Journal of Southeast University (English Edition) Vol. 36, No. 3, pp. 364 − 366 Sept. 2020 ISSN 1003—7985

Sweedler's dual of Hopf algebras in $\overline{{}^{H}_{H}\mathrm{YDQCM}}$

Zhang Tao Wang Shuanhong

(School of Mathematics, Southeast University, Nanjing 211189, China)

Abstract: Firstly, the notion of the left-left Yetter-Drinfeld quasicomodule $M=(M,\cdot,\rho)$ over a Hopf coquasigroup H is given, which generalizes the left-left Yetter-Drinfeld module over Hopf algebras. Secondly, the braided monoidal category $\overline{{}^{H}_{H}\mathrm{YDQCM}}$ is introduced and the specific structure maps are given. Thirdly, Sweedler's dual of infinite-dimensional Hopf algebras in $\overline{{}^{H}_{H}\mathrm{YDQCM}}$ is discussed. It proves that if $(B, m_B, \mu_B, \Delta_B, \varepsilon_B)$ is a Hopf algebra in $\overline{{}^{H}_{H}\mathrm{YDQCM}}$ with antipode S_B, then $(B^0, (m_{B^0})^{\mathrm{op}}, \varepsilon_B^*, (\Delta_{B^0})^{\mathrm{op}}, \mu_B^*)$ is a Hopf algebra in $\overline{{}^{H}_{H}\mathrm{YDQCM}}$ with antipode S_B^*, which generalizes the corresponding results over Hopf algebras.

Key words: Hopf (co)quasigroup; Yetter-Drinfeld quasi(co) module; braided monoidal category; duality

DOI: 10.3969/j.issn.1003 − 7985.2020.03.016

Let H be a Hopf algebra. Schauenburg[1] obtained a braided monoidal category equivalence between the category of right-right Yetter-Drinfeld modules over H and the category of two-sided two-cosided Hopf modules over H under some suitable assumption. This yields new sources of braiding by which one can obtain the solutions to the Yang-Baxter equation, which plays a fundamental role in various areas of mathematics[2-3].

In 1997, Doi[4] studied the duality of any finite-dimensional Hopf modules in the left-left Yetter-Drinfeld category ${}^{L}_{L}\mathrm{YD}$ where L denotes any ordinary Hopf algebra over the ground field k with a bijective antipode.

The most well-known examples of Hopf algebras are the linear spans of (arbitrary) groups. Dually, also the vector space of linear functionals on a finite group carries the structure of a Hopf algebra. In the case of quasigroups (nonassociative groups), however, it is no longer a Hopf algebra, but more generally, a Hopf quasigroup[5-10], which is a specific case of the notion of unital coassociative bialgebra[11].

Motivated by these notions and structures, this paper aims to construct Sweedler's dual of infinite-dimensional Hopf algebras in $\overline{{}^{H}_{H}\mathrm{YDQCM}}$.

Throughout this paper, let k be a fixed field. We will work over k. Let C be a coalgebra with a coproduct Δ. We will use Heyneman-Sweedler's notation[12], $\Delta(c)=\sum c_1\otimes c_2$ for all $c\in C$, for coproduct.

1 Preliminaries

Recall from Ref. [5] that a Hopf coquasigroup is a unital associative algebra H, armed with three linear maps: $\Delta: H\rightarrow H\otimes H$, $\varepsilon: H\rightarrow K$ and $S: H\rightarrow H$ satisfying the following equations for all $a, b\in H$:

$$\begin{aligned}\Delta(ab) &= \Delta(a)\Delta(b)\\ \varepsilon(ab) &= \varepsilon(a)\varepsilon(b)\\ (\mathrm{id}\otimes\varepsilon)\Delta(a) &= a = (\varepsilon\otimes\mathrm{id})\Delta(a)\\ \sum S(a_1)a_{21}\otimes a_{22} &= 1\otimes a = \sum a_1 S(a_{21})\otimes a_{22}\\ \sum a_{11}\otimes S(a_{12})a_2 &= a\otimes 1 = \sum a_{11}\otimes a_{12}S(a_2)\end{aligned}$$

Recall from Ref. [6], the authors gave the notion of a left H-quasimodule over a Hopf quasigroup H. Duality, a left H-quasicomodule over a Hopf coquasigroup H is a vector space M with a linear map $\rho: M\rightarrow H\otimes M$, where $\rho(m) = \sum m_{-1}\otimes m_0$ such that $\sum \varepsilon(m_{-1})m_0 = m$ and

$$\sum S(m_{-1})m_{0-1}\otimes m_{00} = \sum m_{-1}S(m_{0-1})\otimes m_{00} = 1\otimes m$$

for all $m\in M$.

Moreover, the authors studied the notion of the left-left Yetter-Drinfeld quasimodule $M=(M,\cdot,\rho)$ over a Hopf quasigroup H.

Duality, a left-left Yetter-Drinfeld quasicomodule $M=(M,\cdot,\rho)$ over a Hopf coquasigroup H is a left H-module $(M,\cdot)$ and a left H-quasicomodule $(M,\cdot)$ satisfying the following equations:

$$\begin{aligned}\sum (a_1\cdot m)_{-1}a_2\otimes(a_1\cdot m)_0 &= \sum a_1 m_{-1}\otimes a_2\cdot m_0\\ \sum a_1\cdot m\otimes a_{21}\otimes a_{22} &= \sum a_{11}\cdot m\otimes a_{12}\otimes a_2\\ \sum a_1\otimes a_{21}\cdot m\otimes a_{22} &= \sum a_{11}\otimes a_{12}\cdot m\otimes a_2\end{aligned}$$

for all $a\in H$, $m\in M$.

Remark that the first equation is equivalent to the following formula:

$$\rho(a\cdot m) = \sum a_{11}m_{-1}S(a_2)\otimes a_{12}\cdot m_0$$

We use ${}^{H}_{H}\mathrm{YDQCM}$ to denote the category of the left-left Yetter-Drinfeld quasicomodules over a Hopf coquasigroup

---

Received 2019-06-21, **Revised** 2020-01-20.

Biographies: Zhang Tao (1990—), male, Ph. D. candidate; Wang Shuanhong (corresponding author), male, doctor, professor, Shuanwang@seu.edu.cn.

Foundation item: The National Natural Science Foundation of China (No. 11371088, 11571173, 11871144).

Citation: Zhang Tao, Wang Shuanhong. Sweedler's dual of Hopf algebras in $\overline{{}^{H}_{H}\mathrm{YDQCM}}$. [J]. Journal of Southeast University (English Edition), 2020, 36(3): 364 − 366. DOI: 10.3969/j.issn.1003 − 7985.2020.03.016.

H. Moreover, if we assume that M is an ordinary left H-comodule, we say that M is a left-left Yetter-Drinfeld module over H. Obviously, the left-left Yetter-Drinfeld modules with the obvious morphisms is a subcategory of ${}^H_H\mathrm{YDQCM}$. We denote it by ${}^H_H\mathrm{YDCM}$.

Note that if the antipode S of Hopf quasigroup H is bijective, then the category ${}^H_H\mathrm{YDCM}$ is a braided monoidal category with a "pre-braiding" defined as

$$\tau: M\otimes N\to N\otimes M,\ \tau(m\otimes n) = \sum m_{-1}\cdot n\otimes m_0$$

$$\tau^{-1}: N\otimes M\to M\otimes N,\ \tau^{-1}(n\otimes m) = \sum m_0\otimes S^{-1}(m_{-1})\cdot n$$

for any $M, N\in {}^H_H\mathrm{YDCM}$, $m\in M$ and $n\in N$.

One can check the following lemmas and Corollary 1.

Lemma 1 Let H be a Hopf coquasigroup. Then, $({}^H_H\mathrm{YDQCM}, \otimes, k)$ is a monoidal category.

Lemma 2 Let H be a Hopf coquasigroup with a bijective antipode S. Then, the monoidal category $({}^H_H\mathrm{YDQCM}, \otimes, k)$ with the pre-braiding defined above is a braided monoidal category if and only if the following identity holds:

$$\sum m_{-11}\cdot n\otimes m_{-12}\cdot p\otimes m_0 = \sum m_{-1}\cdot n\otimes m_{0-1}\cdot p\otimes m_{00}$$

For any $M, N, P\in {}^H_H\mathrm{YDQCM}$, $m\in M$, $n\in N$ and $p\in P$, this category will be denoted as $({}^H_H\mathrm{YDQC}, \otimes, k)$.

Corollary 1 Let H be a Hopf coquasigroup with a bijective antipode S. If the following equations hold:

$$\sum m_{-11}\cdot n\otimes m_{-12}\otimes m_0 = \sum m_{-1}\cdot n\otimes m_{0-1}\otimes m_{00}$$

$$\sum m_{-11}\otimes m_{-12}\cdot n\otimes m_0 = \sum m_{-1}\otimes m_{0-1}\cdot n\otimes m_{00}$$

for any $M, N\in {}^H_H\mathrm{YDQCM}$, $m\in M$, $n\in N$, then $({}^H_H\mathrm{YDQCM}, \otimes, k)$ is a braided monoidal category, and we denote it as $(\overline{{}^H_H\mathrm{YDQCM}}, \otimes, k)$.

Let H be a Hopf coquasigroup with a bijective antipode S. Under the hypotheses of the above results, we have the relationship:

$$({}^H_H\mathrm{YDCM}, \otimes, k)\subseteq(\overline{{}^H_H\mathrm{YDQCM}}, \otimes, k)\subseteq({}^H_H\mathrm{YDQC}, \otimes, k)$$

In what follows, let L denote a Hopf coquasigroup with a bijective antipode S_L. Let H be a Hopf algebra in $\overline{{}^L_L\mathrm{YDQCM}}$, i.e., explicitly, it is both a L-algebra and a L-coalgebra with comultiplication Δ and counit ε, and the following identities hold:

$$\Delta(xy) = \sum x_1(x_{2-1}\cdot y_1)\otimes x_{20}y_2,\ \Delta(1) = 1\otimes 1$$

$$\varepsilon(xy) = \varepsilon(x)\varepsilon(y),\ \varepsilon(1_H) = 1$$

$$\rho_H(xy) = \sum (xy)_{-1}\otimes(xy)_0 = \sum x_{-1}y_{-1}\otimes x_0y_0,\ \rho_H(1_H) = 1_L\otimes 1_H$$

$$\sum x_{-1}\otimes x_{01}\otimes x_{02} = \sum x_{1-1}x_{2-1}\otimes x_{10}\otimes x_{20}$$

$$\sum x_{-1}\varepsilon_H(x_0) = \varepsilon_H(x)1$$

$$l\cdot(xy) = \sum(l_1\cdot x)(l_2\cdot y),\ l\cdot 1_H = \varepsilon(l)1_H$$

$$\Delta(l\cdot x) = \sum(l_1\cdot x_1)\otimes(l_2\cdot x_2),\ \varepsilon(l\cdot x) = \varepsilon(l)\varepsilon(x)$$

$$S_H(xy) = \sum((S(x))_{-1}\cdot S_H(y))(S(x))_0 = \sum(x_{-1}\cdot S(y))S(x_0),\ S(1) = 1$$

$$S_H(xy) = \sum((S(x))_{-1}\cdot S_H(y))(S(x))_0 = \sum(x_{-1}\cdot S(y))S(x_0),\ S(1) = 1$$

for any $x, y\in H$ and $l\in L$.

2 A Generalization of Sweedler's Dual of Hopf Algebras in ${}^H_H\mathrm{YD}$

In this section, let H be a Hopf coquasigroup with a bijective antipode S, and B an infinite-dimensional Hopf algebra in $\overline{{}^H_H\mathrm{YDQCM}}$.

Let (A, m_A, μ_A) be an associative algebra. Then, we have coalgebra A^0 given in Ref. [13] as

$$A^0 = \{f\in A^*\mid \mathrm{Ker}f\supset \text{an ideal of } A \text{ of cofinite dimension}\}$$

Let $(B, m_B, \mu_B, D_B, \varepsilon_B)$ be a bialgebra in $\overline{{}^H_H\mathrm{YDQCM}}$. Recall that B^0 is the subspace of all $b^*\in B^*$ vanishing on some cofinite ideal I of B. Let $i: B^*\otimes B^*\to(B\otimes B)^*$ be the natural embedding, defined as $(i(f\otimes g))(a\otimes b) = f(a)g(b)$ for $f, g\in B^*$ and $a, b\in B$. For all $f\in B^*$, the following statements are equivalent:

$$f\in B^0 = (m_B^*)^{-1}(A^*\otimes A^*),\ \dim(B\rightharpoonup f)<\infty$$

$$\dim(f\leftharpoonup B)<\infty,\ \dim(B\rightharpoonup f\leftharpoonup B)<\infty$$

For any $f\in B^*$ and $a, b\in B$, we define $(a\rightharpoonup f)(b) = f(ba)$ and $(f\leftharpoonup a)(b) = f(ab)$. This defines a B-B bimodule structure on B^*.

We consider the action of H on B^* given by $(h\cdot f)(b) = f(S(h)\cdot b)$ and the quasicoaction of H on B^* defined by $\rho(f)(b) = S^{-1}(b_{(-1)})\otimes f(b_0)$ for all $h\in H$, $b\in B$ and $f\in B^*$.

Let A, B be algebras in $\overline{{}^H_H\mathrm{YDQCM}}$. Then, we have the braided tensor product algebra $A\underline{\otimes}B$ with the product $(x\underline{\otimes}y)(a\underline{\otimes}b) = \sum x(y_{(-1)}\cdot a)\underline{\otimes}y_0b$ for all $x, a\in A$ and $y, b\in B$.

It is not difficult for one to check the following two lemmas.

Lemma 3 The action $\leftharpoonup: B^*\otimes B\to B$ is a left H-linear and the action $\rightharpoonup: B\otimes B^*\to B$ is a left H^{cop}-linear.

Lemma 4 B^0 is an H-submodule of $(B^*, \Delta_B^*, \varepsilon_B^*)$.

Proposition 1 B^0 is a subalgebra of $(B^*, \Delta_B^*, \varepsilon_B^*)$.

Proof By Lemma 3, for any f, $g\in B^*$ and a, $b\in B$, we obtain

$$((fg)\leftharpoonup a)(b) = (fg)(ab) = (f\otimes g)\Delta(ab) = f(a_1(a_{2(-1)}\cdot b_1)g(a_{20}b_2) = f(a_{2(-1)2}\cdot[(S^{-1}(a_{2(-1)1})\cdot a_1)b_1])g(a_{20}b_2) = (S^{-1}(a_{2(-1)2})\cdot f)[(S^{-1}(a_{2(-1)1})\cdot a_1)b_1]\times$$

$$
\begin{aligned}
g(a_{20}b_2) &= [(S^{-1}(a_{2(-1)2})\cdot f)\leftharpoonup\\
&(S^{-1}(a_{2(-1)1})\cdot a_1)](b_1)(g\leftharpoonup a_{20})(b_2) =\\
&\Delta^*[(S^{-1}(a_{2(-1)})\cdot(f\leftharpoonup a_1)\otimes g\leftharpoonup a_{20})](b)
\end{aligned}
$$

Thus,

$$(fg)\leftharpoonup B\subseteq\Delta^*[H\cdot(f\leftharpoonup B)\otimes g\leftharpoonup B]\subseteq\Delta^*[(H\cdot f)\leftharpoonup B)\otimes g\leftharpoonup B]$$

Combining $f\in B^0$ with Lemma 4, we can conclude that $H\cdot f\in B^0$. Moreover, since $f, g\in B^0$, the left-hand side of the above containment is finite dimensional. Hence, $fg\in B^0$. Finally, it is easy to check that $\varepsilon_B^*(1)\in B^0$.

Lemma 5 We have that $i\circ\tau$: $(B^*)^{\mathrm{op}}\underline{\otimes}(B^*)^{\mathrm{op}}\to(B\underline{\otimes}B)^{*\mathrm{op}}$ is an algebra map in $\overline{{}^H_H\mathrm{YDQCM}}$.

Proof Applying the quasicoaction of H on B^*, the proof is complete.

Now, we obtain the main result of this paper which gives a characterization of Sweedler's dual of Hopf algebras in ${}^H_H\mathrm{YD}$.

Theorem 1 Let H be any Hopf coquasigroup with a bijective antipode S. If $(B, m_B, \mu_B, \Delta_B, \varepsilon_B)$ be a Hopf algebra in $\overline{{}^H_H\mathrm{YDQCM}}$ with antipode S_B, then $(B^0, (m_{B^0})^{\mathrm{op}}, \varepsilon_B^*, (\Delta_{B^0})^{\mathrm{op}}, \mu_B^*)$ is a Hopf algebra in $\overline{{}^H_H\mathrm{YDQCM}}$ with antipode S_B^*.

Proof According to Ref. [13], we check that

1) B^0 is an H-subquasicomodule of B^*.

2) Observe that $(m_{B^0})^{\mathrm{op}}=\Delta_B^*\circ i\circ\tau$: $B^0\otimes B^0\to B^*$. It is morphism in ${}^H_H\mathrm{YDQCM}$. Obviously, $\varepsilon_B^*: k\to B^0$ is. Thus, $(B^0, (m_{B^0})^{\mathrm{op}}, \varepsilon_B^*)$ is an algebra in ${}_H M$.

3) Observe that $(\Delta_{B^0})^{\mathrm{op}}$ is the composite map $B^0\xrightarrow{m_B^*} i(B^0\otimes B^0)\xrightarrow{(i\circ\tau)^{-1}} B^0\otimes B^0$. It is a morphism in ${}^H_H\mathrm{YDQCM}$. Obviously, $\mu_B^*: B^0\to k$. Thus, $(B^0, (\Delta_{B^0})^{\mathrm{op}}, \mu_B^*)$ is a coalgebra in ${}_H M$.

4) $(\Delta_{B^0})^{\mathrm{op}}: (B^0)^{\mathrm{op}}\to(B^0)^{\mathrm{op}}\underline{\otimes}(B^0)^{\mathrm{op}}$ is an algebra map.

5) $S_B^*(B^0)\subseteq B^0$.

6) $(m_{B^0})^{\mathrm{op}}(S_B^*\otimes\mathrm{id}_{B^0})(\Delta_{B^0})^{\mathrm{op}}=\varepsilon_B^*\mu_B^*$ and $(m_{B^0})^{\mathrm{op}}(\mathrm{id}_{B^0}\otimes S_B^*)(\Delta_{B^0})^{\mathrm{op}}=\varepsilon_B^*\mu_B^*$.

In the setting of Hopf coquasigroups, the notion of the left H-module is exactly the same as that for ordinary Hopf algebras since it only depends on the algebra structure of H. Thus, the proof of these assertions is either trivial or will become trivial after acquainting the Hopf coquasigroup calculus developed above.

References

[1] Schauenburg P. Hopf modules and Yetter-Drinfel'd modules[J]. *Journal of Algebra*, 1994, **169**: 874 – 890.

[2] Caenepeel S, Militaru G, Zhu S L. Frobenius and separable functors for entwined modules[C]//*Lecture Notes in Mathematics*. Heidelberg: Springer, 2002: 89 – 157.

[3] Wang S H. Braided monoidal categories associated to yetter-drinfeld categories[J]. *Communications in Algebra*, 2002, **30**(11): 5111 – 5124.

[4] Doi Y. Hopf modules in Yetter-Drinfeld categories[J]. *Communications in Algebra*, 1998, **26**(9): 3057 – 3070.

[5] Klim J, Majid S. Hopf quasigroups and the algebraic 7-sphere[J]. *Journal of Algebra*, 2010, **323**(11): 3067 – 3110.

[6] Alonso Álvarez J N, Fernández Vilaboa J M, González Rodríguez R, et al. Projections and Yetter-Drinfel'd modules over Hopf (co)quasigroups[J]. *Journal of Algebra*, 2015, **443**: 153 – 199.

[7] Brzeziński T. Hopf modules and the fundamental theorem for Hopf (co) quasigroups [J]. *International Electron Journal of Algebra*, 2010, **8**: 114 – 128.

[8] Brzeziński T, Jiao Z M. Actions of Hopf quasigroups[J]. *Communications in Algebra*, 2012, **40**(2): 681 – 696.

[9] Fang X L, Wang S H. Twisted smash product for Hopf quasigroups [J]. *Journal of Southeast University (English Edition)*, 2011, **27**(3): 343 – 346.

[10] Zhang T, Wang S H, Wang D G. A new approach to braided monoidal categories [J]. *Journal of Mathematical Physics*, 2019, **60**(1): 013510. DOI: 10.1063/1.5055707.

[11] Pérez-Izquierdo J M. Algebras, hyperalgebras, nonassociative bialgebras and loops[J]. *Advances in Mathematics*, 2007, **208**(2): 834 – 876.

[12] Sweedler M E. *Hopf algebras*[M]. New York: W A Benjamin Inc, 1969.

[13] Ng S H, Taft E J. Quantum convolution of linearly recursive sequences[J]. *Journal of Algebra*, 1997, **198**(1): 101 – 119.

$\overline{{}^H_H\mathrm{YDQCM}}$范畴上 Hopf 代数的 Sweedler 对偶

张　涛　　王栓宏

(东南大学数学学院, 南京 211189)

摘要: 首先,给出了 Hopf 余拟群 H 上的左-左 Yetter-Drinfeld 拟余模 $M=(M, \cdot, \rho)$ 的概念,其为 Hopf 代数上的左-左 Yetter-Drinfeld 模结构的推广. 其次,介绍了辫子张量范畴$\overline{{}^H_H\mathrm{YDQCM}}$的定义并且给出其具体的结构映射. 最后,讨论辫子张量范畴$\overline{{}^H_H\mathrm{YDQCM}}$上的无限维 Hopf 代数 Sweedler 的对偶问题. 证明了如果$(B, m_B, \mu_B, \Delta_B, \varepsilon_B)$是$\overline{{}^H_H\mathrm{YDQCM}}$上有对极 S_B 的 Hopf 代数, 那么$(B^0, (m_{B^0})^{\mathrm{op}}, \varepsilon_B^*, (\Delta_{B^0})^{\mathrm{op}}, \mu_B^*)$是$\overline{{}^H_H\mathrm{YDQCM}}$上有对极 S_B^* 的 Hopf 代数,从而推广了 Hopf 代数上的相应结果.

关键词: Hopf(余)拟群; Yetter-Drinfeld 拟(余)模; 辫子张量范畴; 对偶

中图分类号: O153.5