldsymbol{B} - \boldsymbol{C} > 0 (\boldsymbol{B} - \boldsymbol{C} \geqslant 0)$. For any Hermitian matrix $\boldsymbol{H}$, its eigenvalues are ordered as $\lambda_1(\boldsymbol{H}) \geqslant \lambda_2(\boldsymbol{H}) \geqslant \ldots \geqslant \lambda_n(\boldsymbol{H})$.

For convenience of discussion, in the sequel, a solution always means a Hermitian positive definite solution unless otherwise noted.

1 Some New Bounds of the Solutions

First, we give some lemmas which are useful for obtaining our main results.

Lemma 1[18] If $\boldsymbol{B} > \boldsymbol{C} > 0$ (or $\boldsymbol{B} \geqslant \boldsymbol{C} > 0$), then $\boldsymbol{B}^\gamma > \boldsymbol{C}^\gamma > 0$ (or $\boldsymbol{B}^\gamma \geqslant \boldsymbol{C}^\gamma > 0$) for all $\gamma \in (0, 1]$ and $0 < \boldsymbol{B}^\gamma < \boldsymbol{C}^\gamma$ (or $0 < \boldsymbol{B}^\gamma \leqslant \boldsymbol{C}^\gamma$) for all $\gamma \in [-1, 0)$.

Lemma 2[19] Let $\boldsymbol{B}$ and $\boldsymbol{C}$ be positive operators on a Hilbert space such that $0 < m_1 \leqslant \boldsymbol{B} \leqslant \boldsymbol{M}_1$, $0 < m_2 \leqslant \boldsymbol{C} \leqslant \boldsymbol{M}_2$, and $0 < \boldsymbol{B} \leqslant \boldsymbol{C}$. Then

$$\boldsymbol{B}^p \leqslant \left(\frac{\boldsymbol{M}_1}{m_1}\right)^{p-1} \boldsymbol{C}^p, \quad \boldsymbol{B}^p \leqslant \left(\frac{\boldsymbol{M}_2}{m_2}\right)^{p-1} \boldsymbol{C}^p$$

hold for any $p \geqslant 1$.

Lemma 3[2] Let $\boldsymbol{B} \in \mathbf{C}^{n \times n}$, $\boldsymbol{U} \in \mathbf{C}^{n \times k}$, $\boldsymbol{V} \in \boldsymbol{C}^{k \times n}$, and $\boldsymbol{B}, \boldsymbol{B} - \boldsymbol{U}\boldsymbol{V}$ be nonsingular, then

$$(\boldsymbol{B} - \boldsymbol{U}\boldsymbol{V})^{-1} = \boldsymbol{B}^{-1} - \boldsymbol{B}^{-1}\boldsymbol{U}(\boldsymbol{V}\boldsymbol{B}^{-1}\boldsymbol{U} - \boldsymbol{I})^{-1}\boldsymbol{V}\boldsymbol{B}^{-1}$$

Lemma 4[13] Let $f(x) = x^t(\theta - x^s)$, $\theta > 0$, $x \geqslant 0$. Then

1) f is increasing on $\left[0, \left(\frac{t}{s+t}\theta\right)^{1/s}\right]$ and decreasing on $\left[\left(\frac{t}{s+t}\theta\right)^{1/s}, +\infty\right)$;

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2) $f_{\max} = f\left(\left(\frac{t}{s+t}\theta \right)^{1/s} \right) = \frac{s}{s+t}\left(\frac{t}{s+t} \right)^{t/s} \theta^{t/s+1}$.

The following theorem presents some solution bounds of Eq. (1) which may help us to obtain rough estimate of the solutions and reduce the computational cost.

Theorem 1 Suppose that X is a solution of Eq. (1). If $s \geqslant t > 0$ and $s \geqslant 1$, then

$$\left(\frac{\lambda_n(B)}{\lambda_1(B)} \right)^{1/t-1/s} B^{1/t} < X < (Q - A^*(Q - A^* Q^{-t/s} A)^{-t/s} A)^{1/s} \tag{2}$$

where $B = AQ^{-1}A^* + AQ^{-1}(\eta(AQ^{-1}A^*)^{-s/t} - Q^{-1})^{-1} Q^{-1} A^*$ with $\eta = \left(\frac{\lambda_1(AQ^{-1}A^*)}{\lambda_n(AQ^{-1}A^*)} \right)^{s/t-1}$.

If $0 < s \leqslant t$ and $t \geqslant 1$, then

$$X > \{AQ^{-1}A^* + AQ^{-1}[(Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)} \right)^{t/s-1} A^* Q^{-t/s} A)^{-1} - Q^{-1}]^{-1} Q^{-1} A^*\}^{1/t} \tag{3}$$

and

$$X < \left(\frac{\mu}{\nu} \right)^{1/s-1/t} \left(Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)} \right)^{t/s-1} A^* Q^{-t/s} A \right)^{1/s} \tag{4}$$

where μ and ν are the maximal and minimal eigenvalues of the matrx $Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)} \right)^{t/s-1} A^* Q^{-t/s} A$.

Proof Suppose that X is a solution of Eq. (1), then $X^s = Q - A^* X^{-t} A < Q$.

If $s \geqslant t > 0$ and $s \geqslant 1$, then according to Lemma 1, we have $X^{-t} = (X^{-s})^{t/s} > Q^{-t/s}$.

From which it follows that

$$X^{-t} = (X^s)^{-t/s} = (Q - A^* X^{-t} A)^{-t/s} > (Q - A^* Q^{-t/s} A)^{-t/s}$$

Then, we have

$$X^s = Q - A^* X^{-t} A < Q - A^* (Q - A^* Q^{-t/s} A)^{-t/s} A$$

which yields

$$X < (Q - A^* (Q - A^* Q^{-t/s} A)^{-t/s} A)^{1/s} \tag{5}$$

On the other hand, since

$$\begin{aligned} X^{-t} &= (A^*)^{-1}(Q - X^s)A^{-1} = \\ &(A(Q - X^s)^{-1}A^*)^{-1} < (AQ^{-1}A^*)^{-1} \end{aligned} \tag{6}$$

according to Lemma 2, we obtain

$$\begin{aligned} X^{-s} &= (X^{-t})^{s/t} < \left(\frac{\lambda_1(AQ^{-1}A^*)}{\lambda_n(AQ^{-1}A^*)} \right)^{s/t-1} (AQ^{-1}A^*)^{-s/t} \equiv \\ &\eta(AQ^{-1}A^*)^{-s/t} \end{aligned} \tag{7}$$

By Lemma 3 and (7), we have

$$\begin{aligned} X^t &= A(Q - X^s)^{-1} A^* = \\ &A(Q^{-1} - Q^{-1}(X^s Q^{-1} - I)^{-1} X^s Q^{-1}) A^* = \\ &A(Q^{-1} + Q^{-1}(X^{-s} - Q^{-1})^{-1} Q^{-1}) A^* > \\ &AQ^{-1}A^* + AQ^{-1}(\eta(AQ^{-1}A^*)^{-s/t} - Q^{-1})^{-1} Q^{-1} A^* \equiv B \end{aligned} \tag{8}$$

Then, according to Lemma 2, we obtain

$$X^s = (X^t)^{s/t} > \left(\frac{\lambda_n(B)}{\lambda_1(B)} \right)^{s/t-1} B^{s/t}$$

which implies $X > \left(\frac{\lambda_n(B)}{\lambda_1(B)} \right)^{1/t-1/s} B^{1/t}$.

If $0 < s \leqslant t$ and $t \geqslant 1$, according to Lemma 2 and the inequality

$$X^s = Q - A^* X^{-t} A < Q$$

we obtain $X^t = (X^s)^{t/s} < \left(\frac{\lambda_1(Q)}{\lambda_n(Q)} \right)^{t/s-1} Q^{t/s}$. Then we have

$$X^s = Q - A^* X^{-t} A < Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)} \right)^{t/s-1} A^* Q^{-t/s} A \tag{9}$$

Hence, by Lemma 2, we have

$$X^t = (X^s)^{t/s} < \left(\frac{\mu}{\nu} \right)^{t/s-1} \left(Q - \frac{\lambda_n(Q)}{\lambda_1(Q)} \right)^{t/s-1} A^* Q^{-t/s} A)^{t/s} \tag{10}$$

where μ and ν are the maximal and minimal eigenvalues of the matrix $Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)} \right)^{t/s-1} A^* Q^{-t/s} A$.

Noting that $t \geqslant 1$, it follows from (10) and Lemma 1 that

$$X < \left(\frac{\mu}{\nu} \right)^{1/s-1/t} \left(Q - (\frac{\lambda_n(Q)}{\lambda_1(Q)} \right)^{t/s-1} A^* Q^{-t/s} A)^{1/s}$$

Combining (9) and Lemma 3, we obtain

$$\begin{aligned} X^t &= A(Q - X^s)^{-1} A^* = \\ &A(Q^{-1} - Q^{-1}(X^s Q^{-1} - I)^{-1} X^s Q^{-1}) A^* = \\ &A(Q^{-1} + Q^{-1}(X^{-s} - Q^{-1})^{-1} Q^{-1}) A^* > \\ &AQ^{-1}A^* + AQ^{-1}[(Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)} \right)^{t/s-1} A^* Q^{-t/s} A)^{-1} - \\ &Q^{-1}]^{-1} Q^{-1} A^* \end{aligned}$$

which yields

$$X > \{AQ^{-1}A^* + AQ^{-1}[(Q - (\frac{\lambda_n(Q)}{\lambda_1(Q)})^{t/s-1} A^* Q^{-t/s} A)^{-1} - Q^{-1}]^{-1} Q^{-1} A^*\}^{1/t}$$

This completes the proof.

Corollary 1 Assume that Eq. (1) has a solution.

If $s \geqslant 1, 0 < t \leqslant 1$, then

$$\left(\frac{\lambda_n(B)}{\lambda_1(B)} \right)^{1/t-1} B^{1/t} < X < (Q - A^*(Q - A^* Q^{-t/s} A)^{-t/s} A)^{1/s} \tag{11}$$

where B is as the same as the one defined in Theorem 1.

If $0 < s \leqslant 1$, $t \geqslant 1$, then

$$X > \{AQ^{-1}A^* + AQ^{-1}[(Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)} \right)^{t/s-1} A^* Q^{-t/s} A)^{-1} - Q^{-1}]^{-1} Q^{-1} A^*\}^{1/t} \tag{12}$$

and

$$X < \left(\frac{\mu}{\nu}\right)^{1/s-1}\left(Q - \frac{\lambda_n(Q)}{\lambda_1(Q)}\right)^{t/s-1} A^* Q^{-t/s} A)^{1/s} \quad (13)$$

where μ and ν are as the same as the ones defined in Theorem 1.

Proof

If $s \geqslant 1, 0 < t \leqslant 1$, then by (8) and Lemma 2, we obtain $X > (\frac{\lambda_n(B)}{\lambda_1(B)})^{1/t-1} B^{1/t}$. Moreover, the upper bound in (11) can be derived by Theorem 1. If $0 < s \leqslant 1, t \geqslant 1$, then by (9) and Lemma 2, we obtain $X < \left(\frac{\mu}{\nu}\right)^{1/s-1} (Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)}\right)^{t/s-1} A^* Q^{-t/s} A)^{1/s}$.

The lower bound in (12) can be derived by Theorem 1. This completes the proof.

Remark 1 Zhou et al.[17] presented the bounds of the solutions of Eq. (1) with the case $s \geqslant 1, 0 < t \leqslant 1$ or $0 < s \leqslant 1, t \geqslant 1$ as follows.

If $s \geqslant 1, 0 < t \leqslant 1$, then

$$\left(\frac{\lambda_n(B)}{\lambda_1(B)}\right)^{1/t-1} B^{1/t} < X < (Q - A^* Q^{-t/s} A)^{1/s} \quad (14)$$

If $0 < s \leqslant 1, t \geqslant 1$, then

$$[AQ^{-1}A^* + AQ^{-1}((AQ^{-1}A^*)^{-s/t} - Q^{-1})^{-1} Q^{-1} A^*]^{1/t} <$$
$$X < \left(\frac{\mu}{\nu}\right)^{1/s-1} \left(Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)}\right)^{t/s-1} A^* Q^{-t/s} A\right)^{1/s} \quad (15)$$

where B, μ and ν are as the same as the ones defined in Theorem 1.

Comparing Corollary 1 with (14) and (15), one can easily find that for the case $s \geqslant 1$, $0 < t \leqslant 1$, the upper bound of X in (11) is smaller than that in (14). For the case $0 < s \leqslant 1, t \geqslant 1$, combining (6) and (9), we have

$$\left(Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)}\right)^{t/s-1} A^* Q^{-t/s} A\right)^{-1} < X^{-s} =$$
$$(X^{-t})^{s/t} < (AQ^{-1}A^*)^{-s/t} \quad (16)$$

which implies that the lower bound of X in (12) is larger than that in (15). Hence, our estimate in Corollary 1 is sharper.

Corollary 2 Assume that Eq. (1) with the case $s, t \in \mathbf{N}^+$ has a solution.

If $s \geqslant t$, then

$$B^{1/t} < X < (Q - A^* (Q - A^* Q^{-t/s} A)^{-t/s} A)^{1/s} \quad (17)$$

where B is as the same as the one defined in Theorem 1.

If $s \leqslant t$, then

$$X > \{AQ^{-1}A^* + AQ^{-1}[(Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)}\right)^{t/s-1} A^* Q^{-t/s} A)^{-1} -$$
$$Q^{-1}]^{-1} Q^{-1} A^*\}^{1/t}$$

and

$$X < \left(Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)}\right)^{t/s-1} A^* Q^{-t/s} A\right)^{1/s} \quad (18)$$

Proof For the case $s, t \in \mathbf{N}^+$, if $s \geqslant t$, then it follows from the right sides of (2), (8) and Lemma 1 that (17) holds. If $s \leqslant t$, it follows from the left sides of (3), (9) and Lemma 1 that (18) holds. This completes the proof.

Remark 2 Duan and Liao[13] obtained the bounds of the solution X of Eq. (1) with the case $s, t \in \mathbf{N}^+$ as follows:

$$X \in (M, N) \quad (19)$$

where

$$M = \{AQ^{-1}A^* + AQ^{-1}\left[\left(\frac{\lambda_1(A^{-*}QA^{-1})}{\lambda_n(A^{-*}QA^{-1})}\right)^{(s-1)/t} \cdot\right.$$
$$\left.(A^{-*}QA^{-1})^{s/t} - Q^{-1}\right]^{-1} Q^{-1} A^*\}^{1/t}$$
$$N = [Q - (\frac{\lambda_n(Q^{-1})}{\lambda_1(Q^{-1})})^{\frac{t-1}{s}} A^* Q^{-t/s} A]^{1/s}$$

For the case $s, t \in \mathbf{N}^+$, if $s \geqslant t$, comparing (17) with (19), we have

$$B^{1/t} = [AQ^{-1}A^* + AQ^{-1}(\eta(AQ^{-1}A^*)^{-s/t} - Q^{-1})^{-1} Q^{-1} A^*]^{1/t} =$$
$$\{AQ^{-1}A^* + AQ^{-1}\left[\left(\frac{\lambda_1(AQ^{-1}A^*)}{\lambda_n(AQ^{-1}A^*)}\right)^{s/t-1} (AQ^{-1}A^*)^{-s/t} -\right.$$
$$\left.Q^{-1}\right]^{-1} Q^{-1} A^*\}^{1/t} > M$$

and $(Q - A^* (Q - A^* Q^{-t/s} A)^{-t/s} A)^{1/s} < (Q - A^* Q^{-t/s} A)^{1/s} < N$.

If $s \leqslant t$, since $s \geqslant 1$, we have

$$\left(Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)}\right)^{t/s-1} A^* Q^{-t/s} A\right)^{1/s} \leqslant$$
$$\left[Q - \left(\frac{\lambda_n(Q^{-1})}{\lambda_1(Q^{-1})}\right)^{(t-1)/s} A^* Q^{-t/s} A\right]^{1/s} = N$$

Furthermore, according to (16), we obtain

$$\{AQ^{-1}A^* + AQ^{-1}[(Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)}\right)^{t/s-1} A^* Q^{-t/s} A)^{-1} -$$
$$Q^{-1}]^{-1} Q^{-1} A^*\}^{1/t} > M$$

So, our estimate in Corollary 2 is sharper than Theorem 2.1 in Duan and Liao[13].

2 Conditions for the Existence of Solutions

The following theorem gives the necessary condition for the existence of solutions.

Theorem 2 If Eq. (1) with the case $s \geqslant t$ has a solution X, then

$$\sigma_n^2(Q^{-t/2s} A Q^{-1/2}) < \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{t/s}$$

and $X^s \leqslant \alpha Q$, where α is a solution of the equation $x^{-t/s}(1 - x^{-1}) = \sigma_n^2(Q^{-t/2s} A Q^{-1/2})$ in $[1, \frac{s+t}{t}]$.

If Eq. (1) with the case $s \leqslant t$ has a solution $\boldsymbol{X}$, then

$$\sigma_n^2((\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/2t}\boldsymbol{Q}^{-1/2}) < \frac{t}{s+t}\left(\frac{s}{s+t}\right)^{s/t}$$

and $\boldsymbol{X}^s \geqslant \beta^{s/t}(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/t}$, where β is a solution of the equation $x^{-s/t}(1-x^{-1}) = \sigma_n^2((\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/2t}\boldsymbol{Q}^{-1/2})$ in $[1, \frac{s+t}{s}]$.

Proof Let $\boldsymbol{X}$ be a solution of Eq. (1). Then $\boldsymbol{X}^{-s}$ is a solution of the following equation:

$$\boldsymbol{Y}^{-1} + \boldsymbol{A}^*\boldsymbol{Y}^{t/s}\boldsymbol{A} = \boldsymbol{Q} \tag{20}$$

From which it follows that

$$\boldsymbol{Q}^{-1/2}\boldsymbol{Y}^{-1}\boldsymbol{Q}^{-1/2} = \boldsymbol{I} - \boldsymbol{Q}^{-1/2}\boldsymbol{A}^*\boldsymbol{Y}^{t/s}\boldsymbol{A}\boldsymbol{Q}^{-1/2}$$

Then, we have

$$\begin{aligned}\lambda_n(\boldsymbol{Q}^{1/2}\boldsymbol{Y}\boldsymbol{Q}^{1/2})^{-1} &= \lambda_1(\boldsymbol{Q}^{-1/2}\boldsymbol{Y}^{-1}\boldsymbol{Q}^{-1/2}) = \\ &\lambda_1(\boldsymbol{I} - \boldsymbol{Q}^{-1/2}\boldsymbol{A}^*\boldsymbol{Y}^{t/s}\boldsymbol{A}\boldsymbol{Q}^{-1/2}) = \\ &1 - \lambda_n(\boldsymbol{Q}^{-1/2}\boldsymbol{A}^*\boldsymbol{Q}^{-t/2s}\boldsymbol{Q}^{t/2s}\boldsymbol{Y}^{t/s}\boldsymbol{Q}^{t/2s}\boldsymbol{Q}^{-t/2s}\boldsymbol{A}\boldsymbol{Q}^{-1/2}) = \\ &1 - \lambda_n(\boldsymbol{Q}^{1/2}\boldsymbol{Y}\boldsymbol{Q}^{1/2})^{t/s}\sigma_n^2(\boldsymbol{Q}^{-t/2s}\boldsymbol{A}\boldsymbol{Q}^{-1/2})\end{aligned}$$

which yields

$$\sigma_n^2(\boldsymbol{Q}^{-t/2s}\boldsymbol{A}\boldsymbol{Q}^{-1/2}) \leqslant \lambda_n(\boldsymbol{Q}^{1/2}\boldsymbol{Y}\boldsymbol{Q}^{1/2})^{-t/s}[1 - \lambda_n(\boldsymbol{Q}^{1/2}\boldsymbol{Y}\boldsymbol{Q}^{1/2})^{-1}] \tag{21}$$

Consider the function $f(x) = x^{-t/s}(1-x^{-1})$. Noting that $\max\limits_{x\in(0,+\infty)} f(x) = f\left(\frac{s+t}{t}\right) = \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{t/s}$, then by (21), we obtain $\sigma_n^2(\boldsymbol{Q}^{-t/2s}\boldsymbol{A}\boldsymbol{Q}^{-1/2}) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{t/s}$.

Define a sequence as follows:

$$\alpha_0 = 1, \alpha_{k+1} = (1 - \alpha_k^{t/s}\sigma_n^2(\boldsymbol{Q}^{-t/2s}\boldsymbol{A}\boldsymbol{Q}^{-1/2}))^{-1} \qquad k = 0, 1, 2, \ldots \tag{22}$$

For the case $s \geqslant t$, we have

$$\boldsymbol{Y} = (\boldsymbol{Q} - \boldsymbol{A}^*\boldsymbol{Y}^{t/s}\boldsymbol{A})^{-1} > \boldsymbol{Q}^{-1} = \alpha_0\boldsymbol{Q}^{-1}$$

Assume that $\boldsymbol{Y} \geqslant \alpha_k\boldsymbol{Q}^{-1}$. By Lemma 1, we have

$$\begin{aligned}\boldsymbol{Y} = (\boldsymbol{Q} - \boldsymbol{A}^*\boldsymbol{Y}^{t/s}\boldsymbol{A})^{-1} &\geqslant (\boldsymbol{Q} - \alpha_k^{t/s}\boldsymbol{A}^*\boldsymbol{Q}^{-t/s}\boldsymbol{A})^{-1} = \\ &[\boldsymbol{Q}^{1/2}(\boldsymbol{I} - \alpha_k^{t/s}\boldsymbol{Q}^{-1/2}\boldsymbol{A}^*\boldsymbol{Q}^{-t/s}\boldsymbol{A}\boldsymbol{Q}^{-1/2})^{-1}\boldsymbol{Q}^{1/2}]^{-1} \geqslant \\ &[\boldsymbol{Q}^{1/2}(1 - \alpha_k^{t/s}\sigma_n^2(\boldsymbol{Q}^{-t/2s}\boldsymbol{A}\boldsymbol{Q}^{-1/2})\boldsymbol{Q}^{1/2}]^{-1} = \\ &[\alpha_{k+1}^{-1}\boldsymbol{Q}]^{-1} = \alpha_{k+1}\boldsymbol{Q}^{-1}.\end{aligned}$$

So, by the principle of induction, we obtain $\boldsymbol{Y} \geqslant \alpha_k\boldsymbol{Q}^{-1}$ $(k = 0, 1, 2, \ldots)$. In addition, it is easy to verify that the sequence $\{\alpha_k\}$ is monotonically increasing and has an upper bound $(s+t)/t$ by induction. Hence, $\{\alpha_k\}$ is convergent.

Let $\lim\limits_{k\to\infty}\alpha_k = \alpha$, then $\boldsymbol{Y} \geqslant \alpha\boldsymbol{Q}^{-1}$. Hence, $\boldsymbol{X}^s \leqslant \alpha\boldsymbol{Q}$, where α is a solution of the equation $x^{-t/s}(1-x^{-1}) = \sigma_n^2(\boldsymbol{Q}^{-t/2s}\boldsymbol{A}\boldsymbol{Q}^{-1/2})$ in $[1, \frac{s+t}{t}]$.

On the other hand, if Eq. (1) with the case $s \leqslant t$ has a solution $\boldsymbol{X}$, then we have $\boldsymbol{X}^t = \boldsymbol{A}(\boldsymbol{Q} - \boldsymbol{X}^s)^{-1}\boldsymbol{A}^*$.

Let $\boldsymbol{Z} = \boldsymbol{X}^s$, the former matrix equation can be reduced to $\boldsymbol{Z} = [\boldsymbol{A}(\boldsymbol{Q} - \boldsymbol{Z}) - 1\boldsymbol{A}^*]^{s/t}$. Then, we obtain

$$\begin{aligned}\lambda_n(\boldsymbol{Q}^{-1/2}\boldsymbol{Z}\boldsymbol{Q}^{-1/2}) &= \lambda_n(\boldsymbol{Q}^{-1/2}[\boldsymbol{A}(\boldsymbol{Q} - \boldsymbol{Z})^{-1}\boldsymbol{A}^*]^{s/t}\boldsymbol{Q}^{-1/2}) = \\ &\lambda_n(\boldsymbol{Q}^{-1/2}[\boldsymbol{A}\boldsymbol{Q}^{-1/2}(\boldsymbol{I} - \boldsymbol{Q}^{-1/2}\boldsymbol{Z}\boldsymbol{Q}^{-1/2})^{-1}\boldsymbol{Q}^{-1/2}\boldsymbol{A}^*]^{s/t}\boldsymbol{Q}^{-1/2}) \geqslant \\ &\lambda_n(\boldsymbol{Q}^{-1/2}(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/t}\boldsymbol{Q}^{-1/2})\lambda_n((\boldsymbol{I} - \boldsymbol{Q}^{-1/2}\boldsymbol{Z}\boldsymbol{Q}^{-1/2})^{-s/t}) \geqslant \\ &\sigma_n^2((\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/2t}\boldsymbol{Q}^{-1/2})[1 - \lambda_n(\boldsymbol{Q}^{-1/2}\boldsymbol{Z}\boldsymbol{Q}^{-1/2})]^{-s/t}\end{aligned}$$

That is

$$\begin{aligned}\sigma_n^2((\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/2t}\boldsymbol{Q}^{-1/2}) \leqslant \lambda_n(\boldsymbol{Q}^{-1/2}\boldsymbol{Z}\boldsymbol{Q}^{-1/2})\cdot \\ [1 - \lambda_n(\boldsymbol{Q}^{-1/2}\boldsymbol{Z}\boldsymbol{Q}^{-1/2})]^{s/t}\end{aligned} \tag{23}$$

Consider the function $g(x) = x(1-x)^{s/t}$. Noting that $\max\limits_{x\in(0,+\infty)} g(x) = f\left(\frac{t}{s+t}\right) = \frac{t}{s+t}\left(\frac{s}{s+t}\right)^{s/t}$, by (23), we obtain $\sigma_n^2((\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/2t}\boldsymbol{Q}^{-1/2}) \leqslant \frac{t}{s+t}\left(\frac{s}{s+t}\right)^{s/t}$.

Define another sequence as follows:

$$\beta_0 = 1, \beta_{k+1} = (1 - \beta_k^{s/t}\sigma_n^2((\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/2t}\boldsymbol{Q}^{-1/2}))^{-1} \quad k = 0, 1, 2, \ldots \tag{24}$$

For the case $s \leqslant t$, we have

$$\boldsymbol{Z} = [\boldsymbol{A}(\boldsymbol{Q} - \boldsymbol{Z})^{-1}\boldsymbol{A}^*]^{s/t} > \beta_0^{s/t}(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/t}$$

Assume that $\boldsymbol{Z} > \beta_k^{s/t}(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/t}$. By Lemma 1 and hypothesis, we have

$$\begin{aligned}\boldsymbol{Z} = [\boldsymbol{A}(\boldsymbol{Q} - \boldsymbol{Z})^{-1}\boldsymbol{A}^*]^{s/t} &= \\ &[\boldsymbol{A}\boldsymbol{Q}^{-1/2}(\boldsymbol{I} - \boldsymbol{Q}^{-1/2}\boldsymbol{Z}\boldsymbol{Q}^{-1/2})^{-1}\boldsymbol{Q}^{-1/2}\boldsymbol{A}^*]^{s/t} \geqslant \\ &[\boldsymbol{A}\boldsymbol{Q}^{-1/2}(\boldsymbol{I} - \beta_k^{s/t}\boldsymbol{Q}^{-1/2}(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/t}\boldsymbol{Q}^{-1/2})^{-1}\boldsymbol{Q}^{-1/2}\boldsymbol{A}^*]^{s/t} \geqslant \\ &[\boldsymbol{A}\boldsymbol{Q}^{-1/2}(\boldsymbol{I} - \beta_k^{s/t}\sigma_n^2((\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/2t}\boldsymbol{Q}^{-1/2})^{-1}\boldsymbol{Q}^{-1/2}\boldsymbol{A}^*]^{s/t} = \\ &\beta_{k+1}^{s/t}(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/t}\end{aligned}$$

Then, by the principle of induction, we conclude that $\boldsymbol{Z} > \beta_k^{s/t}(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/t}$ $(k = 0, 1, 2, \ldots)$. In addition, it is easy to verify that $\{\beta_k\}$ is monotonically increasing and has an upper bound $(s+t)/s$ by induction. Hence $\{\beta_k\}$ is convergent.

Let $\lim\limits_{k\to\infty}\beta_k = \beta$, then $\boldsymbol{Z} > \beta^{s/t}(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/t}$, i. e., $\boldsymbol{X}^s > \beta^{s/t}(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/t}$, where β is a solution of the equation $x^{-s/t}(1-x^{-1}) = \sigma_n^2((\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^*)^{s/2t}\boldsymbol{Q}^{-1/2})$ in $[1, \frac{s+t}{s}]$.

This completes the proof.

3 Conclusion

In this paper, the bounds and existence conditions of the Hermitian positive definite solutions of the nonlinear matrix equation $\boldsymbol{X}^s + \boldsymbol{A}^*\boldsymbol{X}^{-t}\boldsymbol{A} = \boldsymbol{Q}$ with general case $s, t > 0$ are discussed. Sharper bound estimations of the solutions of Eq. (1) are obtained with respect to different cases of s, t. The necessary condition for the existence of solutions is presented. Based on this, some corresponding results of Eq. (1) are generalized and improved.

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非线性矩阵方程 $X^{s}+A^{*}X^{-t}A=Q$ 的 Hermitian 正定解的界的新估计

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摘要: 研究了非线性矩阵方程 $X^{s}+A^{*}X^{-t}A=Q$ 的 Hermitian 正定解的范围和存在条件,其中 A 为 n 阶非奇异复矩阵,Q 为 n 阶 Hermitian 正定矩阵,参数 $s,t>0$. 基于矩阵几何理论、相关矩阵不等式和线性代数技术,针对参数 s,t 的不同取值范围,给出了 Hermitian 正定解的存在区间和方程可解的必要条件. 比较已有的相关结果,所给出的 Hermitian 正定解的上界和下界估计更加精准,适用范围更广.

关键词: 非线性; 矩阵方程; Hermitian 正定解; 解的界; 矩阵不等式

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