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An explicit representation and computation for the outer inverse

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Abstract: First, an explicit representation $A_{T,S}^{(2)}=(GA+E)^{-1}G$ of the outer invers $A_{T,S}^{(2)}$ for a matrix $A\in\mathbf{C}^{m\times n}$ with the prescribed range T and null space S is derived, which is simpler than $A_{T,S}^{(2)}=(GA+E)^{-1}G-V(UV)^{-2}UG$ proposed by Ji in 2005. Next, a new algorithm for computing the outer inverse $A_{T,S}^{(2)}$ based on the improved representation $A_{T,S}^{(2)}=(GA+E)^{-1}G$ through elementary operations on an appropriate partitioned matrix $\begin{bmatrix} GA & I_n \\ I_n & 0\end{bmatrix}$ is proposed and investigated. Then, the computational complexity of the introduced algorithm is also analyzed in detail. Finally, two numerical examples are shown to illustrate that this method is correct.

Key words: outer inverse; explicit representation; elementary operation; computational complexity

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Throughout this paper, the standard notations in Ref. [1] are used. The symbol $\mathbf{C}_r^{m\times n}$ denotes the set of all $m\times n$ complex matrices with rank r, and $\mathbf{C}^n$ represents the n-dimensional complex space. I_n represents an identity matrix of order n. For $A\in\mathbf{C}^{m\times n}$, symbols $R(A)$, $N(A)$, A^*, A^{-1} and $r(A)$ denote its range, null space, the conjugate transpose, inverse and rank, respectively. $R(A)^{\perp}$ and $N(A)^{\perp}$ are orthogonal complement spaces of $R(A)$ and $N(A)$, respectively. The index of $A\in\mathbf{C}^{n\times n}$, denoted as $\operatorname{ind}(A)$, is the smallest nonnegative integer k such that $r(A^k)=r(A^{k+1})$.

The {2} -inverse $A_{T,S}^{(2)}$ of a matrix $A\in\mathbf{C}^{n\times n}$ with the prescribed range T and null space S is defined as follows:

Definition 1[1] If $A\in\mathbf{C}_r^{m\times n}$, T is a subspace of $\mathbf{C}^n$ of dimension $s\leqslant r$, and S is a subspace of $\mathbf{C}^m$ of dimension $m-s$, and then A has a {2}-inverse X such that $R(X)=T$ and $N(X)=S$, if and only if $AT\oplus S=\mathbf{C}^m$.

In such a case, X is unique and denoted as $A_{T,S}^{(2)}$. It is well known that the {2} -inverse $A_{T,S}^{(2)}$ proposes a unified representation for six kinds of generalized inverses, such as the Moore-Penrose inverse $A^{\dagger}$, the weighted Moore-Penrose inverse $A_{MN}^{\dagger}$, the group inverse A_g, the Drazin inverse A_d, the Bott-Duffin inverse $A_{(L)}^{(-1)}$, and the generalized Bott-Duffin inverse $A_{(S)}^{(\dagger)}$. In addition, suppose that the matrix $G\in\mathbf{C}^{n\times m}$ satisfies $R(G)=T$ and $N(G)=S$, then the unified treatment is as follows:

$$A_{T,S}^{(2)}=\begin{cases} A^{\dagger} & \text{if } G=A^* \\ A_{MN}^{\dagger} & \text{if } G=A^{\#}=NA^{-1}M \\ A_g & \text{if } m=n, G=A \text{ with } \operatorname{ind}(A)=1 \\ A_d & \text{if } m=n, G=A^k \text{ with } \operatorname{ind}(A)=k \\ A_{(L)}^{(-1)} & \text{if } R(G)=L \text{ and } N(G)=L^{\perp} \\ A_{(S)}^{(\dagger)} & \text{if } R(G)=S \text{ and } N(G)=S^{\perp} \end{cases}$$

The {2} -inverse plays an important role in a stable approximation of ill-posed problems and in linear and nonlinear problems involving a rank-deficient generalized inverse[2-3]. In particular, {2} -inverse can be used in the iterative methods for solving nonlinear equations[1,4] and in statistics[5-6].

In the past thirty years, numerous experts and scholars investigated the subject of computation and representation for $A_{T,S}^{(2)}$. Some results of the minor of generalized inverse $A_{T,S}^{(2)}$ can be viewed in Refs. [7 − 10]. The use of the iterative method or approximation to compute $A_{T,S}^{(2)}$ can be seen in Refs. [11 − 15]. Some other representations and computations can be found in Refs. [16 − 18]. Recently, some scholars[19-21] used Gauss-Jordan elimination methods to compute $A_{T,S}^{(2)}$. Moreover, the computational complexity of these Gauss-Jordan elimination methods are also analyzed in detail.

In 1998, Wei[16] provided an expression of the generalized inverse $A_{T,S}^{(2)}$ by using group inverse, which employs a new way to study $A_{T,S}^{(2)}$.

Lemma 1[16] Let A, T, S be the same as those in Definition 1. Suppose that $G\in\mathbf{C}^{n\times m}$ such that $R(G)=T$ and $N(G)=S$. If A has a {2} -inverse $A_{T,S}^{(2)}$, then

$$\operatorname{ind}(AG)=\operatorname{ind}(GA)=1 \tag{1}$$

Furthermore, we have

$$A_{T,S}^{(2)}=G(AG)_g=(GA)_gG \tag{2}$$

Soon after, Ji[17] enhanced the results of Wei[16] and established another explicit representation of $A_{T,S}^{(2)}$, which is shown as follows.

Lemma 2[17] Let A, T, S and G be the same as those

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in Lemma 1. Let $\boldsymbol{V}$ and $\boldsymbol{U}^*$ be matrices whose columns form the bases of $N(\boldsymbol{GA})$ and $N((\boldsymbol{GA})^*)$, respectively. Define $\boldsymbol{E}=\boldsymbol{VU}$. Then, $\boldsymbol{E}$ is nonsingular, satisfying

$$R(\boldsymbol{E})=R(\boldsymbol{V})=R(\boldsymbol{GA}),\ N(\boldsymbol{E})=N(\boldsymbol{U})=N(\boldsymbol{GA}) \quad (3)$$

Moreover, $\boldsymbol{GA}+\boldsymbol{E}$ is nonsingular and

$$\boldsymbol{A}^{(2)}_{T,S}=(\boldsymbol{GA}+\boldsymbol{E})^{-1}\boldsymbol{G}-\boldsymbol{V}(\boldsymbol{UV})^{-2}\boldsymbol{UG} \quad (4)$$

In the following lemma, we will prove $\boldsymbol{V}(\boldsymbol{UV})^{-2}\boldsymbol{UG}=\boldsymbol{0}$.

Lemma 3 Let $\boldsymbol{A}$, T, S and $\boldsymbol{G}$ be the same as those in Lemma 1. Let $\boldsymbol{V}$ and $\boldsymbol{U}^*$ be the matrices whose columns form the bases of $N(\boldsymbol{GA})$ and $N((\boldsymbol{GA})^*)$, respectively. Then,

$$\boldsymbol{V}(\boldsymbol{UV})^{-2}\boldsymbol{UG}=\boldsymbol{0} \quad (5)$$

Proof Since the columns of matrix $\boldsymbol{U}^*$ is the basis of $N((\boldsymbol{GA})^*)$, we have $(\boldsymbol{GA})^*\boldsymbol{U}^*=\boldsymbol{0}$, which is also equivalent to $\boldsymbol{UGA}=\boldsymbol{0}$. This implies $R(\boldsymbol{GA})\subset N(\boldsymbol{U})$. From Lemma 1, we know that $r(\boldsymbol{G})=r(\boldsymbol{GA})=s$, which means that $R(\boldsymbol{GA})=R(\boldsymbol{G})$. Then, we have $\boldsymbol{UG}=\boldsymbol{0}$ and Eq. (5) is followed.

In this paper, we first develop the result of Ji[17] and obtain a more brief explicit representation of $\boldsymbol{A}^{(2)}_{T,S}$. Based on the developing explicit representation, a new algorithm for computing the outer inverse $\boldsymbol{A}^{(2)}_{T,S}$ through elementary operations on an appropriate partitioned matrix is proposed and investigated. The computational complexity of the new algorithm is also analyzed in detail.

1 Main Results

In the following theorem, we establish a new explicit expression for $\boldsymbol{A}^{(2)}_{T,S}$, which is simpler than that in Ref. [17].

Theorem 1 Let $\boldsymbol{A}$, T, S and $\boldsymbol{G}$ be the same as those in Lemma 1. Let $\boldsymbol{V}$ and $\boldsymbol{U}^*$ be matrices whose columns form the bases of $N(\boldsymbol{GA})$ and $N((\boldsymbol{GA})^*)$, respectively. Define $\boldsymbol{E}=\boldsymbol{VU}$. Then,

$$\boldsymbol{A}^{(2)}_{T,S}=(\boldsymbol{GA}+\boldsymbol{E})^{-1}\boldsymbol{G} \quad (6)$$

Proof Since $\boldsymbol{V}(\boldsymbol{UV})^{-2}\boldsymbol{UG}=\boldsymbol{0}$, we obtain Eq. (6) immediately from Eq. (4).

Following the same line of Theorem 1, another explicit representation of {2} -inverse $\boldsymbol{A}^{(2)}_{T,S}$ is proposed.

Theorem 2 Let $\boldsymbol{A}$, T, S and $\boldsymbol{G}$ be the same as those in Lemma 1. Let $\boldsymbol{P}$ and $\boldsymbol{Q}^*$ be matrices whose columns form the bases of $N(\boldsymbol{AG})$ and $N((\boldsymbol{AG})^*)$, respectively. Define $\boldsymbol{F}=\boldsymbol{PQ}$. Then,

$$\boldsymbol{A}^{(2)}_{T,S}=\boldsymbol{G}(\boldsymbol{AG}+\boldsymbol{F})^{-1} \quad (7)$$

Based on the two explicit representations (6) and (7), a method to calculate $\boldsymbol{A}^{(2)}_{T,S}$ through elementary operations on an appropriate partitioned matrix is derived and investigated. We first give the case of $m\geqslant n$.

Theorem 3 Let $\boldsymbol{A}$, T, S and $\boldsymbol{G}$ be the same as those in Lemma 1. Then there are two nonsingular elementary matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ of order n, respectively, such that

$$\boldsymbol{U}[\boldsymbol{GA}\quad \boldsymbol{I}_n]=\begin{bmatrix}\boldsymbol{B} & \boldsymbol{U}_1\\ \boldsymbol{0} & \boldsymbol{U}_2\end{bmatrix} \quad (8)$$

$$\begin{bmatrix}\boldsymbol{B}\\ \boldsymbol{0}\\ \boldsymbol{I}_n\end{bmatrix}\boldsymbol{V}=\begin{bmatrix}\boldsymbol{I}_s & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{V}_1 & \boldsymbol{V}_2\end{bmatrix} \quad (9)$$

where matrix $\boldsymbol{B}\in\mathbf{C}^{s\times n}$ and s columns of $\boldsymbol{B}$ are the same as those of $\boldsymbol{I}_s$; furthermore,

1) $R(\boldsymbol{U}_2^*)=N((\boldsymbol{GA})^*)$ and $R(\boldsymbol{V}_2)=N(\boldsymbol{GA})$;

2) $\boldsymbol{A}^{(2)}_{T,S}=(\boldsymbol{GA}+\boldsymbol{V}_2\boldsymbol{U}_2)^{-1}\boldsymbol{G}$.

Proof According to Lemma 1, we have $r(\boldsymbol{GA})=r(\boldsymbol{G})=s$, then there are two elementary matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ satisfying (8) and (9). This means

$$\boldsymbol{UGAV}=\begin{bmatrix}\boldsymbol{U}_1\\ \boldsymbol{U}_2\end{bmatrix}\boldsymbol{GA}[\boldsymbol{V}_1\quad \boldsymbol{V}_2]=$$

$$\begin{bmatrix}\boldsymbol{U}_1\boldsymbol{GAV}_1 & \boldsymbol{U}_1\boldsymbol{GAV}_2\\ \boldsymbol{U}_2\boldsymbol{GAV}_1 & \boldsymbol{U}_2\boldsymbol{GAV}_2\end{bmatrix}=\begin{bmatrix}\boldsymbol{I}_s & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix} \quad (10)$$

Comparing both sides of (10), we drive

$$\left.\begin{array}{l}\boldsymbol{U}_1\boldsymbol{GAV}_1=\boldsymbol{I}_s,\ \boldsymbol{U}_1\boldsymbol{GAV}_2=\boldsymbol{0}\\ \boldsymbol{U}_2\boldsymbol{GAV}_1=\boldsymbol{0},\ \boldsymbol{U}_2\boldsymbol{GAV}_2=\boldsymbol{0}\end{array}\right\} \quad (11)$$

We notice that matrices $\boldsymbol{U}_2$ and $\boldsymbol{V}_2$ are row full rank and column full rank matrices, respectively, and then, the above four equalities imply that

$$\boldsymbol{GAV}_2=\boldsymbol{0},\ \boldsymbol{U}_2\boldsymbol{GA}=\boldsymbol{0} \quad (12)$$

This implies that

$$R(\boldsymbol{U}_2^*)\subset N((\boldsymbol{GA})^*),\ R(\boldsymbol{V}_2)\subset N(\boldsymbol{GA}) \quad (13)$$

According to the fact that $r(\boldsymbol{U})=r(\boldsymbol{V}_2)=n-s=\dim N((\boldsymbol{GA})^*)=\dim N(\boldsymbol{GA})$, we obtain

$$R(\boldsymbol{U}_2^*)=N((\boldsymbol{GA})^*),\ R(\boldsymbol{V}_2)=N(\boldsymbol{GA})$$

From Theorem 1, we derive a representation $\boldsymbol{A}^{(2)}_{T,S}=(\boldsymbol{GA}+\boldsymbol{V}_2\boldsymbol{U}_2)^{-1}\boldsymbol{G}$.

Theorem 4 Let $\boldsymbol{A}$, T, S and $\boldsymbol{G}$ be the same as those in Lemma 1. Then, there exist two nonsingular elementary matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ of order m, respectively, such that

$$\boldsymbol{P}[\boldsymbol{AG}\quad \boldsymbol{I}_m]=\begin{bmatrix}\boldsymbol{C} & \boldsymbol{P}_1\\ \boldsymbol{0} & \boldsymbol{P}_2\end{bmatrix} \quad (14)$$

$$\begin{bmatrix}\boldsymbol{C}\\ \boldsymbol{0}\\ \boldsymbol{I}_m\end{bmatrix}\boldsymbol{V}=\begin{bmatrix}\boldsymbol{I}_s & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{Q}_1 & \boldsymbol{Q}_2\end{bmatrix} \quad (15)$$

where matrix $\boldsymbol{C}\in\mathbf{C}^{s\times m}$ and s columns of $\boldsymbol{C}$ are those of $\boldsymbol{I}_s$; furthermore,

1) $R(\boldsymbol{P}_2^*)=N((\boldsymbol{AG})^*)$ and $R(\boldsymbol{Q}_2)=N(\boldsymbol{AG})$;

2) $\boldsymbol{A}^{(2)}_{T,S}=\boldsymbol{G}(\boldsymbol{AG}+\boldsymbol{Q}_2\boldsymbol{P}_2)^{-1}$.

The proof of Theorem 4 is similar to that of Theorem 3.

2 An Algorithm to $A_{T,S}^{(2)}$ Based on Gaussian Elimination and the Computational Complexity

Let $A \in \mathbf{C}_r^{m\times n}$ with $m \geqslant n$, and Theorem 3 is summarized in the following algorithm.

Algorithm 1

Input matrices $A \in \mathbf{C}_r^{m\times n}$ and $G \in \mathbf{C}_s^{n\times m}$ with $s \leqslant r$ and calculate GA.

Execute elementary row operations on the first n rows and the first n columns of a partitioned matrix $\begin{bmatrix} GA & I_n \\ I_n & 0 \end{bmatrix}$, which is changed into $\begin{bmatrix} \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ [V_1 \quad V_2] & 0 \end{bmatrix}$.

Perform elementary row operations on matrix $[GA + V_2U_2 \quad G]$ until $[I_n \quad A_{T,S}^{(2)}]$ is reached.

In the following theorem, the computational complexity of Algorithm 1 is analyzed, which only focuses on multiplications and divisions.

Theorem 5 The computational complexity of Algorithm 1 to compute $A_{T,S}^{(2)}$ is

$$T(m, n) = n^2\left(2m + \frac{3}{2}n - \frac{1}{2}\right) \tag{16}$$

Proof It requires mn^2 multiplications to form GA. s pivoting steps are needed to transform $[GA \quad I_n]$ into $\begin{bmatrix} B & U_1 \\ 0 & U_2 \end{bmatrix}$ following $r(GA) = s$, where matrix $B \in \mathbf{C}^{s\times n}$ and s columns of B are the same as those of I_s. The first pivoting step involves $n+1$ non-zero columns in $[GA \quad I_n]$. Thus, n divisions and $n(n-1)$ multiplications with a total of n^2 multiplications and divisions are required. For the second pivoting step, the first part of the pair reduces one column to deal with, but the second part increases one column, resulting in $n+1$ columns involved. This pivoting step also requires n^2 operations. Continuing this way, it requires sn^2 multiplications and divisions to reach the matrix $\begin{bmatrix} B & U_1 \\ 0 & U_2 \end{bmatrix}$.

According to the fact that s columns of B are the same as those of I_s, we can directly read V_1 and V_2. In the third step, $n^2(n-s)$ multiplications are required to compute V_2U_2.

In the fourth step, n pivoting steps are needed to transform $[GA + V_2U_2 \quad G]$ into $[I_n \quad A_{T,S}^{(2)}]$. The first pivoting step involves $n+m$ non-zero columns in $[GA + V_2U_2 \quad G]$. Thus, $n+m-1$ divisions and $(n+m-1)(n-1)$ multiplications are required with a total of $n(n+m-1)$ multiplications and divisions. For the second pivoting step, There is one less non-zero column in the first part of the pair. There are $n+m-1$ non-zero columns to deal with. These pivoting steps require $n(n+m-2)$ operations. Following the same idea, the i-th $(1 \leqslant i \leqslant n)$ pivoting step needs $n(n+m-i)$ operations. So, it requires

$$n(n+m-1) + n(n+m-2) + \ldots + n(n+m-n) = n^2\left(m + \frac{1}{2}n - \frac{1}{2}\right)$$

Therefore, the total number of operations for computing $A_{T,S}^{(2)}$ is

$$T(m, n) = n^2m + n^2s + n^2(n-s) + n^2\left(m + \frac{1}{2}n - \frac{1}{2}\right) = n^2\left(2m + \frac{3}{2}n - \frac{1}{2}\right)$$

If $m \leqslant n$, the similar algorithm and computation complexities are given as follows.

Algorithm 2

Input: matrix $A \in \mathbf{C}_r^{m\times n}$ and $G \in \mathbf{C}_s^{n\times m}$ with $s \leqslant r$ and calculate AG.

Execute elementary row operations on the first m rows and the first m columns of a partitioned matrix $\begin{bmatrix} AG & I_m \\ I_m & 0 \end{bmatrix}$, which is changed into $\begin{bmatrix} \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \\ [Q_1 \quad Q_2] & 0 \end{bmatrix}$.

Perform elementary row operations on the matrix $\begin{bmatrix} AG + Q_2P_2 \\ G \end{bmatrix}$ until $\begin{bmatrix} I_m \\ A_{T,S}^{(2)} \end{bmatrix}$ is reached.

Theorem 6 The computational complexity of Algorithm 2 to compute $A_{T,S}^{(2)}$ is

$$T(m, n) = m^2\left(2n + \frac{3}{2}m - \frac{1}{2}\right) \tag{17}$$

3 Numerical Examples

The Gauss-Jordan elimination is a popular method for computing the inverse of a nonsingular matrix of low order by hand. In this section, Algorithm 1 and Algorithm 2 can be used to compute some famous generalized inverses by hand if the size of matrix A is small by choosing the appropriate parameter matrix G.

Example 1[18] Use Algorithm 1 to compute the Drazin inverse A_d of matrix A, where

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

Solution Simple computation leads to

$$A^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This implies $\operatorname{ind}(A) = 2$. Take $G = A^2$ and perform elementary row and column operations on $\begin{bmatrix} AG & I_3 \\ I_3 & 0 \end{bmatrix} = \begin{bmatrix} A^3 & I_3 \\ I_3 & 0 \end{bmatrix}$.

$$\begin{bmatrix} AG & I_3 \\ I_3 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1/8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Thus, we have $s=1$ and

$$V_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From U_2 and V_2, in view of Algorithm 1, we have

$$AG + V_2U_2 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, perform elementary row operations on $[AG + V_2U_2 \quad G] \to [I_3 \quad A_d]$

$$\begin{bmatrix} 8 & 0 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, we obtain

$$A_d = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 2[20] Take $A = \begin{bmatrix} -1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & -3 & 1 & 2 \end{bmatrix}$ and $G = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 4 & -2 \\ -5 & -4 & 1 \\ 0 & 7 & -3 \end{bmatrix}$. It is easy to show that $AR(G) \oplus N(G) = R^3$, then we have $A^{(2)}_{R(G),N(G)}$. Algorithm 2 is used to compute $A^{(2)}_{R(G),N(G)}$ through elementary operations.

Solution By computing, we have

$$AG = \begin{bmatrix} -1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ -2 & 4 & -2 \\ -5 & -4 & 1 \\ 0 & 7 & -3 \end{bmatrix} = \begin{bmatrix} -12 & 3 & -3 \\ -2 & 4 & -2 \\ -2 & -3 & 1 \end{bmatrix}$$

Executing elementary row operations on the first 3 rows and column operations on the first 3 columns of the following partitioned matrix:

$$\begin{bmatrix} AG & I_3 \\ I_3 & 0 \end{bmatrix} = \begin{bmatrix} -12 & 3 & -3 & 1 & 0 & 0 \\ -2 & 4 & -2 & 0 & 1 & 0 \\ -2 & -3 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & -3/14 & -2/7 \\ 0 & 1 & 0 & 0 & -1/7 & 1/7 \\ 0 & 0 & 0 & 1 & -3 & -3 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 \end{bmatrix}$$

This implies

$$P_2 = [1 \quad -3 \quad -3], \quad Q_2 = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}$$

By direct calculation, we can obtain

$$AG + Q_2P_2 = \begin{bmatrix} -13 & 6 & 0 \\ 1 & -5 & -11 \\ 5 & -24 & -20 \end{bmatrix}$$

Then, we perform the elementary column operation transform and change $\begin{bmatrix} AG + Q_2P_2 \\ G \end{bmatrix}$ into $\begin{bmatrix} I_3 \\ A^{(2)}_{R(G),N(G)} \end{bmatrix}$.

$$\begin{bmatrix} -13 & 6 & 0 \\ 1 & -5 & -11 \\ 5 & -24 & -20 \\ 3 & 1 & 0 \\ -2 & 4 & -2 \\ -5 & -4 & 1 \\ 0 & 7 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -17/62 & 10/31 & -11/62 \\ 3/31 & 22/31 & -9/31 \\ 1/2 & -1 & 1/2 \\ -4/31 & 43/31 & -19/31 \end{bmatrix}$$

This yields

$$A^{(2)}_{R(G),N(G)} = \begin{bmatrix} -17/62 & 10/31 & -11/62 \\ 3/31 & 22/31 & -9/31 \\ 1/2 & -1 & 1/2 \\ -4/31 & 43/31 & -19/31 \end{bmatrix} = \frac{1}{62}\begin{bmatrix} -17 & 20 & -11 \\ 6 & 44 & -18 \\ 31 & -62 & 31 \\ -8 & 86 & -38 \end{bmatrix}$$

4 Conclusion

In this paper, we establish two new explicit representations for $A^{(2)}_{T,S}$ and give two algorithms to compute $A^{(2)}_{T,S}$ based on the two expressions through elementary operation on the appropriate partitioned matrices. The computation complexities of the two algorithms are also analyzed.

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外逆的一个显示表示及其计算

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摘要：首先给出了矩阵 $A \in \mathbf{C}^{m\times n}$ 具有指定值域 T 和零空间 S 的外逆 $A_{T,S}^{(2)}$ 的一个显示表示 $A_{T,S}^{(2)} = (GA+E)^{-1}G$，该显示表示式比 Ji 在 2005 提出的表达式 $A_{T,S}^{(2)} = (GA+E)^{-1}G - V(UV)^{-2}UG$ 要简单. 然后，基于该改进的显示表示 $A_{T,S}^{(2)} = (GA+E)^{-1}G$，通过对一个适当的分块矩阵 $\begin{bmatrix} GA & I_n \\ I_n & 0 \end{bmatrix}$ 使用初等变换的方法得出外逆 $A_{T,S}^{(2)}$ 的一个新的算法，并且详细分析了该算法的计算量. 最后，用一个数值例子检验了所提算法的正确性.

关键词：外逆；显示表示；初等变换；计算量

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