

# Diagonal crossed product of multiplier Hopf algebras

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**Abstract:** Let  $A$  and  $B$  be two regular multiplier Hopf algebras. First, the notion of diagonal crossed product  $B\#A$  of multiplier Hopf algebras is constructed using the bimodule algebra, which is a generalization of the diagonal crossed product in the sense of Hopf algebras. The result that the product in  $B\#A$  is non-degenerate is given. Next, the definition of the comultiplication  $\Delta_{\#}$  on  $B\#A$  is introduced, which is composed of the multiplier  $\Delta_B(b)$  on  $B\otimes B$  and the multiplier  $\Delta_A(a)$  on  $A\otimes A$ , and the element  $\Delta_{\#}(b\otimes a)$  is a two-side multiplier of  $B\#A\otimes B\#A$ , for any  $b\in B$  and  $a\in A$ . Then, a sufficient condition for  $B\#A$  to be a regular multiplier Hopf algebra is described. In particular, Delvaux's main theorem in the case of smash products is generalized. Finally, these integrals on a diagonal crossed product of multiplier Hopf algebras are considered.

**Key words:** multiplier Hopf algebra; bimodule algebra; diagonal crossed product

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As is known, multiplier Hopf algebras are a generalization of Hopf algebras<sup>[1-2]</sup>. Differently from Hopf algebra, the underlying algebra is no longer assumed to be a unit, but the product of algebra is non-degenerate. Some studies of multiplier Hopf algebras and their applications can be found in Refs. [3–6]. In Ref. [7], the author considered the module algebra as a regular multiplier Hopf algebra, and the theory of smash products was generalized to multiplier Hopf algebras from Hopf algebras. The definition and properties of diagonal crossed products were introduced in Ref. [8]. In this article, we will consider the bimodule algebra as a regular multiplier Hopf algebra and generalize the theory of diagonal crossed products to the multiplier Hopf algebra case.

## 1 Preliminaries

Throughout this article, let  $k$  be a fixed field of characteristic 0 (i. e., all algebraic systems are over  $k$ ). In or-

der to facilitate our computations, we always omit the summation symbol  $\Sigma$ .

In the following, we recall some definitions.

For an associative algebra  $A$ ,  $A$  has a non-degenerate product with or without identity. We denote its multiplier algebra by  $M(A)$ , and  $M(A)$  always contains a unit 1. In fact,  $M(A)$  can be characterized as the largest algebra with a unit, in which  $A$  is regarded as an essential two-side ideal. Clearly,  $A = M(A)$  if and only if  $A$  has a unit (see the appendix in Ref. [2] for details), which is similar to  $M(A\otimes A)$ .

A comultiplication on  $A$  is a homomorphism  $\Delta_A: A \rightarrow M(A\otimes A)$  such that  $\Delta_A(a)(1\otimes b)$  and  $(a\otimes 1)\Delta_A(b)$  belong to  $A\otimes A$  for all  $a, b \in A$ , and  $\Delta_A$  is coassociative in the sense that

$$(a\otimes 1\otimes 1)(\Delta_A\otimes i)(\Delta_A(b)(1\otimes c)) = (i\otimes \Delta_A)((a\otimes 1)\Delta_A(b))(1\otimes 1\otimes c) \quad (1)$$

for all  $a, b, c \in A$ , where  $i$  is the identity map.

A pair  $(A, \Delta_A)$ , in which  $A$  is an algebra with a non-degenerate product and  $\Delta_A$  is a comultiplication on  $A$ , is called a multiplier Hopf algebra if there are two linear bijections  $T_1^A$  and  $T_2^A: A\otimes A \rightarrow A\otimes A$ , which are defined as

$$T_1^A(a\otimes b) = \Delta_A(a)(1\otimes b), \quad T_2^A(a\otimes b) = (a\otimes 1)\Delta_A(b)$$

We say that  $(A, \Delta_A)$  is regular if  $\sigma\Delta_A$  is again a comultiplication on  $A$  such that a pair  $(A, \sigma\Delta_A)$  is also a multiplier Hopf algebra, where  $\sigma$  is the flip.

**Remark 1** 1) The use of the Sweedler notation for regular multiplier Hopf algebra is discussed in Ref. [1]. Take  $a, b \in A$  as an example and consider  $\Delta_A(a)(1\otimes b)$ . If we choose  $e$  such that  $eb = b$ , where  $e$  is called the local unit (see Ref. [1], Proposition 2.2), then,  $\Delta_A(a)(1\otimes b) = (\Delta_A(a)(1\otimes e))(1\otimes b) = a_1\otimes a_2b$ .

2) By the definition of two linear maps  $T_1^A$  and  $T_2^A$ , Eq. (1) can be replaced by  $(T_2^A\otimes i)(i\otimes T_1^A) = (i\otimes T_1^A)(T_2^A\otimes i)$ .

**Definition 1**<sup>[9]</sup> Let  $A$  be a regular multiplier Hopf algebra. Then, algebra  $B$  is called an  $A$ -bimodule algebra if

1)  $B$  is a unital left  $A$ -module and a unital right  $A$ -module such that  $(a\cdot b)\cdot a' = a\cdot (b\cdot a')$ ;

2)  $a\cdot bb' = (a_1\cdot b)(a_2\cdot b')$ ;

3)  $bb'\cdot a = (b\cdot a_1)(b'\cdot a_2)$  for all  $b, b' \in B$  and  $a, a' \in A$ .

## 2 Main Results

In this section, we will give the construction of the diagonal crossed product in the sense of multiplier Hopf al-

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gebras. More information can be found in Refs. [3–4].

**Definition 2** Let  $A$  be a regular multiplier Hopf algebra and  $B$  an  $A$ -bimodule algebra. Then, the diagonal crossed product  $B\#A$  built on  $B\otimes A$  with multiplication is given as

$$(b\otimes a)(b'\otimes a') = b(a_1 \cdot b' \cdot S^{-1}(a_3)) \otimes a_2 a' \quad (2)$$

for all  $b, b' \in B$  and  $a, a' \in A$ . Note that on the right side, each decomposition is well-covered.

We will further investigate the algebra  $B\#A$ .

**Lemma 1** Let  $A$  be a cocommutative multiplier Hopf algebra and  $B$  is a regular multiplier Hopf algebra such that  $B$  is an  $A$ -bimodule algebra. Then, the product in  $B\#A$  is non-degenerate.

**Proof** Suppose that  $b_i \# a_i \in B\#A$  and that  $(b_i \# a_i)(b\#a) = 0$  for all  $b \in B$  and  $a \in A$ . Then, according to the definition of the product in  $B\#A$  and the non-degeneracy of the product in  $A$ , we have

$$b_i(a_{i1} \cdot b \cdot S^{-1}(a_{i3})) \# a_{i2} = 0$$

Applying  $\Delta$  and  $S$ , multiplying with  $a$  from the right and replacing  $b$  by  $a''b$ , we can obtain

$$b_i(a_{i1} a'' \cdot b \cdot S^{-1}(a_{i4})) \otimes S(a_{i2}) a \otimes a_{i3} = 0$$

for all  $b \in B$  and  $a, a', a'' \in A$ . Replacing  $a''$  by  $S(a_{i2})a$ , we have

$$b_i(a_{i1} S(a_{i2}) a \cdot b \cdot S^{-1}(a_{i4})) \otimes a_{i3} = 0$$

That is

$$b_i(a \cdot b \cdot S^{-1}(a_{i2})) \otimes a_{i1} = 0$$

As that  $A$  is cocommutative, we can obtain

$$b_i(a \cdot b \cdot S^{-1}(a_{i1})) \otimes a_{i2} = 0$$

Applying  $\Delta$  again, multiplying with  $a'$  from the left and replacing  $b$  by  $ba''$ , we obtain

$$b_i(a \cdot b \cdot a'' S^{-1}(a_{i1})) \otimes a' a_{i2} \otimes a_{i3} = 0$$

Replacing  $a''$  by  $a' a_{i2}$ , we can obtain

$$b_i(a \cdot b \cdot a' a_{i2} S^{-1}(a_{i1})) \otimes a_{i3} = 0$$

Hence,  $b_i(a \cdot b \cdot a') \otimes a_i = 0$ . Note that  $B$  has a non-degenerate product and it is a unital left  $A$ -module and a unital right  $A$ -module, and we obtain  $b_i \otimes a_i = 0$ .

On the other hand, suppose that  $(b\#a)(b_i \# a_i) = 0$  for all  $b \in B$  and  $a \in A$ . Similar to the proof of Lemma 5.6 in Ref. [1], we can obtain  $b_i \otimes a_i = 0$ .

Moreover, we will construct a comultiplication for the diagonal crossed product such that it admits a structure of multiplier Hopf algebra. First, we give two maps as follows.

Let  $A$  and  $B$  be regular multiplier Hopf algebras such that  $B$  is an  $A$ -bimodule algebra. The multiplication of diagonal crossed product  $B\#A$  is defined by two twist maps  $R: A\otimes B \rightarrow B\otimes A$  via  $a\otimes b \mapsto a_1 \cdot b\otimes a_2$  and  $R': B\otimes A \rightarrow B\otimes A$  via  $b\otimes a \mapsto b \cdot S^{-1}(a_2) \otimes a_1$ . It can easily be

checked that  $R$  and  $R'$  are two bijections.  $R^{-1}: B\otimes A \rightarrow A\otimes B$  via  $b\otimes a \mapsto a_2 \otimes S^{-1}(a_1) \cdot b$  and  $R'^{-1}: B\otimes A \rightarrow B\otimes A$  via  $b\otimes a \mapsto b \cdot a_2 \otimes a_1$ .

We note that the comultiplication  $\Delta_A$  in a multiplier Hopf algebra  $(A, \Delta_A)$  is determined by two linear bijections  $T_1^A$  and  $T_2^A: A\otimes A \rightarrow A\otimes A$ . Therefore, in order to define the comultiplication of  $B\#A$ , we first need to introduce the corresponding maps  $T_1^\#$  and  $T_2^\#: B\#A\otimes B\#A \rightarrow B\#A\otimes B\#A$ . For all  $b, b' \in B$  and  $a, a' \in A$ , we define

$$\begin{aligned} T_1^\#(b\#a\otimes b'\#a') &= \\ (T_1^B)_{13} \circ R'_{34} \circ R_{34} \circ (T_1^A)_{23} \circ (R^{-1})_{34} \circ (R'^{-1})(b\#a\otimes b'\#a') &= \\ b_1 \# a_1 \otimes b_2 \# (a_2 \cdot b' \cdot S^{-1}(a_4)) \# a_3 a' & \end{aligned}$$

where  $(T_1^B)_{13}$  denotes the operator  $T_1^B$  on the first and the third components, similar to other operators. Observe that  $a_2$  and  $a_4$  are covered by  $b'$ ,  $a_3$  is covered by  $a'$ , and  $b_2$  is covered by  $(a_2 \cdot b' \cdot S^{-1}(a_4))$ .

$$\begin{aligned} T_2^\#(b\#a\otimes b'\#a') &= \\ (T_2^A)_{24} \circ R'_{12} \circ R_{12} \circ (T_2^B)_{23} \circ (R^{-1})_{12} \circ (R'^{-1})(b\#a\otimes b'\#a') &= \\ a_2 \cdot ((S^{-1}(a_1) \cdot b \cdot a_5) b'_1) \cdot S^{-1}(a_4) \# a_3 a'_1 \otimes b'_2 \# a'_2 & \end{aligned}$$

Observe that  $a_1$  and  $a_5$  are covered by  $b$ ,  $b'_1$  is covered by  $(S^{-1}(a_1) \cdot b \cdot a_5)$ ,  $a_2$  and  $a_4$  are covered by  $[(S^{-1}(a_1) \cdot b \cdot a_5) b'_1]$  and  $a'_1$  is covered by  $a_3$ .

**Definition 3** For all  $b, b', b'' \in B$  and  $a, a', a'' \in A$ , we define

$$\begin{aligned} \Delta_\#(b\#a)(b''\#a''\otimes b'\#a') &= \\ T_1^\#(b\#a\otimes b'\#a')(b''\#a''\otimes 1\#1) &= \\ b_1(a_1 \cdot b'' \cdot S^{-1}(a_3)) \# a_2 a'' \otimes b_2(a_4 \cdot b' \cdot S^{-1}(a_6)) \# a_5 a' & \end{aligned}$$

and

$$\begin{aligned} (b''\#a''\otimes b'\#a') \Delta_\#(b\#a) &= \\ (1\#1\otimes b'\#a') T_2^\#(b''\#a''\otimes b\#a) &= \\ a'_2 \cdot ((S^{-1}(a'_1) \cdot b'' \cdot a'_5) b_1) \cdot S^{-1}(a'_4) \# a'_3 a_1 \otimes & \\ b'(a'_1 \cdot b_2 \cdot S^{-1}(a'_3)) \# a'_2 a_2 & \end{aligned}$$

**Lemma 2** For any  $b \in B$  and  $a \in A$ ,  $\Delta_\#(b\#a)$  is a two-sided multiplier of  $B\#A\otimes B\#A$ .

**Proof** According to the definition of the comultiplication<sup>[2]</sup>, we only need to prove that the equation  $T_2^\#(b''\#a''\otimes b\#a)(1\#1\otimes b'\#a') = (b''\#a''\otimes 1\#1) T_1^\#(b\#a\otimes b'\#a')$  holds. For all  $b, b', b'' \in B$  and  $a, a', a'' \in A$ , we have

$$\begin{aligned} T_2^\#(b''\#a''\otimes b\#a)(1\#1\otimes b'\#a') &= \\ a'_2((S^{-1}(a'_1) \cdot b'' \cdot a'_5) b_1) \cdot S^{-1}(a'_4) \# a'_3 a_1 \otimes & \\ b_2(a_2 \cdot b' \cdot S^{-1}(a_4)) \# a_3 a' = b'' \cdot (a'_1 \cdot b_1 \cdot S^{-1}(a'_3)) \# & \\ a'_2 a_1 \otimes b_2(a_2 \cdot b' \cdot S^{-1}(a_4)) \# a_3 a' = & \\ (b''\#a''\otimes 1\#1)(b_1 \# a_1 \otimes b_2(a_2 \cdot b' \cdot S^{-1}(a_4)) \# a_3 a') &= \\ (b''\#a''\otimes 1\#1) T_1^\#(b\#a\otimes b'\#a') & \end{aligned}$$

**Lemma 3** The comultiplication  $\Delta_\#$  is coassociative on  $B\#A$ .

**Proof** To inquire if  $\Delta_\#$  is coassociative in the sense of Definition 2.2 in Ref. [2], we have to check if the linear maps  $T_1^\#$  and  $T_2^\#$  obey the following relationship:

$$(T_2^\# \otimes i_B \otimes i_A)(i_B \otimes i_A \otimes T_1^\#) = (i_B \otimes i_A \otimes T_1^\#)(T_2^\# \otimes i_B \otimes i_A)$$

For all  $b, b', b'' \in B$  and  $a, a', a'' \in A$ , we have

$$\begin{aligned} (T_2^\# \otimes i_B \otimes i_A)(i_B \otimes i_A \otimes T_1^\#)(b' \# a' \otimes b \# a \otimes b'' \# a'') &= \\ (T_2^\# \otimes i_B \otimes i_A)(b' \# a' \otimes b_1 \# a_1 \otimes b_2(a_2 \cdot b'' \cdot S^{-1}(a_4)) \# a_3 a'') &= \\ a'_2 \cdot ((S^{-1}(a'_1) \cdot b' \cdot a'_3) b_{11}) \cdot S^{-1}(a'_4) \# a'_3 a_{11} \otimes & \\ b_{12} \# a_{12} \otimes b_2(a_2 \cdot b'' \cdot S^{-1}(a_4)) \# a_3 a'' & \end{aligned}$$

and

$$\begin{aligned} (i_B \otimes i_A \otimes T_1^\#)(T_2^\# \otimes i_B \otimes i_A)(b' \# a' \otimes b \# a \otimes b'' \# a'') &= \\ (i_B \otimes i_A \otimes T_1^\#)(a'_2 \cdot ((S^{-1}(a'_1) \cdot b' \cdot a'_3) b_1) \cdot S^{-1}(a'_4) \# & \\ a'_3 a_1 \otimes b_2 \# a_2 \otimes b'' \# a'') &= \\ a'_2 \cdot ((S^{-1}(a'_1) \cdot b' \cdot a'_3) b_1) \cdot S^{-1}(a'_4) \# a'_3 a_1 \otimes b_{21} \# a_{21} \otimes & \\ b_{22}(a_{22} \cdot b'' \cdot S^{-1}(a_4)) \# a_3 a'' & \end{aligned}$$

We can easily see that these two terms are equal by using  $(T_2^\# \otimes i)(i \otimes T_1^\#) = (i \otimes T_1^\#)(T_2^\# \otimes i)$  on  $A \otimes A \otimes A$  and  $(T_2^\# \otimes i)(i \otimes T_1^\#) = (i \otimes T_1^\#)(T_2^\# \otimes i)$  on  $B \otimes B \otimes B$ .

Before we proceed to give the main result, we need the following definition and lemma.

**Definition 4** Let  $A$  and  $B$  be regular multiplier Hopf algebras and  $B$  is an  $A$ -bimodule algebra. Then,  $B$  is an  $A$ -bimodule bialgebra if

$$\Delta_\#(a \cdot b \cdot a')(1 \otimes b') = (a \cdot b \cdot a')_1 \otimes (a \cdot b \cdot a')_2 b' = a_1 \cdot b_1 \cdot a'_1 \otimes a_2 \cdot (b_2(S(a_3) \cdot b' \cdot S^{-1}(a'_3))) \cdot a'_2 \quad (3)$$

$$\varepsilon_\#(a \cdot b \cdot a') = \varepsilon_A(a) \varepsilon_B(b) \varepsilon_A(a') \quad (4)$$

for all  $b, b' \in B$  and  $a, a' \in A$ . Observe that on the right side, all decompositions are well-covered.

**Lemma 4**  $A$  and  $B$  are the same as those in Definition 4. We denote the antipode of  $B$  ( $A$ , resp.) by  $S_B$  ( $S_A$ , resp.). Then,

$$a \cdot S_B(b) \cdot a' = S_B(a \cdot b \cdot a')$$

The proof is straightforward.

Now, we can formulate the main result as follows.

**Theorem 1** Let  $A$  be a cocommutative regular multiplier Hopf algebra and  $B$  is a regular multiplier Hopf algebra such that  $B$  is an  $A$ -bimodule bialgebra. Then,  $\Delta_\#$  is a comultiplication on  $B \# A$  such that  $(B \# A, \Delta_\#)$  is a regular multiplier Hopf algebra.

**Proof** According to Proposition 2.9 in Ref. [10], our proof is given as follows.

First, we can easily check that the diagonal crossed product  $B \# A$  is an associative algebra, and we prove that it has a non-degenerate product by Lemma 1.

The comultiplication  $\Delta_\#$  as a multiplier of  $B \# A \otimes B \# A$  is coassociative by Lemma 3. We now show that  $\Delta_\#: B \# A \rightarrow M(B \# A \otimes B \# A)$  is a homomorphism. For all  $b, b', b'' \in B$  and  $a, a', a'' \in A$ , we have

$$\begin{aligned} \Delta_\#(b \# a)(\Delta_\#(b' \# a')(1 \# 1 \otimes b'' \# a'')) &= \\ \Delta_\#(b \# a)(b'_1 \# a'_1 \otimes b'_2(a'_2 \cdot b'' \cdot S^{-1}(a'_4)) \# a'_3 a'') &= \\ b_1(a_1 \cdot b'_1 \cdot S^{-1}(a'_3)) \# a_2 a'_1 \otimes b_2(a_4 \cdot (b'_2(a'_2 \cdot b'' \cdot & \\ S^{-1}(a'_4))) \cdot S^{-1}(a'_6) \# a_5 a'_3 a'' & \end{aligned}$$

$$b_1(a_1 \cdot b'_1 \cdot S^{-1}(a'_6)) \# a_3 a'_1 \otimes b_2(a_2 \cdot (b'_2(a'_2 \cdot b'' \cdot S^{-1}(a'_4))) \cdot S^{-1}(a'_5) \# a_4 a'_3 a'' =$$

$$\begin{aligned} b_1(a_1 \# a_3 a'_1 \otimes b_2(a_1(a_2 a'_2 \cdot b'' \cdot S^{-1}(a'_4) S^{-1}(a'_5)) \# a_4 a'_3 a'') &= \\ b_1(a_1 \# a_2 a'_1 \otimes b_2(a_1(a_3 a'_2 \cdot b'' \cdot S^{-1}(a'_5 a'_4)) \# a_4 a'_3 a'') &= \\ \Delta_\#(b(a_1 \cdot b' \cdot S^{-1}(a'_3)) \# a_2 a') (1 \# 1 \otimes b'' \# a'') &= \\ \Delta_\#((b \# a)(b' \# a')) (1 \# 1 \otimes b'' \# a'') & \end{aligned}$$

We can easily prove that the functions  $T_1^\#$  and  $T_2^\#$  are bijective by the operators  $T_1^A, T_2^A, T_1^B, T_2^B$ , and  $R$  and  $R'$  are bijective.

There is a counit  $\varepsilon_\#$  defined as

$$\varepsilon_\#(b \# a) = \varepsilon_B(b) \varepsilon_A(a)$$

for all  $b \in B$  and  $a \in A$ . It can easily be checked that these two equations  $(\varepsilon_\# \otimes i)(\Delta_\#(b \# a)(1 \# 1 \otimes b' \# a')) = (b \# a)(b' \# a')$  and  $(i \otimes \varepsilon_\#)((b' \# a' \otimes 1 \# 1) \Delta_\#(b \# a)) = (b' \# a')(b \# a)$  are valid. From Eq. (4), it easily follows that  $\varepsilon_\#$  is a homomorphism.

There is an invertible antipode  $S_\#$  defined as

$$\begin{aligned} S_\#(b \# a) &= R' \circ \text{Ro}(S_A \otimes S_B) \circ \sigma(b \# a) = \\ S_A(a_3) \cdot S_B(b) \cdot a_1 \# S_A(a_2) &\in B \# A \end{aligned}$$

For any  $b, b' \in B$  and  $a, a' \in A$ , we obtain

$$\begin{aligned} m_\#(S_\# \otimes i_B \# i_A)(\Delta_\#(b \# a)(1 \# 1 \otimes b' \# a')) &= \\ m_\#(S_\# \otimes i_B \# i_A)(b_1 \# a_1 \otimes b_2(a_2 \cdot b' \cdot S^{-1}(a_4)) \# a_3 a') &= \\ m_\#(S(a_3) \cdot S(b_1) \cdot a_1 \# S(a_2) \otimes b_2(a_4 \cdot b' \cdot S^{-1}(a_6)) \# & \\ a_5 a') &= (S(a_5) \cdot S(b_1) \cdot a_1)(S(a_4) \cdot \\ (b_2(a_6 \cdot b' \cdot S^{-1}(a_8))) \cdot a_2) \# S(a_3) a_7 a' &= \\ S(a_3) \cdot (S(b_1) b_2(a_4 \cdot b' \cdot S^{-1}(a_6))) \cdot a_1 \# S(a_2) a_5 a' &= \\ \varepsilon_B(b) b' \cdot S^{-1}(a_4) \cdot a_1 \# S(a_2) a_3 a' = \varepsilon_B(b) \varepsilon_A(a) b' \# a' &= \\ \varepsilon_{B \# A}(b \# a)(b' \# a') & \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} m_\#(i_B \# i_A \otimes S_\#)((b' \# a' \otimes 1 \# 1) \Delta_\#(b \# a)) &= \\ \varepsilon_{B \# A}(b' \# a')(b \# a) & \end{aligned}$$

Using Lemma 4, we can easily prove that  $S_\#$  is an anti-homomorphism.

Finally, we can easily observe that the multipliers  $\Delta_\#(b \# a)(b' \# a' \otimes 1 \# 1)$  and  $(1 \# 1 \otimes b' \# a') \Delta_\#(b \# a)$  are in  $B \# A \otimes B \# A$ .

The proof is complete.

In Theorem 1, if the right action is trivial, we have the following corollary which is the main result of Ref. [7].

**Corollary 1** Let  $A$  be a cocommutative multiplier Hopf algebra and  $B$  is a regular multiplier Hopf algebra such that  $B$  is an  $A$ -bimodule bialgebra. Then,  $\Delta_\#$  is a comultiplication on  $B \# A$  such that the smash product  $(B \# A, \Delta_\#)$  is a regular multiplier Hopf algebra.

In Ref. [10], for multiplier Hopf algebras, there is a natural notion of left and right invariance for linear functionals (called integrals in the theory of Hopf algebra). We now suppose that  $A$  and  $B$  are multiplier Hopf algebras which are the same as those in Theorem 1. Furthermore, we suppose that  $A$  and  $B$  have invariant functions.

In the following proposition, we will give these integrals on  $(B\#A, \Delta_{\#})$ .

**Proposition 1**  $A$  and  $B$  are the same as those in Theorem 1. Let  $\varphi_B(\varphi_A, \text{ resp. })$  be a left integral of  $B(A, \text{ resp. })$ . Then,  $\varphi_B \otimes \varphi_A$  is the left integral of  $B\#A$ . The same statement yields for the right integral.

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乘子 Hopf 代数上的对角交叉积

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**摘要:** 设  $A$  和  $B$  是 2 个正则乘子 Hopf 代数. 首先, 使用双模代数构造了乘子 Hopf 代数上对角交叉积  $B\#A$  的定义, 推广了 Hopf 代数上的对角交叉积. 给出了  $B\#A$  上的积是非退化的结论. 介绍了对角交叉积  $B\#A$  上的余乘  $\Delta_{\#}$  的概念, 对于任意的  $b \in B$  和  $a \in A$ , 它由  $B \otimes B$  上的乘子  $\Delta_B(b)$  和  $A \otimes A$  上的乘子  $\Delta_A(a)$  构成, 且元素  $\Delta_{\#}(b \otimes a)$  是  $B\#A \otimes B\#A$  上的双边乘子. 然后, 描述了对角交叉积  $B\#A$  成为一个正则乘子 Hopf 代数的充分条件. 特别地, 推广了 Delvaux 在冲积情况下的主要定理. 最后, 考虑了乘子 Hopf 代数上对角交叉积的积分.

**关键词:** 乘子 Hopf 代数; 双模代数; 对角交叉积

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