

Gorenstein dimensions for weak Hopf-Galois extensions

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Abstract: The representation of weak Hopf algebras is studied by investigating the Gorenstein dimensions of weak Hopf algebras and weak Hopf-Galois extensions. Let H be a weak Hopf algebra with a bijective antipode, A a weak right H -comodule algebra and B the H -coinvariant subalgebra of A . First, some properties of Gorenstein projective H -modules in the representation category are studied, and the fact that Gorenstein global dimension of H is the same as the Gorenstein projective dimension of its left unital subalgebra is demonstrated. Secondly, by applying the integral theory of weak Hopf algebras, on the one hand, a sufficient and necessary condition that a projective A -module is a projective B -module is given; on the other hand, the separability of the functor $A \otimes_B -$ and that of the restriction of scalar function ${}_B(-)$ are described, respectively. Finally, as a mean result, the Gorenstein global dimension of a weak Hopf-Galois extension is investigated under the condition that H is both semisimple and cosemisimple.

Key words: weak Hopf algebra; integral; weak Hopf-Galois extension; Gorenstein dimension

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The study of Gorenstein projective modules can be traced back to Ref. [1], where the G -dimension of a finitely generated module over a two-sided Noetherian ring was introduced. Furthermore, the definition of Gorenstein projective modules was given in Ref. [2]. We knew already that a module of G -dimension zero is actually a Gorenstein projective module. Gorenstein projective modules play an important role in many areas. For example, they are widely used in the representation theory of Artin algebras, the theory of stable and singularity categories, and the cohomology theory of commutative rings, and so on.

Weak bialgebras and weak Hopf algebras introduced in Ref. [3] generalized the ordinary bialgebras and Hopf algebras by weakening the comultiplication of unit and the

multiplication of counit. Comultiplication is allowed to be non-unital, but it is still coassociative. In exchange for coassociativity, the multiplicativity of the counit is replaced by a weaker condition, implying that the unit representation is not necessarily one-dimensional and irreducible. Weak Hopf algebras can provide us with a good framework for studying the symmetries of certain quantum field theories. Groupoid algebras, face algebras and generalized Kac algebras are examples of weak Hopf algebras.

The main purpose of this paper is to study the representation of weak Hopf algebras by investigating the Gorenstein dimensions of weak Hopf algebras and weak Hopf-Galois extensions. Let H be a weak Hopf algebra with a bijective antipode, A a weak right H -comodule algebra and B the H -coinvariant subalgebra of A . After recalling some basic definitions and giving a summary of the fundamental properties concerned with weak Hopf algebras, we study some properties of Gorenstein projective H -modules in representation category, and find that the Gorenstein global dimension of H is the same as the Gorenstein projective dimension of its left unital subalgebra. By applying the integral theory of weak Hopf algebras, on the one hand, we give a sufficient and necessary condition that a projective A -module is a projective B -module; on the other hand, we describe the separability of the functor $A \otimes_B -$ and that of the restriction of scalar function ${}_B(-)$. Consequently, we investigate the Gorenstein global dimension of a weak Hopf-Galois extension.

1 Preliminaries

Throughout this article, k denotes a fixed field, and we will always work over k . The tensor product $\otimes := \otimes_k$ and Hom-functor are always assumed to be over k . We also use Sweedler's notations for the terminologies on coalgebras and comodules. For a coalgebra C , we write the comultiplication $\Delta(c) = \sum c_1 \otimes c_2$ for any $c \in C$. For a right C -comodule M , we denote its coaction by $\rho(m) = \sum m_{(0)} \otimes m_{(1)}$ for any $m \in M$. For any unexplained definitions and notations, one may refer to Ref. [4] or Ref. [5].

Definition 1^[3] A weak Hopf algebra H is an algebra (H, μ, η) and a coalgebra (H, Δ, ε) such that

$$\Delta(xy) = \Delta(x)\Delta(y) \quad (1)$$

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$$\varepsilon(xyz) = \sum \varepsilon(xy_1) \varepsilon(y_2z) = \sum \varepsilon(xy_2) \varepsilon(y_1z) \quad (2)$$

$$\Delta^2(1_H) = (\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H)) = (1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H) \quad (3)$$

and there exists a linear map $S: H \rightarrow H$, called an antipode, satisfying

$$\left. \begin{aligned} \sum x_1 S(x_2) &= \sum \varepsilon(1_1 x) 1_2 \\ \sum S(x_1) x_2 &= \sum 1_1 \varepsilon(x 1_2) \\ \sum S(x_1) x_2 S(x_3) &= S(x) \end{aligned} \right\} \quad (4)$$

for all $x, y, z \in H$, where $\Delta(1_H) = \sum 1_1 \otimes 1_2$, $\Delta^2 = (\Delta \otimes \text{id}_H) \circ \Delta$.

Let H be a weak Hopf algebra. The images $H_t = \varepsilon_t(h)$ and $H_s = \varepsilon_s(h)$ of the projections $\varepsilon_t, \varepsilon_s: H \rightarrow H$ defined by

$$\varepsilon_t(h) = \sum \varepsilon(1_1 h) 1_2, \quad \varepsilon_s(h) = \sum 1_1 \varepsilon(h 1_2)$$

are both separable subalgebras of H and commute with each other^[3]. H_t and H_s are called left and right unital subalgebras, respectively.

Definition 2^[3] A left integral in H is an element $\ell \in H$ satisfying $h\ell = \varepsilon_t(h)\ell$ for all $h \in H$. Furthermore, if $\varepsilon_t(\ell) = 1_H$, then it is said to be normalized. A normalized right integral is defined similarly. A left or right integral in H is called non-degenerate if it defines a non-degenerate functional on H^* , the dual space of H .

The spaces of left and right integrals in H are denoted as \int_H^l and \int_H^r , respectively.

The left integral $l \in \int_H^l$ satisfies $S(l) \in \int_H^r$ ^[3] and

$$\sum S(h\ell_1) \otimes \ell_2 = \sum \ell_1 \otimes h\ell_2 \quad (5)$$

for all $h \in H$. Meanwhile, according to Ref. [6], the left integral $\lambda \in \int_H^l$ satisfies the following invariant properties for all $h, g \in H$,

$$\sum h_1 \lambda(gh_2) = \sum \lambda(g_2 h) S(g_1) \quad (6)$$

If a weak Hopf algebra H is finite dimensional, then it has a bijective antipode S ^[6], in which inverse is denoted as S^{-1} in the following, and there exist non-degenerate left integrals $l \in \int_H^l$ and $\lambda \in \int_H^r$ such that $\sum \ell_1 \lambda(\ell_2) = 1_H$ ^[7]. Furthermore, the quasi-basis of λ is $\sum \ell_2 \otimes S^{-1}(\ell_1)$ ^[3]. Therefore, for all $h \in H$,

$$\sum \lambda(S^{-1}(\ell)h) = \varepsilon(h) \quad (7)$$

Let H be a weak Hopf algebra with an antipode S . According to Refs. [8–9], we have

$$\varepsilon_t \circ S = \varepsilon_t \circ \varepsilon_s = S \circ \varepsilon_s, \quad \varepsilon_s \circ S = \varepsilon_s \circ \varepsilon_t = S \circ \varepsilon_t \quad (8)$$

$$\left. \begin{aligned} \sum h_1 \otimes \varepsilon_s(h_2) &= \sum h 1_1 \otimes S(1_2) \\ \sum \varepsilon_t(h_1) \otimes h_2 &= \sum S(1_1) \otimes 1_2 h \end{aligned} \right\} \quad (9)$$

$$\left. \begin{aligned} \sum h_1 \otimes \varepsilon_t(h_2) &= \sum 1_1 h \otimes 1_2 \\ \sum \varepsilon_s(h_1) \otimes h_2 &= \sum 1_1 \otimes h 1_2 \end{aligned} \right\} \quad (10)$$

$$\left. \begin{aligned} \sum \varepsilon(h_1 g) h_2 h &= \varepsilon_t(g) \\ \sum \varepsilon(h g_2) g_1 &= \varepsilon_s(h) g \end{aligned} \right\} \quad (11)$$

Definition 3^[10] Let H be a weak Hopf algebra, and A a right H -comodule, which is also an associative algebra. We call A a weak right H -comodule algebra if

$$\rho(ab) = \rho(a)\rho(b) \quad (12)$$

$$\sum a 1_{(0)} \otimes 1_{(1)} = \sum a_{(0)} \otimes \varepsilon_t(a_{(1)}) \quad (13)$$

for all $a, b \in A$.

Let H be a weak Hopf algebra, and A a weak right H -comodule algebra. Then, according to Ref. [8], we obtain that

$$\begin{aligned} \sum a_{(0)} \otimes \varepsilon_s(a_{(1)}) &= \sum a 1_{(0)} \otimes S(1_{(1)}) \stackrel{(8)}{=} \\ &= \sum a 1_{(0)} \otimes \varepsilon_s(1_{(1)}) \end{aligned} \quad (14)$$

Define the H -coinvariant subalgebra of A as

$$B := A^{\text{co}H} = \left\{ a \in A \mid \rho(a) = \sum a_{(0)} \otimes \varepsilon_t(a_{(1)}) \right\}$$

Then, we know from Ref. [11] that

$$B = \left\{ a \in A \mid \rho(a) = \sum a 1_{(0)} \otimes 1_{(1)} = \sum 1_{(0)} a \otimes 1_{(1)} \right\}$$

Definition 4^[12] Let A be a weak right H -comodule algebra and B the H -coinvariant subalgebra of A . The extension A/B is said to be weak Hopf-Galois, if the canonical map

$$\beta: A \otimes_B A \rightarrow A \otimes_s H, \quad \beta(a \otimes_B b) = \sum ab_{(0)} \otimes_s b_{(1)}$$

is an isomorphism, where $\otimes_s := \otimes_{H_s}$ is the relative tensor product over H_s (A is a right H_s -module via $y \cdot a = \sum \varepsilon(y a_{(1)}) a_{(0)}$ for all $y \in H_s$, $a \in A$, and H is a left H_s -module via its multiplication).

2 Gorenstein Global Dimensions for Weak Hopf Algebras

For a ring R , we denote the class of projective left R -modules by \mathcal{P} , and for an object M in ${}_R M$, the category of left R -modules, we denote the projective and injective dimension of M by $p.\dim_R M$ and $i.\dim_R M$, respectively.

Definition 5^[13] A left R -module M is Gorenstein pro-

jective if there exists an ${}_R\text{Hom}(-, \mathcal{P})$ -exact sequence

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$$

such that $M = \text{Ker}(P^0 \rightarrow P^1)$, where every P_i and P^i are projective.

Dually, Gorenstein injective left R -modules can be defined.

Definition 6 For a left R -module M , the Gorenstein projective dimension $\text{G. p. dim}_R M$ is at most n if there is an exact sequence

$$0 \rightarrow G_n \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$$

where every G_i is Gorenstein projective.

Dually, the Gorenstein injective dimension $\text{G. i. dim}_R M$ can be defined.

For any ring R , Ref. [14] shows us that

$$\sup \{ \text{G. p. dim}_R M \mid M \in {}_R M \} = \sup \{ \text{G. i. dim}_R M \mid M \in {}_R M \}$$

The common value is called the left Gorenstein global dimension of R and denoted as $\text{G. gl. dim}(R)$.

We know that $\text{G. gl. dim}(R) = 0$ if and only if the ring R is quasi-Frobenius^[15]. In addition, R is left Gorenstein hereditary if every submodule of a projective left R -module is Gorenstein projective^[16], i. e., $\text{G. gl. dim}(R) \leq 1$.

In what follows, we always assume that H is a weak Hopf algebra with a bijective antipode S . Then, the left H -module category ${}_H M$, called the representation category, is an abelian monoidal category^[17-18] such that

- H_1 is the unit object with a left H -action via $h \cdot x = \varepsilon_1(hx)$ for all $h \in H$ and $x \in H_1$;
- For $M, N \in {}_H M$, the tensor product is $M \otimes_1 N := M \otimes_{H_1} N$, where the right H_1 -module structure on M is defined by $m \cdot x = S^{-1}(x) \cdot m$ for all $m \in M$ and $x \in H_1$;
- The left H -module structure on $M \otimes_1 N$ is defined by the following diagnosing action $h \cdot (m \otimes_1 n) = h_1 \cdot m \otimes_1 h_2 \cdot n$, for all $h \in H$, $m \in M$, $n \in N$;
- For $M, N \in {}_H M$, a homomorphism between M and N is left H -linear.

Proposition 1 If P is a Gorenstein projective left H -module, then so is $P \otimes_1 X$ for any left H -module X .

Proof If P is a Gorenstein projective left H -module, then there is an ${}_H\text{Hom}(-, \mathcal{P})$ -exact sequence

$$P : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$$

such that $M = \text{Ker}(P^0 \rightarrow P^1)$, where every P_i and P^i are projective. As H_1 is separable, we can obtain an exact sequence

$$P \otimes_1 X : \cdots \rightarrow P_1 \otimes_1 X \rightarrow P_0 \otimes_1 X \rightarrow P^0 \otimes_1 X \rightarrow P^1 \otimes_1 X \rightarrow \cdots$$

such that $M \otimes_1 X = \text{Ker}(P^0 \otimes_1 X \rightarrow P^1 \otimes_1 X)$. Meanwhile, every $P_i \otimes_1 X$ and $P^i \otimes_1 X$ are projective left H -modules^[19]. For any projective left H -module Q ,

$${}_H\text{Hom}(P \otimes_1 X, Q) \cong {}_{H_1}\text{Hom}(X, {}_H\text{Hom}(P, Q))$$

Hence, ${}_H\text{Hom}(P \otimes_1 X, Q)$ is exact, as desired.

Theorem 1 Let H be a weak Hopf algebra with a bijective antipode S . Then, $\text{G. gl. dim}(H) = \text{G. p. dim}_H H_1$.

Proof Obviously, $\text{G. gl. dim}(H) \geq \text{G. p. dim}_H H_1$. Hence, we shall prove the reverse inequality. Assume that $\text{G. p. dim}_H H_1 = n < +\infty$. Then, there is an exact sequence

$$0 \rightarrow G_n \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow H_1 \rightarrow 0$$

where every G_i is Gorenstein projective. For any left H -module X , we obtain an exact sequence

$$0 \rightarrow G_n \otimes_1 X \rightarrow G_{n-1} \otimes_1 X \rightarrow \cdots \rightarrow G_1 \otimes_1 X \rightarrow G_0 \otimes_1 X \rightarrow H_1 \otimes_1 X \rightarrow 0$$

As $H_1 \otimes_1 X \cong X$, and every $G_i \otimes_1 X$ is Gorenstein projective by Proposition 1, we obtain $\text{G. gl. dim}(H) \leq n$. This shows that $\text{G. gl. dim}(H) \leq \text{G. p. dim}_H H_1$.

3 Gorenstein Global Dimensions for Weak Hopf-Galois Extensions

Throughout this section, H is always assumed to be a weak Hopf algebra with a bijective antipode S , unless otherwise stated. Then, let $l \in \int_H^l$ and $\lambda \in \int_{H^*}^l$ be two non-degenerate left integrals such that $\sum \lambda_1(\lambda_2) = 1_H$, and we fix a weak Hopf-Galois extension A/B and use the following formal notation for the inverse of β for all $h \in H$,

$$\beta^{-1}(1_A \otimes_s h) = \sum_i l_i(h) \otimes_B r_i(h) \in A \otimes_B A$$

such that

$$\sum \sum_i l_i(h) r_i(h)_{(0)} \otimes_s r_i(h)_{(1)} = 1_A \otimes_s h \quad (15)$$

by definition. Such elements enjoy the following properties.

Lemma 1^[20] For all $a \in A$ and $h \in H$, we have

$$\sum_i l_i(h) r_i(h) = \sum \varepsilon(h 1_{(1)}) 1_{(0)}$$

$$\sum \sum_i l_i(h) \otimes_B r_i(h)_{(0)} \otimes_s r_i(h)_{(1)} = \sum \sum_i l_i(h_1) \otimes_B r_i(h_1) \otimes_s h_2$$

$$\sum \sum_i l_i(h)_{(0)} \otimes_B r_i(h) \otimes_s l_i(h)_{(1)} = \sum \sum_i l_i(h_2) \otimes_B r_i(h_2) \otimes_s S(h_1)$$

Consider the two functors:

$$A \otimes_B - : {}_B M \rightarrow {}_A M, \quad N \mapsto A \otimes_B N$$

$${}_B(-) : {}_A M \rightarrow {}_B M, \quad M \mapsto {}_B M$$

where ${}_B(-)$ is the restriction of the scalars functor.

Lemma 2 $(A \otimes_B -, {}_B(-))$ and $({}_B(-), A \otimes_B -)$ are

double adjunctions.

Proof Note that the element $\sum \lambda(a_{(1)})a_{(0)} \in B^{[21]}$ for $a \in A$. Then, for all $N \in {}_B M$, we have a well-defined map $\phi: A \otimes_B N \rightarrow {}_B \text{Hom}(A, N)$ given by

$$\phi(a \otimes_B n)(b) = \sum \lambda(b_{(1)}a_{(1)})b_{(1)}a_{(0)} \cdot n$$

$$a, b \in A, n \in N$$

Then, it is easy to check that ϕ is a morphism of left A -modules, where ${}_B \text{Hom}(A, N) \in {}_A M$ via $(a \cdot f)(b) = f(ba)$ for all $a, b \in A$ and $f \in {}_B \text{Hom}(A, N)$. Meanwhile, we claim that ϕ is a bijection with the inverse given by

$$\varphi: {}_B \text{Hom}(A, N) \rightarrow A \otimes_B N$$

$$\varphi(f) = \sum l_i(S^{-1}(\ell)) \otimes_B f(r_i(S^{-1}(\ell)))$$

Indeed, on the one hand, for all $a, b \in A$ and $n \in N$, by Lemma 1, we have

$$\begin{aligned} \phi\phi(a \otimes_B n) &= \\ \sum \sum l_i(S^{-1}(\ell)) \otimes_B \lambda(r_i(S^{-1}a_{(1)}))r_i(S^{-1}a_{(0)}) \cdot n &= \\ \sum \sum l_i(S^{-1}(\ell))r_i(S^{-1}a_{(0)})\lambda(r_i(S^{-1}a_{(1)})) \otimes_B n &= \\ \sum \sum l_i(S^{-1}(\ell_2))r_i(S^{-1}(1_2))a_{(0)}\lambda(S^{-1}(\ell_1)a_{(1)}) \otimes_B n &= \\ \sum 1_{(0)}a_{(0)}\varepsilon(S^{-1}(\ell_2)1_{(1)})\lambda(S^{-1}(\ell_1)a_{(1)}) \otimes_B n &= \\ \sum 1_{(0)}a_{(0)}\lambda(S^{-1}(\ell)1_{(1)}a_{(1)}) \otimes_B n &= \\ \sum 1_{(0)}a_{(0)}\lambda(S^{-1}(\ell)1_{(1)}a_{(1)}) \otimes_B n &= \end{aligned}$$

On the other hand, for all $b \in A$ and $f \in {}_B \text{Hom}(A, N)$, by Lemma 1, we have

$$\begin{aligned} \phi\phi(f)(b) &= \\ \sum \sum l_i(b_{(1)}l_i(S^{-1}(\ell))_{(1)})b_{(0)}l_i(S^{-1}(\ell))_{(0)} \cdot & \\ f(r_i(S^{-1}(\ell)))n &= \\ \sum \sum l_i(b_{(1)}\ell_2)f(b_{(0)}l_i(S^{-1}(\ell_1))r_i(S^{-1}(\ell_1))) &= \\ \sum \lambda(b_{(1)}\ell_2)\varepsilon(S^{-1}(\ell_1)1_{(1)})f(b_{(0)}1_{(0)}) &= \\ \sum \lambda(\ell_2)\varepsilon(S^{-1}(\ell_1)b_{(1)}1_{(1)})f(b_{(0)}1_{(0)}) &= \\ \sum \lambda(\ell_2)\varepsilon(S^{-1}(\ell_1)b_{(1)})f(b_{(0)})n = f(b) & \end{aligned}$$

Hence, the adjoint isomorphism theorem, $(A \otimes_B -, {}_B(-))$ and $({}_B(-), A \otimes_B -)$ are double adjunctions^[12].

By Lemma 2, we have the following assertion.

Corollary 1 If $P \in {}_B M$ is projective, then $A \otimes_B P \in {}_A M$ is projective.

Lemma 3 If H is semisimple and $P \in {}_A M$, then P is projective as a left A -module if and only if P is projective as a left B -module.

Proof As H is semisimple, we know that H is finite dimensional and there is $\mathcal{J} \in \int_H^f$, which is normalized^[3], i. e., $\varepsilon_s(\mathcal{J}) = 1_H$. Then, for all $a \in A$,

$$\sum_i al_i(\mathcal{J}) \otimes_B r_i(\mathcal{J}) = \sum_i l_i(\mathcal{J}) \otimes_B r_i(\mathcal{J})a \quad (16)$$

In fact, by Lemma 1, we have

$$\begin{aligned} \beta\left(\sum_i al_i(\mathcal{J}) \otimes_B r_i(\mathcal{J})\right) &= \\ \sum \sum_i al_i(\mathcal{J})r_i(\mathcal{J})_{(0)} \otimes_s r_i(\mathcal{J})_{(1)} &= \\ \sum \sum_i al_i(\mathcal{J}_1)r_i(\mathcal{J}_1) \otimes_{\mathcal{J}_2} \mathcal{J}_2 &= \\ \sum a1_{(0)}\varepsilon(\mathcal{J}_11_{(1)}) \otimes_{\mathcal{J}_2} \mathcal{J}_2 &= \\ \sum a1_{(0)} \otimes_s \mathcal{J}_{1(1)} &= \sum a_{(0)} \otimes_s \mathcal{J}\varepsilon_s(a_{(1)}) = \\ \sum a_{(0)} \otimes_s \mathcal{J}a_{(1)} &= \beta\left(\sum_i l_i(\mathcal{J}) \otimes_B r_i(\mathcal{J})a\right) \end{aligned}$$

as needed.

Consider that P is a projective left B -module and let $M, N \in {}_A M$. Suppose that $\delta: M \rightarrow N$ and $\xi: P \rightarrow N$ are left A -module morphisms such that δ is surjective. Since P is projective as a left B -module, there is a left B -module morphism $f: P \rightarrow M$ such that $\delta \circ f = \xi$. Define

$$\tilde{f}: P \rightarrow M, \quad \tilde{f}(p) = \sum_i l_i(\mathcal{J}) \cdot f(r_i(\mathcal{J}) \cdot p)$$

for all $p \in P$.

Obviously, \tilde{f} is a morphism of left A -modules by (16). Moreover,

$$\begin{aligned} \delta\tilde{f}(p) &= \\ \sum_i l_i(\mathcal{J})r_i(\mathcal{J}) \cdot \xi(p) &= \sum \varepsilon(\mathcal{J}_{(1)})1_{(0)} \cdot \xi(p) = \\ \sum \varepsilon(\varepsilon_s(\mathcal{J})1_{(1)})1_{(0)} \cdot \xi(p) &= \sum \varepsilon(1_{(1)})1_{(0)} \cdot \xi(p) = \\ 1_A \cdot \xi(p) &= \xi(p) \end{aligned}$$

where the third equality follows^[8]. Hence, $\delta \circ \tilde{f} = \xi$.

Thus, P is projective as a left A -module.

Conversely, suppose that P is projective as a left A -module. It follows from Lemma 2 that $(A \otimes_B -, {}_B(-))$ is an adjoint pair. Since A is projective as a right B -module^[22], we obtain that the functor $A \otimes_B -$ is exact. Hence, ${}_B(-)$ preserves projective objects. Therefore, P is projective as B -module.

Definition 7^[23] Let \mathcal{C} and \mathcal{D} be categories. A covariant functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is separable if for all objects $M, N \in \mathcal{C}$, there are maps $\varphi_{M,N}^F: \text{Hom}_{\mathcal{D}}(F(M), F(N)) \rightarrow \text{Hom}_{\mathcal{C}}(M, N)$, satisfying the following conditions:

- 1) For $\alpha \in \text{Hom}_{\mathcal{C}}(M, N)$, we have $\varphi_{M,N}^F(F(\alpha)) = \alpha$.
- 2) Given $M', N' \in \mathcal{C}$, $f \in \text{Hom}_{\mathcal{D}}(F(M), F(N))$, $g \in \text{Hom}_{\mathcal{D}}(F(M'), F(N'))$, $\alpha \in \text{Hom}_{\mathcal{C}}(M, M')$, $\beta \in \text{Hom}_{\mathcal{C}}(N, N')$, we have $F(\beta) \circ f = g \circ F(\alpha)$ and $\beta \circ \varphi_{M,N}^F(f) = \varphi_{M',N'}^F(g) \circ \alpha$.

Lemma 4 1) If H is semisimple, then ${}_B(-)$ is separable.

2) If H^* is semisimple, then $A \otimes_B -$ is separable.

Proof 1) Let $\mathcal{J} \in \int_H^r$ such that $\varepsilon_s(\mathcal{J}) = 1_H$. By the proof of Lemma 3, we can know that the element $\sum_i l_i(\mathcal{J}) \otimes_{B^*} r_i(\mathcal{J})$ is exactly the separability idempotent, i. e., the extension A/B is separable. Hence, the restriction of scalars functor ${}_B(-)$ is separable^[23].

2) Note that A is a weak left H^* -module algebra under the action $\phi \rightarrow a = \sum \phi(a_{(1)})a_{(0)}$ for $a \in A$, $\phi \in H^*$ ^[24]. As H^* is semisimple, there exists a normalized left integral $\lambda \in \int_{H^*}^l$. Then, $\lambda \rightarrow 1_A = 1_A$, and the map $F: A \rightarrow B$ defined by $F(a) = \lambda \rightarrow a$ is a B -bimodule projection^[21]. This assumes that B is a direct summand of A as a B -bimodule. Hence, the unit $1_{A \otimes_B (-)} \circ (A \otimes_B -)$ of the adjunction $(A \otimes_B -, {}_B(-))$ is a split monomorphism^[25]. So, the induction function $A \otimes_B -$ is separable^[11].

Corollary 2 1) If H is semisimple, then M is a left A -module direct summand of $A \otimes_B M$ for all $M \in {}_A M$.

2) If H^* is semisimple, then N is a left B -module direct summand of $A \otimes_B N$ for all $N \in {}_B M$.

Proof 1) As H is semisimple, we know from Lemma 4 that ${}_B(-)$ is separable. Hence, the counit $\epsilon: (A \otimes_B -) \circ {}_B(-) \rightarrow 1_{A \otimes_B M}$ of the adjunction $(A \otimes_B -, {}_B(-))$ is a cosplit epimorphism^[11], that is, there is a natural transformation $\bar{\epsilon}: 1_{A \otimes_B M} \rightarrow (A \otimes_B -) \circ {}_B(-)$ such that $\epsilon_M \circ \bar{\epsilon}_M = \text{id}$ for all $M \in {}_A M$. Thus, M is a left A -module direct summand of $A \otimes_B M$.

2) As H^* is semisimple, we have that $A \otimes_B -$ is separable by Lemma 4. Hence, we also know that the counit ${}_B(-) \circ (A \otimes_B -) \rightarrow 1_{A \otimes_B N}$ of the adjunction $({}_B(-), A \otimes_B -)$ is a cosplit epimorphism^[11]. Thus, N is a left B -module direct summand of $A \otimes_B N$ for all $N \in {}_B M$.

Lemma 5 If $N \in {}_B M$ is Gorenstein projective, then $A \otimes_B N \in {}_A M$ is Gorenstein projective.

Proof Suppose that N is a Gorenstein projective left B -module. Then, there is a ${}_B \text{Hom}(-, \mathcal{P})$ -exact sequence

$$\mathcal{P}: \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$$

such that $N = \text{Ker}(P^0 \rightarrow P^1)$, where every P_i and P^i are projective. Since A is projective as a right B -module, we obtain that $A \otimes_B P$ is exact and $A \otimes_B N = \text{Ker}(A \otimes_B P^0 \rightarrow A \otimes_B P^1)$. We also obtain that $A \otimes_B P^i$ is projective for every i because of the assertion that ${}_B(-)$ is exact and $A \otimes_B -$ is a left adjoint^[12].

Let us suppose finally that $Q \in {}_A M$ is Gorenstein projective. Then,

$${}_A \text{Hom}(A \otimes_B P, Q) \cong {}_B \text{Hom}(P, Q)$$

However, ${}_B Q$ is projective since ${}_B(-)$ is a left adjoint of $A \otimes_B -$ which is exact. Thus, ${}_B \text{Hom}(P, Q)$ is exact since N is Gorenstein projective, and so, ${}_A \text{Hom}(A \otimes_B P,$

$Q)$ is also exact, which implies that $A \otimes_B N$ is Gorenstein projective.

Theorem 2 If H is a semisimple and cosemisimple weak Hopf algebra, then $\text{G. gl. dim}(A) = \text{G. gl. dim} B$.

Proof Assume that $\text{G. gl. dim} B = n < +\infty$. For any $M \in {}_A M$, as a left B -module, there is a Gorenstein projective resolution:

$$0 \rightarrow G_n \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$$

where every G_i is Gorenstein projective. Since the functor $A \otimes_B -$ is exact, it induces a left A -module exact sequence

$$0 \rightarrow A \otimes_B G_n \rightarrow A \otimes_B G_{n-1} \rightarrow \cdots \rightarrow A \otimes_B G_1 \rightarrow A \otimes_B G_0 \rightarrow A \otimes_B M \rightarrow 0$$

From Lemma 5, we know that every $A \otimes_B G_i$ is Gorenstein projective, and thus, $\text{G. gl. dim}_A(A \otimes_B M) \leq n$. Since M is a direct summand of $A \otimes_B M$ as left A -modules by Corollary 2, $\text{G. gl. dim}_A M \leq n$ ^[26]. Hence, $\text{G. gl. dim}(A) \leq n$.

Suppose that $\text{G. gl. dim}(A) = q \leq n = \text{G. gl. dim} B$. Then, $\text{G. gl. dim}_B(A \otimes_B N) \leq q$ for any $N \in {}_B M$. Therefore, ${}_B \text{Ext}^i(A \otimes_B N, U) = 0$ for all $i > q$ and all projective left B -module U ^[26]. Since N is a left B -module direct summand of $A \otimes_B N$ and the functor ${}_B \text{Ext}(-, U)$ preserves finite direct sums, ${}_B \text{Ext}(N, U) = 0$ for all $i > q$ and all projective left B -module U , and, hence, $\text{G. gl. dim}_B N \leq q$. This implies that $\text{G. gl. dim}(A) \leq \text{G. gl. dim} B$. Therefore, in view of the above discussion, we have $\text{G. gl. dim}(A) = \text{G. gl. dim} B$.

Corollary 3 Let H be a semisimple and cosemisimple weak Hopf algebra. Then, A is quasi-Frobenius (rep. left Gorenstein hereditary) if and only if so is B .

In particular, if $R \# H$ is a weak smash product of a weak left H -module algebra R (see Ref. [24] for the definitions), then $R \# H$ is quasi-Frobenius (rep. left Gorenstein hereditary) if and only if so is R .

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弱 Hopf-Galois 扩张的 Gorenstein 维数

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摘要:通过讨论弱 Hopf 代数和弱 Hopf-Galois 扩张的 Gorenstein 整体维数,研究了弱 Hopf 代数的表示. 设 H 是带有双射对积的弱 Hopf 代数, A 是弱右 H -余模代数, B 是 A 的余不变子代数. 首先,通过讨论 Gorenstein 投射 H -模在表示范畴中的性质,证明了 H 的 Gorenstein 整体维数等于其左单位子代数的 Gorenstein 投射维数;其次,利用弱 Hopf 代数的积分理论,给出了投射 A -模是投射 B -模的刻画,并讨论了函子 $A \otimes_B -$ 和标量函子的限制 $_B(-)$ 的可分性;最后,研究了 H 是半单且余半单弱 Hopf 代数条件下的弱 Hopf-Galois 扩张的 Gorenstein 整体维数.

关键词:弱 Hopf 代数;积分;弱 Hopf-Galois 扩张;Gorenstein 维数

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