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Galois linear maps and their construction

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Abstract: The condition of an algebra to be a Hopf algebra or a Hopf (co) quasigroup can be determined by the properties of Galois linear maps. For a bialgebra H, if it is unital and associative as an algebra and counital coassociative as a coalgebra, then the Galois linear maps T_1 and T_2 can be defined. For such a bialgebra H, it is a Hopf algebra if and only if T_1 is bijective. Moreover, T_1^{-1} is a right H-module map and a left H-comodule map (similar to T_2). On the other hand, for a unital algebra (no need to be associative), and a counital coassociative coalgebra A, if the coproduct and counit are both algebra morphisms, then the sufficient and necessary condition of A to be a Hopf quasigroup is that T_1 is bijective, and T_1^{-1} is left compatible with $\Delta^r_{T_1^{-1}}$ and right compatible with $m^1_{T_1^{-1}}$ at the same time (The properties are similar to T_2). Furthermore, as a corollary, the quasigroups case is also considered.

Key words: Galois linear map; antipode; Hopf algebra; Hopf (co) quasigroup

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1 Preliminaries

An algebra (A, m) is a vector space A over a field k equipped with a map m: $A \otimes A \to A$. A unital algebra (A, m, μ) is a vector space A over a field k equipped with two maps m: $A \otimes A \to A$ and μ: $k \to A$ such that $m(\mathrm{id} \otimes \mu) = \mathrm{id} = m(\mu \otimes \mathrm{id})$, where the natural identification $A \otimes k \cong k \cong k \otimes A$ is assumed. Generally, we write $1 \in A$ for $\mu(1_k)$.

The algebra (A, m, μ) is called associative if $m(\mathrm{id} \otimes m) = m(m \otimes \mathrm{id})$. It is customary to write

$$m(x \otimes y) = xy \qquad \forall\, x, y \in C$$

A coalgebra (C, Δ) is a vector space C over a field k equipped with a map Δ: $C \to C \otimes C$. A counital coalgebra (C, Δ, ε) is a vector space C over a field k equipped with two maps Δ: $C \to C \otimes C$ and ε: $C \to k$ such that $(\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id})\Delta$, where the natural identification $C \otimes k \cong k \cong k \otimes C$ is assumed.

The coalgebra (C, Δ, ε) is called coassociative if $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$. By using the Sweedler's notation in Ref. [1], it is customary to write

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \qquad \forall\, x \in C$$

Given acounital coalgebra (C, Δ, ε) and a unital algebra (A, m, μ), the vector space $\mathrm{Hom}(C, A)$ is a unital algebra with the product given by the convolution

$$(f * g)(x) = \sum f(x_{(1)})\, g(x_{(2)}) \tag{1}$$

for all $x \in C$, and unit element $\mu\varepsilon$. This algebra is denoted as $C * A$.

In particular, we have the algebra $\mathrm{End}(C)$ of endomorphisms on a given counital coalgebra (C, Δ, ε). Then, we have the convolution algebra $C * \mathrm{End}(C)$ with the unit element id: $x \mapsto \varepsilon(x)\,\mathrm{id}_C$. In the case that the coalgebra C is coassociative, then $C * \mathrm{End}(C)$ is an associative algebra.

Anonunital noncounital bialgebra (B, Δ, m) is an algebra (B, m) and a coalgebra (B, Δ) such that

$$\Delta(xy) = \Delta(x)\Delta(y) \qquad \forall\, x, y \in B$$

A counital bialgebra $(B, \Delta, \varepsilon, m)$ is a counital coalgebra (B, Δ, ε) and an algebra (B, m) such that

$$\Delta(xy) = \Delta(x)\Delta(y), \varepsilon(xy) = \varepsilon(x)\varepsilon(y) \qquad \forall\, x, y \in B$$

The multiplicative structure of a counital bialgebra $(B, \Delta, \varepsilon, m)$ is determined by the elements of $\mathrm{Hom}(B, \mathrm{End}(B))$:

$$L: B \to \mathrm{End}(B), \quad a \mapsto L_a (L_a(x) = ax)$$

and

$$R: B \to \mathrm{End}(B), \quad a \mapsto R_a (R_a(x) = xa)$$

Obviously, it satisfies one of these maps to determine the multiplicative structure.

A unital bialgebra (B, Δ, m, μ) is a coalgebra (B, Δ) and a unital (B, m, μ) such that

$$\Delta(xy) = \Delta(x)\Delta(y), \Delta(1) = 1 \qquad \forall\, x, y \in B$$

A unital counital bialgebra $(B, \Delta, \varepsilon, m, \mu)$ is both a unital bialgebra (B, Δ, m, μ) and a counital bialgebra $(B,$

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$\Delta, \varepsilon, m)$ such that $\varepsilon(1)=1$.

Given aunital bialgebra (B, Δ, m, μ), we define the following two Galois linear maps[2-3]:

$$T_1: B\otimes B \to B\otimes B, \quad T_1(x\otimes y) = \Delta(x)(1\otimes y) \quad (2)$$
$$T_2: B\otimes B \to B\otimes B, \quad T_2(x\otimes y) = (x\otimes 1)\Delta(y) \quad (3)$$

for all $x, y \in B$.

It is easy to check that $B\otimes B$ is a left B-module and a right B-module with the respective module structure:

$$a(x\otimes y) = ax\otimes y, \quad (x\otimes y)a = x\otimes ya$$

for all $a, x, y \in B$.

Similarly, $B\otimes B$ is a left B-comodule and a right B-comodule with the respective comodule structure:

$$\rho^l_{H\otimes H}(x\otimes y) = \sum x_{(1)} \otimes (x_{(2)} \otimes y)$$

and

$$\rho^r_{H\otimes H}(x\otimes y) = \sum (x\otimes y_{(1)}) \otimes y_{(2)}$$

for all $a, x, y \in B$.

2 Hopf Algebras

A Hopf algebra H is a unital associative counital coassociative bialgebra $(H, \Delta, \varepsilon, m, \mu)$ equipped with a linear map $S: H\to H$ such that

$$\sum S(h_{(1)})h_{(2)} = \sum h_{(1)}S(h_{(2)}) = \varepsilon(h)1 \quad (4)$$

for all $h, g \in H$.

We have the main result of this section as follows.

Theorem 1 Let $H := (H, \Delta, \varepsilon, m, \mu)$ be a unital associative counital coassociative bialgebra.

Then, the following statements are equivalent:

1) H is a Hopf algebra;

2) There is a linear map $S: H\to H$ such that S and id are invertible to each other in the convolution algebra $H * H$;

3) The linear map $T_1: H\otimes H\to H\otimes H$ is bijective, moreover, T_1^{-1} is a right H-module map and a left H-comodule map;

4) The linear map $T_2: H\otimes H\to H\otimes H$ is bijective, moreover, T_2^{-1} is a left H-module map and a right H-comodule map;

5) The element L is invertible in the convolution algebra $H * \mathrm{End}(H)$;

6) The element R is invertible in the convolution algebra $H * \mathrm{End}(H)$.

Proof 1)$\Leftrightarrow$2). It follows Refs. [1, 4] that 1) is equivalent to 2).

2)$\Leftrightarrow$3). If 2) holds, then it follows Ref. [3] that T_1 has the inverse $T_1^{-1}: A\otimes A\to A\otimes A$ defined as

$$T_1^{-1}(a\otimes b) = \sum a_{(1)} \otimes S(a_{(2)})b$$

for all $a, b\in H$.

It is not difficult to check that T_1^{-1} is a right H-module map and a left H-comodule map.

Conversely, if 2) holds, then we introduce the notation, for all $a\in H$.

$$\sum a^{(1)} \otimes a^{(2)} := T_1^{-1}(a\otimes 1)$$

Define a linear map $S: H\to H$ as

$$S(a) = (\varepsilon\otimes 1)\sum a^{(1)}\otimes a^{(2)} = \sum \varepsilon(a^{(1)})a^{(2)}$$

Since T_1^{-1} is a left H-comodule map, one has $(\rho^l_{H\otimes H}\otimes \mathrm{id})T_1^{-1} = (\mathrm{id}\otimes T_1^{-1})\rho^l_{H\otimes H}$. That implies that, for all $a\in H$,

$$\sum {a^{(1)}}_{(1)} \otimes {a^{(1)}}_{(2)} \otimes a^{(2)} = \sum a_{(1)} \otimes {a_{(2)}}^{(1)} \otimes {a_{(2)}}^{(2)}$$

Applying $(\mathrm{id}\otimes\varepsilon\otimes\mathrm{id})$ to the above equation, one obtains that

$$T_1^{-1}(a\otimes 1) = \sum a^{(1)}\otimes a^{(2)} = \sum a_{(1)}\otimes S(a_{(2)})$$

Since T_1^{-1} is the inverse of T_1 and it is a right H-module map, one can conclude that

$$a\otimes b = T_1^{-1}T_1(a\otimes b) = T_1^{-1}(\Delta(a)(1\otimes b)) = \sum a_{(1)}\otimes S(a_{(2)})a_{(3)}b$$

and

$$a\otimes b = T_1T_1^{-1}(a\otimes b) = T_1(\sum a_{(1)}\otimes S(a_{(2)})b) = \sum a_{(1)}\otimes a_{(2)}S(a_{(3)})b$$

Applying the counit to the first factor and taking $b=1$, we obtain Eq. (4).

Thus, S is the required antipode on H.

2)$\Leftrightarrow$4). Similarly, it follows Ref. [3] that T_2 has the inverse $T_2^{-1}: A\otimes A\to A\otimes A$ given as

$$T_2^{-1}(a\otimes b) = aS(b_{(1)})\otimes b_{(2)}$$

or all $a, b\in H$. Obviously, T_2^{-1} is a left H-module map and a right H-comodule map.

One introduces the notation, for all $a\in H$,

$$\sum a^{[1]}\otimes a^{[2]} := T_2^{-1}(1\otimes a)$$

Define a linear map $S': H\to H$ as

$$S'(a) = (1\otimes\varepsilon)\sum a^{[1]}\otimes a^{[2]} = \sum a^{[1]}\varepsilon(a^{[2]})$$

Following the program of arguments on S, we have S' that satisfies Eq. (4).

Furthermore, we now calculate, for all $a\in H$,

$$S'(a) = \sum S'(a_{(1)})\varepsilon(a_{(2)}) = \sum S'(a_{(1)})a_{(2)}S(a_{(3)}) = \sum \varepsilon(a_{(1)})S(a_{(2)}) = S(a)$$

Therefore, we have $S=S'$ and they are the required antipodes on H.

1)$\Leftrightarrow$5). By hypothesis $B * \mathrm{End}(B)$ is an associative algebra. The element L is invertible in this algebra if and only if there exists $L': B\to \mathrm{End}(B)$ such that

$$\sum L'(a_{(1)})L(a_{(2)}) = \varepsilon(a)\mathrm{id} = \sum L(a_{(1)})L'(a_{(2)})$$

This implies that, for all $a, b\in H$,

$$\sum L'(a_{(1)})(a_{(2)}b) = \varepsilon(a)b = \sum a_{(1)}L'(a_{(2)})(b)$$

and in this case the inverse L' is unique.

Defining $S: B\to B$ by $S(a) = L'(a)(e)$ and taking $b=1$ and comparing this equation with (4), we obtain the desired result about the existence and uniqueness of S.

Similarly for 1)$\Leftrightarrow$6).

This completes the proof.

Let G be a semigroup with unit e. Then, $(G, G) = \{(g, h) \mid g, h\in G\}$ is also a semigroup with the product:

$$(x, y)(g, h) = (xg, yh)$$

for all $x, y, g, h\in G$.

Corollary 1 Let G be a semigroup with unit e. Then, the following statements are equivalent:

1) G is a group;

2) There is a map $S: G\to G$ such that $S(g)g = e = gS(g)$ for all $g\in G$;

3) The map $T_1: (G, G)\to(G, G), (g, h)\mapsto(g, gh)$ is bijective;

4) The map $T_2: (G, G)\to(G, G), (g, h)\mapsto(gh, h)$ is bijective;

5) There is a map $Q: G\to \mathrm{End}(G)$ such that the element $L: G\to\mathrm{End}(G), L(g) = L_g$, for all $g\in G$, satisfies $Q(g)(g) = e = gQ(g)(e)$;

6) There is a map $P: G\to\mathrm{End}(G)$ such that the element $R: G\to\mathrm{End}(G)$, $R(g) = R_g$, for all $g\in G$, satisfies $P(g)(e)g = e = P(g)(g)$.

3 Hopf (co) Quasigroups

Recall from Ref. [5] that an inverse property of quasigroup (or IP loop) is defined as set G with a product, unit e and the property for each $u\in G$, there is $u^{-1}\in G$ such that

$$u^{-1}(uv) = v,\ (vu)u^{-1} = v \qquad \forall v\in G$$

A quasigroup[6] is flexible if $u(vu) = (uv)u$ for all $u, v\in G$ and alternative if also $u(uv) = (uu)v$, $u(vv) = (uv)v$ for all $u, v\in G$.

It is called Moufang if $u(v(uw)) = ((uv)u)w$ for all $u, v, w\in G$.

Recall from Ref. [7] that a Hopf quasigroup is a unital algebra H (possibly nonassociative) equipped with algebra homomorphisms $\Delta: H\to H\otimes H$, $\varepsilon: H\to k$ forming a coassociative coalgebra and a map $S: H\to H$ such that

$$\sum S(h_{(1)})(h_{(2)}g) = \sum h_{(1)}(S(h_{(2)})g) = \varepsilon(h)g \qquad (5)$$

$$\sum (gS(h_{(1)}))h_{(2)} = \sum (gh_{(1)})S(h_{(2)}) = \varepsilon(h)g \qquad (6)$$

for all $h, g\in H$. Furthermore, a Hopf quasigroup H is called flexible if

$$\sum h_{(1)}(gh_{(2)}) = \sum (h_{(1)}g)h_{(2)} \qquad \forall h, g\in H$$

and Moufang if

$$\sum h_{(1)}(g(h_{(2)}f)) = \sum ((h_{(1)}g)h_{(2)})f \qquad \forall h, g, f\in H$$

Hence, a Hopf quasigroup is a Hopf algebra iff its product is associative.

Dually, we have that a Hopf coquasigroup[8] is a unital associative algebra H equipped with counital algebra homomorphisms $\Delta: H\to H\otimes H$, $\varepsilon: H\to k$ and linear map $S: H\to H$ such that

$$\sum S(h_{(1)})h_{(2)(1)}\otimes h_{(2)(2)} = 1\otimes h = \sum h_{(1)}S(h_{(2)(1)})\otimes h_{(2)(2)} \qquad (7)$$

$$\sum h_{(1)(1)}\otimes S(h_{(1)(2)})h_{(2)} = h\otimes 1 = \sum h_{(1)(1)}\otimes h_{(1)(2)}S(h_{(2)}) \qquad (8)$$

for all $h\in H$. Furthermore, a Hopf coquasigroup H is called flexible if

$$\sum h_{(1)}h_{(2)(2)}\otimes h_{(2)(1)} = \sum h_{(1)(1)}h_{(2)}\otimes h_{(1)(2)} \qquad \forall h\in H$$

and Moufang if

$$\sum h_{(1)}h_{(2)(2)(1)}\otimes h_{(2)(1)}\otimes h_{(2)(2)(2)} = \sum h_{(1)(1)(1)}h_{(1)(2)}\otimes h_{(1)(1)(2)}\otimes h_{(2)} \qquad \forall h\in H$$

Let (A, m, μ) be a unital algebra. Assume that $T: A\otimes A\to A\otimes A$ is a map. Then, we can define the following two coproduct maps:

$$\Delta^r_T: A\to A\otimes A,\quad a\mapsto T(a\otimes 1)$$
$$\Delta^l_T: A\to A\otimes A,\quad a\mapsto T(1\otimes a)$$

Definition 1 With the above notation, we say that T is left (resp. right) compatible with Δ^r_T, if $T(a\otimes b) = \Delta^r_T(a)(1\otimes b)$ (resp. $T(a\otimes b) = (a\otimes 1)\Delta^r_T(b)$), for all $a, b\in A$.

Similarly, one says that T is left (resp. right) compatible with Δ^l_T, if $T(a\otimes b) = \Delta^l_T(a)(1\otimes b)$ (resp. $T(a\otimes b) = (a\otimes 1)\Delta^l_T(b)$), for all $a, b\in A$.

Dually, let (C, Δ, ε) be a counital coalgebra. Let $T: A\otimes A\to A\otimes A$ be a map. Then, one can define the following two product maps:

$$m^r_T: A\otimes A\to A,\quad a\otimes b\mapsto(1\otimes\varepsilon)T(a\otimes b)$$
$$m^l_T: A\otimes A\to A,\quad a\otimes b\mapsto(\varepsilon\otimes 1)T(a\otimes b)$$

Definition 2 With the above notation, we say that T is left (resp. right) compatible with m^r_T, if $T(a\otimes b) = (m^r_T\otimes 1)(1\otimes\Delta)(a\otimes b)$ (resp. $T(a\otimes b) = (1\otimes m^r_T)(\Delta\otimes 1)(a\otimes b)$), for all $a, b\in A$.

Similarly, one says that T is left (resp. right) compatible with m_T^l, if $T(a\otimes b)=(m_T^l\otimes 1)(1\otimes\Delta)(a\otimes b)$ (resp. $T(a\otimes b)=(1\otimes m_T^l)(\Delta\otimes 1)(a\otimes b)$), for all a, $b\in A$.

We now have the main result of this section as follows.

Theorem 2 Let $H:=(H,\Delta,\varepsilon,m,\mu)$ be a unital counital coassociative bialgebra. Then, the following statements are equivalent:

1) H is a Hopf quasigroup.

2) The linear map $T_1, T_2: H\otimes H\rightarrow H\otimes H$ is bijective, and T_1^{-1} is left compatible with $\Delta^r_{T_1^{-1}}$ and right compatible with $m^l_{T_1^{-1}}$. At the same time, the map $T_2: H\otimes H\rightarrow H\otimes H$ is bijective. Moreover, T_2^{-1} is right compatible with $\Delta^l_{T_2^{-1}}$ and left compatible with $m^r_{T_2^{-1}}$.

3) The elements L and R are invertible in the convolution algebra $H*\mathrm{End}(H)$.

Proof 1)$\Leftrightarrow$2). If (2) holds, similar to Theorem 1, it is easy to check whether T_1 has the inverse $T_1^{-1}: A\otimes A\rightarrow A\otimes A$ defined as $T_1^{-1}(a\otimes b)=\sum a_{(1)}\otimes S(a_{(2)})b$ for all $a, b\in H$. Then, we have

$$\Delta^r_{T_1^{-1}}: A\rightarrow A\otimes A$$
$$a\mapsto T_1^{-1}(a\otimes 1)=\sum a_{(1)}\otimes S(a_{(2)})$$
$$m^l_{T_1^{-1}}: A\otimes A\rightarrow A$$
$$a\otimes b\mapsto(\varepsilon\otimes 1)T_1^{-1}(a\otimes b)=S(a)b$$

It is not difficult to check whether T_1^{-1} is left compatible with $\Delta^r_{T_1^{-1}}$ and right compatible with $m^l_{T_1^{-1}}$.

Conversely, if (2) holds, then, we define a linear map $S: H\rightarrow H$ as

$$S(a)=(\varepsilon\otimes 1)\Delta^r_{T_1^{-1}}(a)$$

Since T_1^{-1} is a right compatible with $m^l_{T_1^{-1}}$ and it is left compatible with $\Delta^r_{T_1^{-1}}$, one has, for all $a, b\in H$,

$$\begin{aligned}T_1^{-1}(a\otimes b)&=(1\otimes m^l_{T_1^{-1}})(\Delta\otimes 1)(a\otimes b)=\\ &\sum a_{(1)}\otimes(\varepsilon\otimes 1)T_1^{-1}(a_{(2)}\otimes b)=\\ &\sum[a_{(1)}\otimes(\varepsilon\otimes 1)\Delta^r_{T_1^{-1}}(a_{(2)})](1\otimes b)=\\ &\sum a_{(1)}\otimes S(a_{(2)})b\end{aligned}$$

Since T_1^{-1} is the inverse of T_1, we can conclude that

$$\begin{aligned}\sum a_{(1)}\otimes a_{(2)}[S(a_{(3)})b]&=T_1(\sum a_{(1)}\otimes S(a_{(2)})b)=\\ &T_1T_1^{-1}(a\otimes b)=a\otimes b=T_1^{-1}T_1(a\otimes b)=\\ &T_1^{-1}(\Delta(a)(1\otimes b))=\sum a_{(1)}\otimes[S(a_{(2)})a_{(3)}]b\end{aligned}$$

Applying the counit to the first factor, we can obtain Eq. (5). We define another linear map $S': H\rightarrow H$ as

$$S'(a)=(1\otimes\varepsilon)\Delta^l_{T_2^{-1}}(a)\qquad\forall a\in H$$

Similar to discussing S, we can obtain Eq. (6). By doing some calculation, we have

$$\begin{aligned}S(a)&=\sum S(a_{(1)})\varepsilon(a_{(2)})=\sum[S(a_{(1)})a_{(2)}]S'(a_{(2)})=\\ &\sum\varepsilon(a_{(1)})S'(a_{(2)})=S'(a)\end{aligned}$$

for all $a\in H$.

Thus, H is a Hopf quasigroup.

1)$\Leftrightarrow$3). Similar to 1)$\Leftrightarrow$5) and 1)$\Leftrightarrow$6) in Theorem 1, it is not difficult to complete the proof.

This completes the proof.

Corollary 2 Let G be nonempty with a product and with unit e. Then, the following statements are equivalent:

1) G is a quaigroup;

2) There is a map $S: G\rightarrow G$ such that $S(g)(gh)=h=(hg)S(g)$ for all $g, v\in G$;

3) The map $T_1:(G,G)\rightarrow(G,G),(\mathrm{g},\mathrm{h})\mapsto(g,gh)$ is bijective;

4) The map $T_2:(G,G)\rightarrow(G,G),(g,h)\mapsto(gh,h)$ is bijective;

5) There is a map $Q: G\rightarrow\mathrm{End}(G)$ such that the element $L: G\rightarrow\mathrm{End}(G), L(g)=L_g$, for all $g, h\in G$, satisfies $Q(g)(g)h=h=hgQ(g)(e)$;

6) There is a map $P: G\rightarrow\mathrm{End}(G)$ such that the element $R: G\rightarrow\mathrm{End}(G), R(g)=R_g$, for all $g\in G$, satisfies $P(g)(e)(g)h=h=P(g)(gh)$.

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Galois 线性映射及其构造

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摘要: 一个代数构成 Hopf 代数或 Hopf(余)拟群的条件可由 Galois 线性映射的性质来确定. 对于一个双代数 H, 如果其作为代数是结合有单位的, 且作为余代数是余结合有余单位的, 则可以定义 Galois 线性映射 T_1 和 T_2. 对于一个结合余结合的双代数 H(有单位和余单位), 则 H 为一个 Hopf 代数当且仅当 Galois 线性映射 T_1 是双射, 且进一步地, T_1^{-1} 是右 H-模和右 H-余模映射. 另一方面, 对于一个有单位的代数 A(不一定是结合的), A 作为余代数是余结合有余单位的, 如果 A 的余乘法和余单位均为代数同态, 则 A 为一个 Hopf 拟群当且仅当 Galois 线性映射 T_1 是双射且 T_1^{-1} 与右余积映射 $\Delta^r_{T_1^{-1}}$ 左相容, 同时与左积映射 $m^l_{T_1^{-1}}$ 右相容(相似的性质也适用于 Galois 线性映射 T_2). 作为推论, 拟群的情形也得到了讨论.

关键词: Galois 线性映射; 对极; Hopf 代数; Hopf (余)拟群
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