

Crossed products for Hopf group-algebras

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Abstract: First, the group crossed product over the Hopf group-algebras is defined, and the necessary and sufficient conditions for the group crossed product to be a group algebra are given. The cleft extension theory of the Hopf group algebra is introduced, and it is proved that the crossed product of the Hopf group algebra is equivalent to the cleft extension. The necessary and sufficient conditions for the crossed product equivalence of two Hopf groups are then given. Finally, combined with the equivalence theory of the Hopf group crossed product and cleft extension, the group crossed product constructed by the general 2-cocycle as algebra is determined to be isomorphic to the group crossed product of the 2-cocycle with a convolutional invertible map of the 2-cocycle. The unit property of a general 2-cocycle is equivalent to the convolutional invertible map of the 2-cocycle, and the combination condition of the weak action is equivalent to the convolutional invertible map of the 2-cocycle and the combination condition of the weak action. Similarly, crossed product algebra constructed by the general 2-cocycle is isomorphic to the Hopf π -crossed product algebra constructed by the 2-cocycle with a convolutional invertible map.

Key words: Hopf π -algebra; cleft extension theorem; π -comodule-like algebra; group crossed products

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Hopf crossed products were introduced independently by Yukio et al.^[1] and Blattner et al.^[2] as a Hopf algebraic generalization of group crossed products. In particular, a Hopf crossed product is, in fact, always a Hopf cleft extension, provided that the cocycle that appeared in a Hopf crossed product is convolution-invertible^[3-5].

Hopf group-algebras were related to homotopy quantum field theories, which are generalizations of ordinary topological quantum field theories^[3,6-8]. In 2007, Wang et al.^[9-11] introduced group smash products of Hopf group-

algebras. Group crossed products of Hopf group-coalgebras were introduced^[12-13]. Other related works can be found in Refs. [14 – 17].

In this article, we introduce and study the notions of a group crossed product and a group cleft extension. We then characterize group crossed products by the group cleft extension. Finally, we prove the equivalences of the group crossed products for the Hopf group-algebras.

1 Group Cleft Extensions and Existence of Group Crossed Products

Definition 1 Let $A = (\{A_\alpha\}_{\alpha \in \pi}, \Delta, \varepsilon)$ be a Hopf π -algebra with the bijective antipode S and J as algebra. We say that A acts weakly on J if there exists a family of maps:

- $a \otimes x \mapsto a \rightharpoonup_\alpha x, \forall \alpha \in \pi, a \in A_\alpha, x \in J, \text{ such that}$
- 1) $1_\alpha \rightharpoonup_\alpha x = x, \forall x \in J; \alpha \in \pi;$
- 2) $a \rightharpoonup_\alpha (xy) = (a_{(1, \alpha)} \rightharpoonup_\alpha x)(a_{(2, \alpha)} \rightharpoonup_\alpha y), \forall a \in A_\alpha, x, y \in J;$
- 3) $a \rightharpoonup_\alpha 1_J = \varepsilon_\alpha(a) 1_J, \forall x \in J.$

Furthermore, if J is an A_α module for each $\alpha \in \pi$ and satisfies 2) and 3), we call J a left π - A -module-like algebra.

Definition 2 Let $A = (\{A_\alpha\}_{\alpha \in \pi}, \Delta, \varepsilon)$ be a Hopf π -algebra and J a left π - A -module-like algebra. Let $\chi_{\alpha, \beta}: A_\alpha \# A_\beta \rightarrow J$ be a family of k -linear maps and suppose that χ is an invertible map. Suppose that J acts weakly on each A_α with $\alpha \in \pi$. For any $\alpha \in \pi$, there is a π -crossed product $J \#_\chi A_\alpha$ with the multiplication given by $(x \#_\chi a)(y \#_\chi b) = x(a_{(1, \alpha)} \rightharpoonup_\alpha y) \chi_{\alpha, \beta}(a_{(2, \alpha)}, b_{(1, \beta)}) \#_{\alpha\beta} a_{(3, \alpha)} b_{(2, \beta)}$, for all $a, b \in A_\alpha, A_\beta, x, y \in J, \alpha, \beta \in \pi$, and the unit is $1_J 1_\alpha$.

Proposition 1 With the above notations, $J \#_\chi A_\alpha$ is a Hopf π -crossed product if and only if the following conditions hold: $\forall a \in A_\alpha, b \in A_\beta, c \in A_\gamma, \forall \alpha, \beta \in \pi$, and $x, y \in J$.

$$\begin{aligned} \chi_{\alpha, \beta}(a, 1_\beta) &= \chi_{\alpha, \gamma}(a, 1_\gamma) = \varepsilon_\alpha(a) 1_J & (1) \\ \chi_{\alpha, \beta}(a_{(1, \alpha)}, b_{(1, \beta)}) \chi_{\alpha\beta, \gamma}(a_{(2, \alpha)} b_{(2, \beta)}, c) &= \\ (a_{(1, \alpha)} \rightharpoonup_\alpha \chi_{\beta, \gamma}(b_{(1, \beta)}, c_{(1, \gamma)})) \chi_{\alpha, \beta\gamma}(a_{(2, \alpha)}, b_{(2, \beta)} c_{(2, \gamma)}) & (2) \\ \chi_{\alpha, \beta}(a_{(1, \alpha)}, b_{(1, \beta)}) (a_{(2, \alpha)} b_{(2, \beta)} \rightharpoonup_{\alpha\beta} x) &= \\ a_{(1, \alpha)} \rightharpoonup_\alpha (b_{(1, \beta)} \rightharpoonup_\beta x) \chi_{\alpha, \beta}(a_{(2, \alpha)}, b_{(2, \beta)}) & (3) \end{aligned}$$

Proposition 2 If $J \#_\chi^\pi A = \{J \#_\chi^\alpha A_\alpha\}_{\alpha \in \pi}$ is a family of ordinary Hopf crossed algebras, then $J \#_\chi^\pi A_\alpha$ is a Hopf π -

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crossed product and $J\#_{\chi}^{\pi}A = \{J\#_{\chi}^{\alpha}A_{\alpha}\}_{\alpha \in \pi}$ is a family of associative algebras.

Remark 1 1) If $\pi = 1$, the Hopf π -crossed product is then the ordinary Hopf crossed product.

2) If we take $\chi_{\alpha,\beta}(a, b) = \varepsilon_{\alpha}(a) \varepsilon_{\beta}(b) 1_J$, $\forall \alpha, \beta \in \pi$, $a \in A_{\alpha}$, $b \in A_{\beta}$, the Hopf π -crossed product becomes the Hopf π -smash product.

Let A be a Hopf algebra. For any $\alpha \in \pi$, denote δ_{α} as the one-dimensional linear space generated by α . Then we have a Hopf group algebra $H = \{H_{\alpha} = A \otimes \delta_{\alpha}\}_{\alpha \in \pi}$ with the structure $(a \otimes \alpha)_{(1,\alpha)} \otimes (a \otimes \alpha)_{(2,\alpha)} = a_1 \otimes \alpha \otimes a_2 \otimes \alpha$, $\varepsilon_{\alpha}(a \otimes \alpha) = \varepsilon(a)$, $S_{\alpha}(a \otimes \alpha) = S(a) \otimes \alpha^{-1}$.

If $J\#_{\sigma}A$ is a crossed product with $\sigma: A \otimes A \rightarrow J$. Define $\chi_{\alpha}: H_{\alpha} \otimes H_{\beta} \rightarrow A$, $\chi_{\alpha}(a \otimes \alpha, b \otimes \beta) = \sigma(a, b)$ for all $x \in J$, we then have the crossed product $J\#_{\chi}^{\pi}H$.

Definition 3 Let J be a left π - A_{α} -module-like algebra.

1) We say that $H \subset J$ is a π - A_{α} -extension if J is a right π - A_{α} -comodule algebra with a family of k -linear maps $\rho = \{\rho_{\alpha}: J \rightarrow J \otimes A_{\alpha}\}$,

$J^{\text{co}A_{\alpha}} = \{x \in J \mid \rho_{\alpha}(x) = x \otimes 1_{A_{\alpha}} \in J \otimes A_{\alpha}, \exists x \in J, \forall \alpha \in \pi\}$ which is called a π -subalgebra of the right π -co-invariants.

2) A π - A_{α} -extension $H \subset J$ is a π - A_{α} -cleft if there exists a family of right π - A_{α} -comodule maps $\gamma = \{\gamma_{\alpha}: A_{\alpha} \rightarrow J\}_{\alpha \in \pi}$ such that γ is convolution-invertible in the sense that there exists a family of maps $\gamma^{-1} = \{\gamma_{\alpha}^{-1}: A_{\alpha} \rightarrow J\}_{\alpha \in \pi}$ satisfying

$$\gamma_{\alpha}(a_{(1,\alpha)}) \gamma_{\alpha}^{-1}(a_{(2,\alpha^{-1})}) = \gamma_{\alpha}^{-1}(a_{(1,\alpha^{-1})}) \gamma_{\alpha}(a_{(2,\alpha)}) = \varepsilon_{\alpha}(a) 1_J \quad \forall a \in A_{\alpha}, \alpha \in \pi$$

Lemma 1 Let $H \subset J$ be a π - A_{α} -cleft extension with a right π - A_{α} -comodule structure map: $\rho = \{\rho_{\alpha}: J \rightarrow J \otimes A_{\alpha}\}$ via $x \mapsto x_{(0,0)} \otimes x_{(0,\alpha)}$ for $\alpha \in \pi$ and a π - A_{α} -cleft structure map: $\gamma = \{\gamma_{\alpha}: A_{\alpha} \rightarrow J\}_{\alpha \in \pi}$ such that $\gamma_{\alpha}(1_{A_{\alpha}}) = 1_J$ with We then have

$$(L1) \quad \rho_{\alpha} \circ \gamma_{\alpha}^{-1} = (\gamma_{\alpha}^{-1} \otimes S_{\alpha^{-1}}) \circ T^{\circ} \Delta_{\alpha^{-1}};$$

$$(L2) \quad x_{(0,\alpha)} \gamma_{\alpha}^{-1}(x_{(1,\alpha^{-1})}) \in H = J^{\text{co}A_{\alpha}} \text{ for any } x \in J.$$

Proof First, observe that since ρ is an algebra map, $\rho_{\alpha} \circ \gamma_{\alpha}^{-1}$ is the inverse of $\rho_{\alpha} \circ \gamma_{\alpha} = (\gamma_{\alpha} \otimes id) \circ \Delta_{\alpha}$. Let $\theta = (\gamma_{\alpha}^{-1} \otimes S_{\alpha^{-1}}) \circ T^{\circ} \Delta_{\alpha^{-1}}$, $\forall x \in J$. Then, $[(\rho_{\alpha} \circ \gamma_{\alpha}) * \theta](x) = 1_J \otimes 1_{A_{\alpha}}$. Thus, θ is a right inverse of $\rho_{\alpha} \circ \gamma_{\alpha}$, and so, $\theta = \rho_{\alpha} \circ \gamma_{\alpha}^{-1}$ by the uniqueness of the inverse.

As for (L2), we compute $\rho_{\alpha}(x_{(0,0)} \gamma_{\alpha}^{-1}(x_{(1,\alpha^{-1})})) = x_{(0,0)} \gamma_{\alpha}^{-1}(x_{(1,\alpha^{-1})}) \otimes 1_{A_{\alpha}}$. This finishes the proof.

Proposition 3 Let $H \subset J$ be a π - A_{α} -cleft via $\gamma = \{\gamma_{\alpha}: A_{\alpha} \rightarrow J\}_{\alpha \in \pi}$ such that $\gamma_{\alpha}(1_{A_{\alpha}}) = 1_J$ with $\alpha \in \pi$. Then, there is a Hopf π -crossed product with a weak action of A_{α} on J given by

$$a \rightharpoonup_{\alpha} x = \gamma_{\alpha}(a_{(1,\alpha)}) x \gamma_{\alpha}^{-1}(a_{(2,\alpha^{-1})}) \quad \forall a \in A_{\alpha}, x \in J$$

and a family of convolution-invertible maps $\chi = \{\chi_{\alpha,\beta}: A_{\alpha} \otimes A_{\beta} \rightarrow J\}_{\alpha,\beta \in \pi}$ given by

$$\chi_{\alpha,\beta}(a, b) = \gamma_{\alpha}(a_{(1,\alpha)}) \gamma_{\beta}(b_{(1,\beta)}) \gamma_{\alpha\beta}^{-1}(a_{(2,\alpha^{-1})} b_{(2,\beta^{-1})}) \quad \forall a \in A_{\alpha}, \forall b \in A_{\beta}$$

Furthermore, there is an algebra isomorphism $\Phi_{\alpha}: J\#_{\chi}^{\alpha}A_{\alpha} \rightarrow H$ given by $x \otimes a \mapsto x \gamma_{\alpha}(a)$ with $\alpha \in \pi$ such that $\Phi = \{\Phi_{\alpha}\}_{\alpha \in \pi}$ is both a left π - J -module and a right π - A_{α} -comodule map, where the right π - A_{α} -comodule structure map of $J\#_{\chi}^{\alpha}A_{\alpha}$ is given by $x\#a \rightarrow x\#a_{(1,\alpha)} \otimes a_{(2,\alpha)}$.

Proof First, we compute for $x \in J$, $a \in A_{\alpha}$,

$$\rho_{\alpha}(a \rightharpoonup_{\alpha} x) = [(\gamma_{\alpha}(a_{(1,\alpha)}) \otimes a_{(2,\alpha)})(x \otimes 1_{A_{\alpha}})](\gamma_{\alpha}^{-1}(a_{(4,\alpha^{-1})}) \otimes S_{\alpha^{-1}}(a_{(3,\alpha^{-1})})) = a \rightharpoonup_{\alpha} x \otimes 1_{A_{\alpha}} \in J \otimes A_{\alpha}$$

and thus, $a \rightharpoonup_{\alpha} x \in H = J^{\text{co}A_{\alpha}}$. Furthermore, it is easy to see that Definition 1 2) and 3) hold.

Similarly, we can prove that $\chi = \{\chi_{\alpha}\}_{\alpha \in \pi}$ has values in A . In fact, $\forall a, b \in A_{\alpha}, A_{\beta}$,

$$\rho_{\alpha}(\chi_{\alpha,\beta}(a, b)) = \rho_{\alpha} \gamma_{\alpha}(a_{(1,\alpha)}) \rho_{\alpha} \gamma_{\beta}(b_{(1,\beta)}) \cdot \rho_{\alpha} \gamma_{\alpha\beta}^{-1}(a_{(2,\alpha^{-1})} b_{(2,\beta^{-1})}) = \chi_{\alpha,\beta}(a, b) \otimes 1_{A_{\alpha\beta}}$$

Now, for $\alpha \in \pi$, we define $\Psi_{\alpha}: H \rightarrow J\#_{\chi}^{\alpha}A_{\alpha}$ by $h \mapsto h_{(0,0)} \gamma_{\alpha}^{-1}(h_{(1,\alpha^{-1})}) \# h_{(2,\alpha)}$.

It is easy to show that Ψ_{α} is the inverse of Φ_{α} with $\alpha \in \pi$. Furthermore, Φ is an algebra-like map: $\Phi_{\alpha}(x\#a) \Phi_{\beta}(y\#b) = \Phi_{\alpha\beta}((x\#a)(y\#b))$. Therefore, we have $H \cong \{J\#_{\chi}^{\alpha}\}_{\alpha \in \pi}$.

Finally, it is easy to check that $\Phi = \{\Phi_{\alpha}\}_{\alpha \in \pi}$ is a left π - J -module-like map and is a right π - A_{α} -comodule map.

Proposition 4 Let $J\#_{\chi}^{\pi}A_{\alpha} = \{J\#_{\chi}^{\alpha}A_{\alpha}\}_{\alpha \in \pi}$ be a Hopf π -crossed product and define $\gamma = \{\gamma_{\alpha}: A_{\alpha} \rightarrow J\#A_{\alpha}\}_{\alpha \in \pi}$ by $\gamma_{\alpha}(a) = 1_J \# a$. Then $\gamma = \{\gamma_{\alpha}\}_{\alpha \in \pi}$ is a family of convolution invertible maps with the inverse $\gamma_{\alpha}^{-1}(a) = \chi_{\alpha}^{-1}(S(a_{(2,\alpha)}), a_{(3,\alpha)}) \# S_{\alpha^{-1}}(a_{(1,\alpha^{-1})})$.

In particular, $J \subset J\#_{\chi}^{\pi}A_{\alpha} = \{J \subset J\#_{\chi}^{\alpha}A_{\alpha}\}_{\alpha \in \pi}$ is π - A_{α} -cleft.

Proof Let $\nu_{\alpha^{-1}}(a) = \chi_{\alpha\alpha^{-1}}^{-1}(S_{\alpha}(a_{(2,\alpha)}), a_{(3,\alpha^{-1})}) \# S_{\alpha^{-1}}(a_{(1,\alpha^{-1})})$. It is then straightforward to verify that ν is a left inverse for γ , and now we have $\nu_{\alpha^{-1}}(a_{(1,\alpha^{-1})}) \gamma_{\alpha}^{-1}(a_{(2,\alpha^{-1})}) = \varepsilon_{\alpha^{-1}}(a) 1_J \# 1_{A_{\alpha}}$.

To check that ν is a right inverse for γ is more complicated. By a computation similar to the above, we have

$$\gamma_{\alpha}(a_{(1,\alpha)}) \nu_{\alpha^{-1}}(a_{(2,\alpha^{-1})}) = [a_{(1,\alpha)} \rightharpoonup_{\alpha} \chi_{\alpha\alpha}^{-1}(S(a_{(4,\alpha)}), a_{(5,\alpha)})] \chi_{\alpha}(\alpha_{(2,\alpha)}, S(a_{(3,\alpha)})) \# 1_{A_{\alpha}} \quad (4)$$

and hence, ν is a right inverse for γ if and only if

$$[a_{(1,\alpha)} \rightharpoonup_{\alpha} \chi_{\alpha\alpha}^{-1}(S(a_{(4,\alpha)}), a_{(5,\alpha)})] \chi_{\alpha}(\alpha_{(2,\alpha)}, S(a_{(3,\alpha)})) = \varepsilon(a) 1_J \quad (5)$$

Since $\chi = \{\chi_{\alpha,\beta}: A_{\alpha} \otimes A_{\beta} \rightarrow J\}$ is invertible, Eq. (2) gives

$$\chi_{\alpha,\beta}(a_{(1,\alpha)}, b_{(1,\beta)}) \chi_{\alpha\beta,\gamma}(a_{(2,\alpha)} b_{(2,\beta)}, c) \chi_{\alpha,\beta\gamma}^{-1}(a_{(3,\alpha)}, b_{(3,\beta)} c_{(3,\gamma)}) = a \rightharpoonup_{\alpha} \chi_{\alpha\beta,\gamma}(b, c) \quad (6)$$

for any $a \in A_{\alpha}$, $b \in A_{\beta}$, $c \in A_{\gamma}$.

Let $a \in A_\alpha$ act on the identity $\chi_{\alpha,\beta}(a_{(1,\alpha)}, b_{(1,\beta)})\chi_{\alpha,\beta}^{-1}(a_{(2,\alpha)}, b_{(2,\beta)}) = 1_J$. We have

$$(a_{(1,\alpha)} \rightarrow \chi_{\alpha,\beta}(b_{(1,\beta)}, c_{(1,\gamma)}))(a_{(2,\alpha)} \rightarrow \chi_{\alpha,\beta}^{-1}(b_{(2,\beta)}, c_{(2,\gamma)})) = \varepsilon_\alpha(a)\varepsilon_\beta(b)\varepsilon_\gamma(c)1_J \quad (7)$$

Hence, from Eq. (7), we obtain

$$a \rightarrow \chi_{\alpha,\beta}^{-1}(b, c) = \chi_{\alpha,\beta\gamma}(a_{(1,\alpha)}, b_{(1,\beta)}c_{(1,\gamma)})\chi_{\alpha\beta,\gamma}^{-1}(a_{(2,\alpha)}b_{(2,\beta)}, c_{(2,\gamma)})\chi_{\alpha,\beta}^{-1}(a_{(3,\alpha)}, b_{(3,\beta)}) \quad (8)$$

We may now verify Eq. (6) using Eq. (8):

$$[a_{(1,\alpha)} \rightarrow \chi_{\alpha,\alpha}^{-1}(S(a_{(4,\alpha)}), a_{(5,\alpha)})]\chi_\alpha(\alpha_{(2,\alpha)}, S(a_{(3,\alpha)})) = \varepsilon_\alpha(a)1_J$$

By Proposition 3 and Proposition 4, we can now get the main result of this section as follows.

Theorem 1 With the above notations, a π - A_α -extension $H \subset J = \{H_\alpha \subset J_\alpha\}_{\alpha \in \pi}$ is a π - A_α -cleft if and only if $\{H_\alpha \cong J_\alpha \#_{\pi}^\alpha A_\alpha\}_{\alpha \in \pi}$.

2 Equivalences of Group Crossed Products

In this section, we will study the equivalences of group crossed products in the setting of Hopf group-coalgebras. Let J be a Hopf π -algebra, A_α a family of coalgebras $A = \{A_\alpha, m_\alpha, 1_{A_\alpha}\}_{\alpha \in \pi}$ over k , and $\gamma = \{\gamma_\alpha: A_\alpha \rightarrow J\}_{\alpha \in \pi}$ a family of convolution-invertible linear maps. Define $\chi^{\gamma_\alpha} = \{\chi_\alpha^{\gamma_\alpha}: A_\alpha \otimes A_\alpha \rightarrow J\}_{\alpha \in \pi}$ and the weak action of A_α on J by $\chi_{\alpha,\beta}^{\gamma_\alpha}(a, b) = \gamma_\alpha(a_{(1,\alpha)})(a_{(2,\alpha)} \rightarrow \gamma_\beta(b_{(1,\beta)}))\chi_{\alpha,\beta}(a_{(3,\alpha)}, b_{(2,\beta)})\gamma_{\alpha^{-1},\beta^{-1}}^{-1}(a_{(4,\alpha^{-1})}, b_{(3,\beta^{-1})})$ and $a \rightarrow x = \gamma_\alpha(a_{(1,\alpha)})(a_{(2,\alpha)} \rightarrow x)\gamma_{\alpha^{-1}}^{-1}(a_{(3,\alpha^{-1})})$ for any $x \in J$ and $\forall a, b \in A_\alpha, A_\beta$.

Lemma 2 Let $J \#_{\chi}^\alpha A_\alpha = \{J \#_{\chi}^\alpha A_\alpha\}_{\alpha \in \pi}$ be a Hopf π -crossed algebra. Then, $\chi^{\gamma_\alpha} = (\chi^{\gamma_\alpha})\mu_\alpha$ and $\rightarrow^{\gamma_\alpha} = (\rightarrow^{\gamma_\alpha})\mu_\alpha$ where $\gamma = \{\gamma_\alpha: A_\alpha \rightarrow J\}_{\alpha \in \pi}$ and $\mu = \{\mu_\alpha: A_\alpha \rightarrow J\}_{\alpha \in \pi}$ are a family of convolution-invertible linear maps.

The proof is clear.

Theorem 2 Let J be a Hopf π -algebra, A_α a family of coalgebras $A = \{A_\alpha, m_\alpha, 1_{A_\alpha}\}_{\alpha \in \pi}$, and $\gamma = \{\gamma_\alpha: A_\alpha \rightarrow J\}_{\alpha \in \pi}$ a family of convolution-invertible linear maps. If $\chi = \{\chi_\alpha: A_\alpha \otimes A_\alpha \rightarrow J\}_{\alpha \in \pi}$ is a family of k -linear maps, we then have the following assertions with the above notations χ^{γ_α} for any $\alpha, \beta \in \pi$:

- 1) As algebras, $J \#_{\chi}^\alpha A_\alpha \cong J \#_{\chi^{\gamma_\alpha}}^\alpha A_\alpha$;
- 2) χ satisfies Eq. (1) if and only if χ^{γ_α} satisfies Eq. (1);
- 3) (χ, \rightarrow) satisfies Eq. (2) if and only if $(\chi^{\gamma_\alpha}, \rightarrow^{\gamma_\alpha})$ satisfies Eq. (2);

4) If (χ, \rightarrow) satisfies Eq. (2), (χ, \rightarrow) satisfies Eq. (3) if and only if $(\chi^{\gamma_\alpha}, \rightarrow^{\gamma_\alpha})$ satisfies Eq. (3);

5) $J \#_{\chi}^\alpha A_\alpha$ is a Hopf π -crossed algebra if and only if $J \#_{\chi^{\gamma_\alpha}}^\alpha A_\alpha$ is a Hopf π -crossed algebra, and they are isomorphic.

Proof 1) Define $\Phi: J \#_{\chi}^\alpha A_\alpha \rightarrow J \#_{\chi^{\gamma_\alpha}}^\alpha A_\alpha$ by $x \otimes a \rightarrow x\gamma_\alpha(a_{(1,\alpha)}) \otimes a_{(2,\alpha)}$, $\forall x, y \in J$, $\forall a, b \in A_\alpha, A_\beta$, $\Phi_{\alpha,\beta}((x \otimes a)(y \otimes b)) = x(a_{(1,\alpha)} \rightarrow^{\gamma_\alpha} y)\chi^{\gamma_\alpha}(a_{(2,\alpha)}, b_{(1,\beta)})\gamma_{\alpha,\beta}(a_{(3,\alpha)}$

$b_{(2,\beta)}) \otimes a_{(4,\alpha)}b_{(3,\beta)} = \Phi_\alpha(x \otimes a)\Phi_\beta(y \otimes b)$. Φ is clearly bijective, $\Phi^{-1}(x \otimes a) = x\gamma_{\alpha^{-1}}^{-1}(a_{(1,\alpha^{-1})}) \otimes a_{(2,\alpha)}$, $\forall x \in J, a \in A_\alpha$, since

$$\Phi\Phi^{-1}(x \otimes a) = \Phi(x\gamma_{\alpha^{-1}}^{-1}(a_{(1,\alpha^{-1})}) \otimes a_{(2,\alpha)}) = \Phi(x\gamma_{\alpha^{-1}}^{-1}(a_{(1,\alpha^{-1})})\gamma_\alpha(a_{(2,\alpha)})) \otimes a_{(3,\alpha)} = x \otimes a$$

3) If (χ, \rightarrow) satisfies Eq. (3), then

$$(a_{(1,\alpha)} \rightarrow^{\gamma_\alpha}(b_{(1,\beta)} \rightarrow^{\gamma_\beta} x))\chi^{\gamma_\alpha}\alpha_{\beta}(a_{(2,\alpha)}, b_{(2,\beta)}) = \gamma_\alpha(a_{(1,\alpha)})(a_{(2,\alpha)} \rightarrow \gamma_\beta(b_{(1,\beta)}))\chi_{\alpha,\beta}(a_{(3,\alpha)}, b_{(2,\beta)})\gamma_{\alpha^{-1},\beta^{-1}}^{-1}(a_{(4,\alpha^{-1})}b_{(3,\beta^{-1})})\gamma_{\alpha\beta}(a_{(5,\alpha)}b_{(4,\beta)})(a_{(6,\alpha)}b_{(5,\beta)} \rightarrow x) \cdot \gamma_{\alpha^{-1},\beta^{-1}}^{-1}(a_{(7,\alpha^{-1})}b_{(6,\beta^{-1})}) = \chi_{\alpha,\beta}^{\gamma_\alpha}(a_{(1,\alpha)}, b_{(1,\beta)})(a_{(2,\alpha)}b_{(2,\beta)} \rightarrow^{\gamma_\alpha} x)$$

Conversely, we get it from Lemma 2.

4) If (χ, \rightarrow) satisfies Eq. (2) and Eq. (3), then, for $a \in A_\alpha$, $b \in A_\beta$, $c \in A_\gamma$,

$$(a_{(1,\alpha)} \rightarrow^{\gamma_\alpha}\chi_{\beta,\gamma}^{\gamma_\beta}(b_{(1,\beta)}, c_{(1,\gamma)}))\chi_{\alpha,\beta\gamma}^{\gamma_\alpha}(a_{(2,\alpha)}, b_{(2,\beta)}c_{(2,\gamma)}) = \gamma_\alpha(a_{(1,\alpha)})(a_{(2,\alpha)} \rightarrow [\gamma_\beta(b_{(1,\beta)})(b_{(2,\beta)} \rightarrow \gamma_\gamma(c_{(1,\gamma)})) \cdot \chi_{\beta,\gamma}(b_{(3,\beta)}, c_{(2,\gamma)})\gamma_{\beta^{-1},\gamma^{-1}}^{-1}(b_{(4,\beta^{-1})}, c_{(3,\gamma^{-1})})])\gamma_{\alpha^{-1}}^{-1}(a_{(3,\alpha^{-1})}) \cdot \gamma_\alpha(a_{(4,\alpha)})(a_{(5,\alpha)} \rightarrow \gamma_{\beta\gamma}(b_{(5,\beta)}c_{(4,\gamma)}))\chi_{\alpha,\beta\gamma}(a_{(5,\alpha)}, b_{(6,\beta)}c_{(5,\gamma)}) \cdot \gamma_{\alpha^{-1},\beta^{-1},\gamma^{-1}}^{-1}(a_{(6,\alpha^{-1})}b_{(7,\beta^{-1})}c_{(6,\gamma^{-1})}) = \chi_{\alpha,\beta}^{\gamma_\alpha}(a_{(1,\alpha)}, b_{(1,\beta)})\chi_{\alpha\beta,\gamma}^{\gamma_\alpha}(a_{(2,\alpha)}b_{(2,\beta)}, c)$$

2) and 5) of Theorem 2 are clearly proved.

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Hopf 群代数上的交叉积

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摘要:首先给出了 Hopf 群代数的群交叉积定义,并给出了群交叉积是群代数的充分必要条件. 引入了 Hopf 群代数的 cleft 扩张理论,并证明了 Hopf 群代数的交叉积与 cleft 扩张等价. 然后,给出了 2 个 Hopf 群交叉积等价的充分必要条件. 最后,结合 Hopf 群交叉积与 cleft 扩张的等价理论得到,群交叉积一般由 2-余循环构造,作为代数同构于带有卷积可逆映射的 2-余循环的群交叉积. 一般 2-余循环的余单位性质等价于带有卷积可逆映射的 2-余循环余单位性质,通常意义下的 2-余循环和弱作用结合条件等价于带有卷积可逆映射的 2-余循环及其弱作用结合条件;同时得到,由一般 2-余循环构造的 Hopf π -交叉积代数同构于带有卷积可逆映射的 2-余循环构造的 Hopf π -交叉积代数.

关键词:Hopf π -代数; cleft 扩张理论; π -余模像代数; 群交叉积

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