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Fundamentals of quasigroup Hopf group-coalgebras

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Abstract: A large class of algebras (possibly nonassociative) with group-coalgebraic structures, called quasigroup Hopf group-coalgebras, is introduced and studied. Quasigroup Hopf group-coalgebras provide a unifying framework for the classical Hopf algebras and Hopf group-coalgebras as well as Hopf quasigroups. Then, basic results similar to those in Hopf algebras H are proved, such as anti-(co) multiplicativity of the antipode S: $H \rightarrow H$, and $S^2 = id$ if H is commutative or cocommutative.

Key words: Hopf quasigroup; group-coalgebra; quasigroup Hopf group-coalgebra; convolution algebra

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In 2000, Turaev[1] introduced, for group π, the notion of a modular crossed π-category and showed that such a category gives rise to a three-dimensional homotopy quantum field theory with target space $K(\pi, 1)$. Examples of π-categories can be constructed from the so-called Hopf π-coalgebras also introduced in Ref. [1]. The notion of a Hopf π-coalgebra generalizes that of a Hopf algebra. Some mathematicians have contributed to the subject[1-7]. The subject of Hopf π-coalgebras continues to grow in some directions. More recent considerations have led to generalizations of the notion of Hopf π-coalgebras.

The authors in Refs. [8 − 10] proved that, for every Malcev algebra L, there is an algebra $U(L)$ and a monomorphism ι: $L \rightarrow \mathrm{U}(L)$ of L into the commutator algebra $U(L)$ such that the image of L lies into the alternative center of $U(L)$, and $U(L)$ is a universal object with respect to such homomorphisms.

The algebra $U(L)$, in general, is not alternative, but it has a basis of Poincaré-Birkhoff-Witt type over L and inherits some good properties of universal enveloping algebras of Lie algebras. If G is a smooth Moufang loop with Malcev algebra $(L, [\,,\,])$ not of characteristic 2, 3, as tangent spaces of G, then the authors in Ref. [11] showed that the enveloping algebra $U(L)$ is a Moufang Hopf quasigroup with the structure maps Δ: $U(L) \rightarrow U(L) \otimes U(L)$, ε: $U(L) \rightarrow k$ defined by $\Delta(x) \rightarrow x \otimes 1 + 1 \otimes x$ and $\varepsilon(x) = 0$ for all $x \in L$ extended to $U(L)$ as algebra homomorphisms, and S: $U(L) \rightarrow U(L)$ defined by $S(x) = -x$ extended as an antialgebra homomorphism. Let G act on $U(L)$ by Hopf quasigroup endomorphisms induced by the quasigroup conjugation. Let $U(L)^G = \{U(L)_\alpha\}_{\alpha \in G}$, where the algebra $U(L)_\alpha$ is a copy of $U(L)$ for each $\alpha \in G$. Fix an identification isomorphism of algebras i_α: $U(L) \rightarrow U(L)$.

For any $\alpha, \beta \in G$, one defines a comultiplication $\Delta_{\alpha,\beta}$: $U(L)_{\alpha\beta} \rightarrow U(L)_\alpha \otimes U(L)_\beta$ by $\Delta_{\alpha,\beta}(i_{\alpha\beta}(h)) = \sum i_\alpha(h_1) \otimes i_\alpha(h_2)$ for any $h \in U(L)$. The counit ε: $U(L)_1 \rightarrow k$ is defined by $\varepsilon(i_1(h)) = \varepsilon(h)$ for $h \in U(L)$. For any $\alpha \in G$, the antipode S_α: $U(L)_\alpha \rightarrow U(L)_{\alpha^{-1}}$ is given by $S_\alpha(i_\alpha(h)) = i_{\alpha^{-1}}(S(h))$. The constructions above give a so-called quasigroup Hopf G-coalgebra in this paper.

The aim of the present paper is to establish the existence of integrals for a quasigroup Hopf π-coalgebras.

Throughout the paper, we let π be a discrete group (with neutral element 1) and k be a field (although much of what we do is valid over any commutative ring). We use the Sweedler notation to express the coproduct of a coalgebra C as $\Delta(c) = \sum c_1 \otimes c_2$ [12]. We set $k^* = k\{0\}$. All algebras are supposed to be over k and unitary, but not necessarily associative. The tensor product $\otimes = \otimes_k$ is always assumed to be over k. If U and V are k-spaces, $\sigma_{U,V}$: $U \otimes V \rightarrow V \otimes U$ will denote the flip map defined by $\sigma_{U,V}(u \otimes v) = v \otimes u$.

1 Preliminaries

1.1 Group-coalgebras

We recall from Ref. [2] that a π-coalgebra (over k) is a family $C = \{C_\alpha\}_{\alpha \in \pi}$ of k-spaces endowed with a family $\Delta = \{\Delta_{\alpha,\beta}: C_{\alpha\beta} \rightarrow C_\alpha \otimes C_\beta\}_{\alpha,\beta \in \pi}$ of k-linear maps (the comultiplication) and a k-linear map ε: $C_1 \rightarrow k$ (the counit) such that Δ is coassociative in the following sense that, for any $\alpha, \beta, \gamma \in \pi$,

The coassociativity axiom

$$(\Delta_{\alpha,\beta} \otimes id_{C_\gamma})\Delta_{\alpha\beta,\gamma} = (id_{C_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}$$

The counit axiom

$$(id_{C_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} = id_{C_\alpha} = (\varepsilon \otimes id_{C_\alpha})\Delta_{1,\alpha}$$

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Note that $(C_1, \Delta_{1,1}, \varepsilon)$ is the usual coalgebra.

We extend the Sweedler notation for a comultiplication in the following way: for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, we write

$$\Delta_{\alpha,\beta}(c) = \sum c_{(1,\alpha)} \otimes c_{(2,\beta)} \in C_\alpha \otimes C_\beta$$

or shortly, if we leave the summation implicit, $\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}$. The coassociativity axiom gives that, for any $\alpha, \beta, \gamma \in \pi$, and $c \in C_{\alpha\beta\gamma}$,

$$c_{(1,\alpha\beta)(1,\alpha)} \otimes c_{(1,\alpha\beta)(2,\beta)} \otimes c_{(2,\gamma)} = c_{(1,\alpha)} \otimes c_{(2,\beta\gamma)(1,\beta)} \otimes c_{(2,\beta\gamma)(2,\gamma)}$$

The element of $C_\alpha \otimes C_\beta \otimes C_\gamma$ is written as $c_{(1,\alpha)} \otimes c_{(2,\beta)} \otimes c_{(3,\gamma)}$. By iterating the procedure, we define inductively $c_{(1,\alpha_1)} \otimes \ldots \otimes c_{(n,\alpha_n)}$, for any $c \in C_{\alpha_1 \ldots \alpha_n}$.

1.2 Convolution algebras

Let $C = (\{C_\alpha\}, \Delta, \varepsilon)_{\alpha \in \pi}$ be a π-coalgebra and $(A, m, 1_A)$ be an (not necessarily associative) algebra with multiplication m and unit element 1_A. For any $f \in \mathrm{Hom}_k(C_\alpha, A)$ and $g \in \mathrm{Hom}_k(C_\beta, A)$, we define their convolution product by

$$f * g = m(f \otimes g)\Delta_{\alpha,\beta} \in \mathrm{Hom}_k(C_{\alpha\beta}, A)$$

Using the coassociativity axiom and counit axiom, one verifies that the k-space

$$\mathrm{Conv}(C, A) = \bigoplus_{\alpha \in \pi} \mathrm{Hom}_k(C_\alpha, A)$$

endowed with the convolution product $*$ and the unit element $\varepsilon 1_A$, is a not necessarily associative π-graded algebra, called convolution algebra.

In particular, for $A = k$, the associative π-graded algebra $\mathrm{Conv}(C, A) = \bigoplus_{\alpha \in \pi} C_\alpha^*$ is called dual to C and is denoted by C^*.

1.3 Hopf quasigroups

Recall from Definition 4.1 in Ref. [11] that a Hopf quasigroup is a unital algebra H(possibly nonassociative) equipped with algebra homomorphisms $\Delta: H \to H \otimes H$, $\varepsilon: H \to k$ forming a coassociative coalgebra and a map $S: H \to H$ such that

$$m(S \otimes m)(\Delta \otimes id) = m(id \otimes m)(id \otimes S \otimes id)(\Delta \otimes id) = \varepsilon \otimes id$$

$$m(m \otimes id)(id \otimes S \otimes id)(id \otimes \Delta) = m(m \otimes S)(id \otimes \Delta) = id \otimes \varepsilon$$

Remark 1 A Hopf quasigroup is a Hopf algebra iff its product is associative.

2 Quasigroup Hopf Group-Coalgebras

Definition 1 A quasigroup Hopf group-coalgebra over π is a π-coalgebra $H = (\{H_\alpha\}_{\alpha \in \pi}, \Delta = \{\Delta_{\alpha,\beta}: H_{\alpha\beta} \to H_\alpha \otimes H_\beta\}_{\alpha,\beta \in \pi}, \varepsilon)$, endowed with a family $S = \{S_\alpha: H_\alpha \to H_{\alpha^{-1}}\}_{\alpha \in \pi}$ of k-linear maps (the antipode) such that the following conditions hold:

Axiom 1 Each $(H_\alpha, m_\alpha, 1_\alpha)$ is not a necessarily associative algebra with multiplication m_α and unit element $1_\alpha \in H_\alpha$.

Axiom 2 For all $\alpha, \beta \in \pi$, $\Delta_{\alpha,\beta}$ and ε are algebra homomorphisms .

Axiom 3 For all $\alpha \in \pi$,

$$m_\alpha(id_{H_\alpha} \otimes m_\alpha)(S_{\alpha^{-1}} \otimes id_{H_\alpha} \otimes id_{H_\alpha})(\Delta_{\alpha^{-1},\alpha} \otimes id_{H_\alpha}) = \varepsilon \otimes id_{H_\alpha} = m_\alpha(id_{H_\alpha} \otimes m_\alpha)(id_{H_\alpha} \otimes S_{\alpha^{-1}} \otimes id_{H_\alpha})(\Delta_{\alpha,\alpha^{-1}} \otimes id_{H_\alpha})$$

Axiom 4 For all $\alpha \in \pi$,

$$m_\alpha(m_\alpha \otimes id_{H_\alpha})(id_{H_\alpha} \otimes S_{\alpha^{-1}} \otimes id_{H_\alpha})(id_{H_\alpha} \otimes \Delta_{\alpha^{-1},\alpha}) = id_{H_\alpha} \otimes \varepsilon = m_\alpha(m_\alpha \otimes id_{H_\alpha})(id_{H_\alpha} \otimes id_{H_\alpha} \otimes S_{\alpha^{-1}})(id_{H_\alpha} \otimes \Delta_{\alpha,\alpha^{-1}})$$

In this paper, a quasigroup Hopf group-coalgebra over π is called a quasigroup Hopf π-coalgebra. We note that the notion of a quasigroup Hopf π-coalgebra is not self-dual and that $(H_1, m_1, 1_1, \Delta_{1,1}, \varepsilon, S_1)$ is a (classical) Hopf quasigroup. One can easily verify that a quasigroup Hopf π-coalgebra is a Hopf π-coalgebra if and only if its product is associative.

Definition 2 1) A quasigroup Hopf π-coalgebra H is commutative if each m_α is commutative.

2) A quasigroup Hopf π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon)_{\alpha \in \pi}$ is cocommutative if, for any $\alpha \in \pi$, $\Delta_{\alpha,\alpha^{-1}} = \sigma_{H_{\alpha^{-1}}, H_\alpha}\Delta_{\alpha^{-1},\alpha}$, i. e. for any $h \in H_1$, $h_{(1,\alpha)} \otimes h_{(2,\alpha^{-1})} = h_{(2,\alpha)} \otimes h_{(1,\alpha^{-1})}$.

3) A quasigroup Hopf π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon)_{\alpha \in \pi}$ is the first flexible if

$$S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})(gh_{(2,\alpha)}) = (S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})g)h_{(2,\alpha)}$$
$$\forall \alpha \in \pi, \quad h \in H_1, \quad g \in H_\alpha$$

and the second flexible if

$$h_{(1,\alpha)}(gS_{\alpha^{-1}}(h_{(2,\alpha^{-1})})) = (h_{(1,\alpha)}g)S_{\alpha^{-1}}(h_{(2,\alpha^{-1})})$$
$$\forall \alpha \in \pi, \quad h \in H_1, \quad g \in H_\alpha.$$

4) A quasigroup Hopf π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon)_{\alpha \in \pi}$ is the first alternative if, for any $\alpha \in \pi$, $h \in H_1$, $g \in H_\alpha$,

$$S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})(h_{(2,\alpha)}g) = (S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})h_{(2,\alpha)})g$$
$$g(S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})h_{(2,\alpha)}) = (gS_{\alpha^{-1}}(h_{(1,\alpha^{-1})}))h_{(2,\alpha)}$$

and the second alternative if, for any $\alpha \in \pi$, $h \in H_1$, $g \in H_\alpha$,

$$h_{(1.\alpha)}(S_{\alpha^{-1}}(h_{(2,\alpha^{-1})})g) = (h_{(1,\alpha)}S_{\alpha^{-1}}(h_{(2,\alpha^{-1})}))g$$
$$g(h_{(1,\alpha)}S_{\alpha^{-1}}(h_{(2,\alpha^{-1})})) = (gh_{(1,\alpha)})S_{\alpha^{-1}}(h_{(2,\alpha^{-1})})$$

5) A quasigroup Hopf π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon)_{\alpha \in \pi}$ is called the first Moufang if, for any $\alpha \in \pi$, $h \in H_1$, $g, f \in H_\alpha$,

$$S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})(g(h_{(2,\alpha)}f)) = ((S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})g)h_{(2,\alpha)})f$$

and the second Moufang if, for any $\alpha \in \pi$, $h \in H_1$, $g, f \in H_\alpha$,

$$h_{(1,\alpha)}(g(S_{\alpha^{-1}}(h_{(2,\alpha^{-1})})f)) = ((h_{(1,\alpha)}g)S_{\alpha^{-1}}(h_{(2,\alpha^{-1})}))f$$

A quasigroup Hopf π-coalgebra $H = (\{H_{\alpha}\}, \Delta, \varepsilon)_{\alpha \in \pi}$ is said to be of finite type if, for all $\alpha \in \pi$, H_{α} is finite dimensional (over k). Note that it does not mean that $\bigoplus_{\alpha \in \pi} H_{\alpha}$ is finite-dimensional (unless $H_{\alpha} \neq 0$, for all but a finite number of $\alpha \in \pi$).

The antipode $S = \{S_{\alpha}\}_{\alpha \in \pi}$ of H is said to be bijective if each S_{α} is bijective. We will later show that it is bijective whenever H is of finite type[12].

Example 1 Let $(H, m, \Delta, \varepsilon, S)$ be a Hopf quasigroup and the group π act on H by Hopf quasigroup endomorphisms.

1) Set $H^{\pi} = \{H_{\alpha}\}_{\alpha \in \pi}$ where the algebra H_{α} is a copy of H for each $\alpha \in \pi$. Fix an identification isomorphism of algebras $i_{\alpha}: H \to H_{\alpha}$. For $\alpha, \beta \in \pi$, one defines a comultiplication $\Delta_{\alpha,\beta}: H_{\alpha\beta} \to H_{\alpha} \otimes H_{\beta}$ by $\Delta_{\alpha,\beta}(i_{\alpha\beta}(h)) = \sum i_{\alpha}(h_1) \otimes i_{\beta}(h_2)$ for any $h \in H$. The counit $\varepsilon: H_1 \to k$ is defined by $\varepsilon(i_1(h)) = \varepsilon(h)$ for $h \in H$. For any $\alpha \in \pi$, the antipode $S_{\alpha}: H_{\alpha} \to H_{\alpha^{-1}}$ is given by $S_{\alpha}(i_{\alpha}(h)) = i_{\alpha^{-1}}(S(h))$. All the axioms of a quasigroup Hopf π-coalgebra for H^{π} follow directly from definitions. Let $\bar{H}^{\pi}$ be the same family of algebras $\{H_{\alpha} = H\}_{\alpha \in \pi}$ with the same counit, the comultiplication $\bar{\Delta}_{\alpha,\beta}: H_{\alpha\beta} \to H_{\alpha} \otimes H_{\beta}$ and the antipode $\bar{S}_{\alpha}: H_{\alpha} \to H_{\alpha^{-1}}$ defined by $\bar{\Delta}_{\alpha,\beta}(i_{\alpha\beta}(h)) = \sum i_{\alpha}(\beta(h_1)) \otimes i_{\beta}(h_2)$ and $\bar{S}_{\alpha}(i_{\alpha}(h)) = i_{\alpha^{-1}}(\alpha(S(h))) = i_{\alpha^{-1}}(S(\alpha(h))$ where $h \in H$. The axioms of a quasigroup Hopf π-coalgebra for $\bar{H}^{\pi}$ follow from definitions. Both H^{π} and $\bar{H}^{\pi}$ are extensions of H since $H_1^{\pi} = \bar{H}_1^{\pi} = H_1$ as Hopf quasigroups.

2) In particular, if G is a smooth Moufang loop with Malcev algebra $(L, [\,,\,])$ not of characteristic 2, 3, as tangent spaces of G, then the enveloping algebra $U(L)$ in Ref. [12] is a Moufang Hopf quasigroup with the structure maps $\Delta: U(L) \to U(L) \otimes U(L)$, $\varepsilon: U(L) \to k$ defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\varepsilon(x) = 0$ for all $x \in L$ extended to $U(L)$ as algebra homomorphisms, and $S: U(L) \to U(L)$ defined by $S(x) = -x$ extended as an antialgebra homomorphism. Let G act on $U(L)$ by Hopf algebra endomorphisms induced by the quasigroup conjugation. The constructions above give a quasigroup Hopf G-coalgebras $(U(L))^G = \{U(L)_{\alpha}\}_{\alpha \in G}$ and $\overline{(U(L))}^G = \{U(L)_{\alpha}\}_{\alpha \in G}$ where each $U(L)_{\alpha}$ is a copy of $U(L)$ sitting at $\alpha \in G$.

Theorem 1 Let H be a quasigroup Hopf π-coalgebra. Then

① $m_{\alpha}(S_{\alpha^{-1}} \otimes id_{H_{\alpha}})\Delta_{\alpha^{-1},\alpha} = 1_{\alpha}\varepsilon = m_{\alpha}(id_{H_{\alpha}} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}$, $\forall \alpha \in \pi$.

② $S_{\alpha}(ab) = S_{\alpha}(b)S_{\alpha}(a)$, $\forall \alpha \in \pi, a, b \in H_{\alpha}$.

③ $S_{\alpha}(1_{\alpha}) = 1_{\alpha^{-1}}$, $\forall \alpha \in \pi$.

④ $\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta} = \sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_{\alpha} \otimes S_{\beta})\Delta_{\alpha,\beta}$, $\forall \alpha, \beta \in \pi$.

⑤ $\varepsilon S_1 = \varepsilon$.

Proof ① is obtained by applying Axiom 3 in the definition of a quasigroup Hopf π-coalgebra to $h \otimes 1_{\alpha}$, $\forall \alpha \in \pi, h \in H_1$. We now show 4) as follows:
for all $h \in H_1$,

$$\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta} * \Delta_{\beta^{-1},\alpha^{-1}}(h) = m_{\beta^{-1}\otimes\alpha^{-1}}(\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta} \otimes \Delta_{\beta^{-1},\alpha^{-1}})\Delta_{\alpha\beta,\beta^{-1}\alpha^{-1}}(h) = \Delta_{\beta^{-1},\alpha^{-1}}(S_{\alpha\beta}(h_{(1,\alpha\beta)}))\Delta_{\beta^{-1},\alpha^{-1}} \cdot (h_{(2,\beta^{-1}\alpha^{-1})}) = \Delta_{\beta^{-1},\alpha^{-1}}(S_{\alpha\beta}(h_{(1,\alpha\beta)})h_{(2,\beta^{-1}\alpha^{-1})}) = \Delta_{\beta^{-1},\alpha^{-1}}(S_{\alpha\beta} * id_{\beta^{-1}\alpha^{-1}}(h)) = \varepsilon(h)\Delta_{\beta^{-1},\alpha^{-1}}(1_{\beta^{-1}\alpha^{-1}}) = \varepsilon(h)(1_{\beta^{-1}} \otimes 1_{\alpha^{-1}})$$

and also, one has

$$\Delta_{\beta^{-1},\alpha^{-1}} * \sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_{\alpha} \otimes S_{\beta})\Delta_{\alpha,\beta}(h) = m_{\beta^{-1}\otimes\alpha^{-1}}(\Delta_{\beta^{-1},\alpha^{-1}} \otimes \sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_{\alpha} \otimes S_{\beta})\Delta_{\alpha,\beta})\Delta_{\beta^{-1}\alpha^{-1},\alpha\beta}(h) = h_{(1,\beta^{-1})}S_{\beta}(h_{(4,\beta)}) \otimes h_{(2,\alpha^{-1})}S_{\alpha}(h_{(3,\alpha)}) = h_{(1,\beta^{-1})}S_{\beta}(h_{(3,\beta)}) \otimes \varepsilon(h_{(2,1)})1_{\alpha^{-1}} = h_{(1,\beta^{-1})}S_{\beta}(h_{(2,\beta)}) \otimes 1_{\alpha^{-1}} = \varepsilon(h)(1_{\beta^{-1}} \otimes 1_{\alpha^{-1}})$$

Furthermore, for any $h \in H_{\alpha\beta}$,

$$\sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_{\alpha} \otimes S_{\beta})\Delta_{\alpha,\beta}(h) = \varepsilon(1_{\beta^{-1}} \otimes 1_{\alpha^{-1}}) * \sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_{\alpha} \otimes S_{\beta})\Delta_{\alpha,\beta}(h) = (\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta} * \Delta_{\beta^{-1},\alpha^{-1}}) * \sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_{\alpha} \otimes S_{\beta}) \cdot \Delta_{\alpha,\beta}(h) = (S_{\alpha\beta}(h_{(1,\alpha\beta)})_{(1,\beta^{-1})}h_{(2,\beta^{-1})})S_{\beta}(h_{(5,\beta)}) \otimes (S_{\alpha\beta}(h_{(1,\alpha\beta)})_{(2,\alpha^{-1})}h_{(3,\alpha^{-1})})S_{\alpha}(h_{(4,\alpha)}) = (S_{\alpha\beta}(h_{(1,\alpha\beta)})_{(1,\beta^{-1})}h_{(2,\beta^{-1})})S_{\beta}(h_{(4,\beta)}) \otimes (S_{\alpha\beta}(h_{(1,\alpha\beta)})_{(2,\alpha^{-1})}\varepsilon(h_{(3,1)}))\ \text{(by Axiom 4)} = (S_{\alpha\beta}(h_{(1,\alpha\beta)})_{(1,\beta^{-1})}h_{(2,\beta^{-1})})S_{\beta}(h_{(3,\beta)}) \otimes S_{\alpha\beta}(h_{(1,\alpha\beta)})_{(2,\alpha^{-1})} = \varepsilon(h_{(2,1)})(S_{\alpha\beta}(h_{(1,\alpha\beta)})_{(1,\beta^{-1})} \otimes S_{\alpha\beta}(h_{(1,\alpha\beta)})_{(2,\alpha^{-1})}) = \varepsilon(h_{(2,1)})\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta}(h_{(1,\alpha\beta)}) = \Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta}(h)$$

Thus, $\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta} = \sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_{\alpha} \otimes S_{\beta})\Delta_{\alpha,\beta}$, $\forall \alpha, \beta \in \pi$.

To show ②, we observe that

$$S_{\alpha}(ab) = \varepsilon(b_{(1,1)})S_{\alpha}(ab_{(2,\alpha)}) = S_{\alpha}(b_{(1,1)(1,\alpha)})(b_{(1,1)(2,\alpha^{-1})}S_{\alpha}(ab_{(2,\alpha)})) = S_{\alpha}(b_{(1,1)(1,\alpha)})(\varepsilon(a_{(1,1)})(b_{(1,1)(2,\alpha^{-1})})S_{\alpha}((a_{(2,\alpha)})b_{(2,\alpha)})) = S_{\alpha}(b_{(1,1)(1,\alpha)})((S_{\alpha}(a_{(1,1)(1,\alpha)})(a_{(1,1)(2,\alpha^{-1})}b_{(1,1)(2,\alpha^{-1})}))S_{\alpha}(a_{(2,\alpha)}b_{(2,\alpha)})) = S_{\alpha}(b_{(1,\alpha)})((S_{\alpha}(a_{(1,\alpha)})(a_{(2,\alpha^{-1})}b_{(2,\alpha^{-1})}))S_{\alpha}(a_{(3,\alpha)}b_{(3,\alpha)})) = S_{\alpha}(b_{(1,\alpha)})((S_{\alpha}(a_{(1,\alpha)})(a_{(2,1)}b_{(2,1)})_{(1,\alpha^{-1})})S_{\alpha}((a_{(2,1)}b_{(2,1)})_{(2,\alpha)}) = S_{\alpha}(b_{(1,\alpha)})(S_{\alpha}(a_{(1,\alpha)})\varepsilon(a_{(2,1)}b_{(2,1)})) = S_{\alpha}(b_{(1,\alpha)})S_{\alpha}(a_{(1,\alpha)})\varepsilon(a_{(2,1)})\varepsilon(b_{(2,1)}) = S_{\alpha}(b)S_{\alpha}(a)$$

Thus, $S_{\alpha} \circ m_{\alpha} = m_{\alpha} \circ \sigma_{H_{\alpha},H_{\alpha}} \circ (S_{\alpha} \otimes S_{\alpha})$, $\forall \alpha \in \pi, a, b \in H_{\alpha}$.

Finally, it is easy to obtain ③ and ⑤ by ①.

This completes the proof.

Corollary 1 Let $H = (\{H_{\alpha}\}_{\alpha \in \pi}, \Delta, \varepsilon)$ be a quasigroup Hopf π-coalgebra with the antipode $S = \{S_{\alpha}\}_{\alpha \in \pi}$. Then, S_{α} is the unique convolution inverse of $id_{H_{\alpha^{-1}}}$ in the convolution algebra $\mathrm{Conv}(H, H_{\alpha^{-1}})$, for all $\alpha \in \pi$.

Proof Theorem 1① says that S_{α} is a convolution inverse of $id_{H_{\alpha^{-1}}}$ in the convolution algebra $\mathrm{Conv}(H, H_{\alpha^{-1}})$, for all $\alpha \in \pi$.

Fix $\alpha \in \pi$. Let T_{α} be a right convolution inverse of $id_{H_{\alpha^{-1}}}$ in the convolution algebra $\mathrm{Conv}(H, H_{\alpha^{-1}})$. For all $h \in H_{\alpha}$,

$$
\begin{aligned}
S_\alpha(h) = {} & S_\alpha * (id_{H_{\alpha^{-1}}} * T_\alpha)(h) = \\
& S_\alpha(h_{(1,\alpha)})(id_{H_{\alpha^{-1}}} * T_\alpha)(h_{(2,1)}) = \\
& S_\alpha(h_{(1,\alpha)})(h_{(2,\alpha^{-1})}T_\alpha(h_{(3,\alpha)})) = \varepsilon(h_{(1,1)})T_\alpha(h_{(2,\alpha)})) = \\
& T_\alpha(\varepsilon(h_{(1,1)})h_{(2,\alpha)})) = T_\alpha(h)
\end{aligned}
$$

and so $T_\alpha = S_\alpha$.

Fix $\alpha \in \pi$. Let T_α now be a left convolution inverse of $id_{H_{\alpha^{-1}}}$ in the convolution algebra Conv$(H, H_{\alpha^{-1}})$. For all $h \in H_\alpha$, similarly we have,

$$
\begin{aligned}
S_\alpha(h) = {} & (T_\alpha * id_{H_{\alpha^{-1}}}) * S_\alpha(h) = \\
& (T_\alpha * id_{H_{\alpha^{-1}}})(h_{(1,1)})S_\alpha(h_{(2,\alpha)}) = \\
& (T_\alpha(h_{(1,\alpha)})h_{(2,\alpha^{-1})})S_\alpha(h_{(3,\alpha)}) = T_\alpha(h_{(1,\alpha)})\varepsilon(h_{(2,1)}) = \\
& T_\alpha(h_{(1,\alpha)}\varepsilon(h_{(2,1)})) = T_\alpha(h)
\end{aligned}
$$

and thus, $T_\alpha = S_\alpha$. Therefore, S_α is the unique convolution inverse of $id_{H_{\alpha^{-1}}}$ in the convolution algebra Conv$(H, H_{\alpha^{-1}})$, for all $\alpha \in \pi$. This completes the proof.

Corollary 2 Let H be a quasigroup Hopf π-coalgebra with the antipode $S = \{S_\alpha\}_{\alpha \in \pi}$. Then, $\Delta_{\beta^{-1},\alpha^{-1}} S_{\alpha\beta}$ is the unique convolution inverse of $\Delta_{\beta^{-1},\alpha^{-1}}$ in the convolution algebra Conv$(H, H_{\beta^{-1}} \otimes H_{\alpha^{-1}})$, for all $\alpha, \beta \in \pi$.

Proof One can see directly from the proof of Theorem 1(4) that $\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta}$ is a convolution inverse of $\Delta_{\beta^{-1},\alpha^{-1}}$ in the convolution algebra Conv$(H, H_{\beta^{-1}} \otimes H_{\alpha^{-1}})$, for all $\alpha, \beta \in \pi$. Fix $\alpha, \beta \in \pi$. Let $T_{\beta^{-1},\alpha^{-1}}$ be a right convolution inverse of $\Delta_{\beta^{-1},\alpha^{-1}}$ in the convolution algebra Conv$(H, H_{\beta^{-1}} \otimes H_{\alpha^{-1}})$.

We write $T_{\beta^{-1},\alpha^{-1}}(h) := T_{\beta^{-1}}(h)_1 \otimes T_{\alpha^{-1}}(h)_2$. For all $h \in H_{\alpha\beta}$, we have

$$
\begin{aligned}
\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta}(h) = {} & \Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta} * (\Delta_{\beta^{-1},\alpha^{-1}} * T_{\beta^{-1},\alpha^{-1}})(h) = \\
& (S_\beta(h_{(2,\beta)}) \otimes S_\alpha(h_{(1,\alpha)}))((h_{(3,\beta^{-1})} \otimes h_{(4,\alpha^{-1})})T_{\beta^{-1},\alpha^{-1}} \\
& (h_{(5,\alpha\beta)})) = (S_\beta(h_{(2,\beta)}) \otimes \\
& S_\alpha(h_{(1,\alpha)}))((h_{(3,\beta^{-1})} \otimes h_{(4,\alpha^{-1})})(T_{\beta^{-1}}(h_{(5,\alpha\beta)})_1 \otimes \\
& T_{\alpha^{-1}}(h_{(5,\alpha\beta)})_2)) = S_\beta(h_{(2,\beta)})(h_{(3,\beta^{-1})}T_{\beta^{-1}}(h_{(5,\alpha\beta)})_1) \otimes \\
& S_\alpha(h_{(1,\alpha)})(h_{(4,\alpha^{-1})}T_{\alpha^{-1}}(h_{(5,\alpha\beta)})_2) = \\
& \varepsilon(h_{(2,1)})T_{\beta^{-1}}(h_{(4,\alpha\beta)})_1 \otimes S_\alpha(h_{(1,\alpha)})(h_{(3,\alpha^{-1})}T_{\alpha^{-1}}(h_{(4,\alpha\beta)})_2) \\
& (\text{by Axiom 3}) = \\
& T_{\beta^{-1}}(h_{(3,\alpha\beta)})_1 \otimes S_\alpha(h_{(1,\alpha)})(h_{(2,\alpha^{-1})}T_{\alpha^{-1}}(h_{(3,\alpha\beta)})_2) = \\
& T_{\beta^{-1}}(h_{(2,\alpha\beta)})_1 \otimes \varepsilon(h_{(1,1)})T_{\alpha^{-1}}(h_{(2,\alpha\beta)})_2 = \\
& \varepsilon(h_{(1,1)})T_{\beta^{-1},\alpha^{-1}}(h_{(2,\alpha\beta)}) = T_{\beta^{-1},\alpha^{-1}}(h)
\end{aligned}
$$

which implies that $\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta} = T_{\beta^{-1},\alpha^{-1}}$. Fix $\alpha, \beta \in \pi$. Let $T_{\beta^{-1},\alpha^{-1}}$ be a left convolution inverse of $\Delta_{\beta^{-1},\alpha^{-1}}$ in the convolution algebra Conv$(H, H_{\beta^{-1}} \otimes H_{\alpha^{-1}})$. In a similar manner, we can deduce by Axiom 4 that $\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta} = T_{\beta^{-1},\alpha^{-1}}$. $\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta}$ is thus the unique convolution inverse of $\Delta_{\beta^{-1},\alpha^{-1}}$ in the convolution algebra Conv$(H, H_{\beta^{-1}} \otimes H_{\alpha^{-1}})$, for all $\alpha, \beta \in \pi$. This completes the proof.

Corollary 3 According to Theorem 1①, each quasigroup Hopf π-coalgebra is both the first alternative and the second alternative.

Corollary 4 Let $H = \{H_\alpha\}_{\alpha \in \pi}$ be a quasigroup Hopf π-coalgebra. Then, $\{\alpha \in \pi \mid H_\alpha \neq 0\}$ is a subgroup of π.

Proof Set $G = \{\alpha \in \pi \mid H_\alpha \neq 0\}$. Since $\varepsilon(1_1) = 1_k \neq 0$, we first have $1_1 \neq 0$, i.e. $H_1 \neq 0$, and so $1 \in G$. Then, let $\alpha, \beta \in G$. Using Axiom 2, one can also see that $\Delta_{\alpha,\beta}(1_{\alpha\beta}) = 1_\alpha \otimes 1_\beta \neq 0$. Then, $1_{\alpha\beta} \neq 0$ and so $\alpha\beta \in G$. Finally, let $\alpha \in G$. By Theorem 1③, $S_{\alpha^{-1}}(1_{\alpha^{-1}}) = 1_\alpha \neq 0$. Thus, $1_{\alpha^{-1}} \neq 0$ and hence, $\alpha^{-1} \in G$. This completes the proof.

Theorem 2 Let H be a quasigroup Hopf π-coalgebra. Then, for any $\alpha \in \pi$, $S_{\alpha^{-1}}S_\alpha = id_{H_\alpha}$ if H is commutative or cocommutative.

Proof For any $\alpha \in \pi$. Let $h \in H_\alpha$. If H is commutative, we have

$$
\begin{aligned}
S_{\alpha^{-1}}S_\alpha(h) = {} & S_{\alpha^{-1}}S_\alpha(h_{(1,\alpha)}\varepsilon(h_{(2,1)})) = \\
& S_{\alpha^{-1}}S_\alpha(h_{(1,\alpha)})(S_{\alpha^{-1}}(h_{(2,\alpha^{-1})})h_{(3,\alpha)}) = \\
& S_{\alpha^{-1}}(S_\alpha(h_{(1,1)(1,\alpha)}))(S_{\alpha^{-1}}(h_{(1,1)(2,\alpha^{-1})})h_{(2,\alpha)}) = \\
& S_{\alpha^{-1}}(S_1(h_{(1,1)})_{(2,\alpha^{-1})})(S_1(h_{(1,1)})_{(1,\alpha)}h_{(2,\alpha)}) = \\
& (h_{(2,\alpha)}S_1(h_{(1,1)})_{(1,\alpha)})S_{\alpha^{-1}}(S_1(h_{(1,1)})_{(2,\alpha^{-1})}) = \\
& \varepsilon(S_1(h_{(1,1)}))h_{(2,\alpha)} = \varepsilon(h_{(1,1)})h_{(2,\alpha)} = h
\end{aligned}
$$

It follows that $S_{\alpha^{-1}}S_\alpha = id_{H_\alpha}$.

If H is cocommutative, we find that

$$
\begin{aligned}
S_{\alpha^{-1}}S_\alpha(h) = {} & S_{\alpha^{-1}}S_\alpha(h_{(1,\alpha)}\varepsilon(h_{(2,1)})) = \\
& S_{\alpha^{-1}}S_\alpha(h_{(1,\alpha)})\varepsilon(h_{(2,1)}) = \\
& S_{\alpha^{-1}}S_\alpha(h_{(1,\alpha)})(S_{\alpha^{-1}}(h_{(2,\alpha^{-1})})h_{(3,\alpha)}) = \\
& S_{\alpha^{-1}}(S_\alpha(h_{(1,1)(1,\alpha)}))(S_{\alpha^{-1}}(h_{(1,1)(2,\alpha^{-1})})h_{(2,\alpha)}) = \\
& S_{\alpha^{-1}}(S_1(h_{(1,1)})_{(2,\alpha^{-1})})(S_1(h_{(1,1)})_{(1,\alpha)}h_{(2,\alpha)}) = \\
& S_{\alpha^{-1}}(S_1(h_{(1,1)})_{(1,\alpha^{-1})})(S_1(h_{(1,1)})_{(2,\alpha)}h_{(2,\alpha)}) = \\
& \varepsilon(S_1(h_{(1,1)}))h_{(2,\alpha)} = \varepsilon(h_{(1,1)})h_{(2,\alpha)} = h
\end{aligned}
$$

It also follows that $S_{\alpha^{-1}}S_\alpha = id_{H_\alpha}$. This completes the proof.

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拟群 Hopf 群余代数基本原理

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摘要:引入并研究一大类具有群余代数结构的(可能非结合的)代数族,称之为拟群 Hopf 群余代数. 拟群 Hopf 群余代数为经典 Hopf 代数和 Hopf 群余代数以及 Hopf 拟群提供了统一的框架. 接着,证明拟群 Hopf 群余代数中类似于 Hopf 代数理论中的基本结果. 例如,Hopf 代数理论中对极 $S:H\to H$ 的反(余)乘法性;如果 H 是交换的或余交换的,$S^2=id$.

关键词: Hopf 拟群;群余代数;拟群 Hopf 群余代数;卷积代数

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