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A second-order convergent and linearized difference scheme for the initial-boundary value problem of the Korteweg-de Vries equation

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Abstract: To numerically solve the initial-boundary value problem of the Korteweg-de Vries equation, an equivalent coupled system of nonlinear equations is obtained by the method of reduction of order. Then, a difference scheme is constructed for the system. The new variable introduced can be separated from the difference scheme to obtain another difference scheme containing only the original variable. The energy method is applied to the theoretical analysis of the difference scheme. Results show that the difference scheme is uniquely solvable and satisfies the energy conservation law corresponding to the original problem. Moreover, the difference scheme converges when the step ratio satisfies a constraint condition, and the temporal and spatial convergence orders are both two. Numerical examples verify the convergence order and the invariant of the difference scheme. Furthermore, the step ratio constraint is unnecessary for the convergence of the difference scheme. Compared with a known two-level nonlinear difference scheme, the proposed difference scheme has more advantages in numerical calculation.

Key words: Korteweg-de Vries (KdV) equation; linearized difference scheme; conservation; convergence

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The Korteweg-de Vries (KdV) equation was derived by Korteweg and de Vries[1] in 1895 and has a history of more than 130 years. The KdV equation is one of the classical mathematical physics equations. The KdV equation plays an important role in nonlinear dispersive waves and has a wide range of applications. Many scholars have investigated its solutions from the point of view of analysis and numerical value. Moreover, the general solutions of the KdV equation are difficult to obtain. Therefore, many numerical methods, such as the finite difference method[2-6], finite element method[7-10], spectral method[11-12], and meshless method[13-16], have been applied to solve the KdV equation. Among them, the finite difference method is simple and easy to implement on computers. The finite difference method is an important method for solving nonlinear evolution equations. Meanwhile, the theoretical analysis of the difference scheme is relatively difficult, particularly for the initial-boundary value problem. Consequently, we will use the finite difference method to solve the initial-boundary value problem of the KdV equation in this study.

When solving nonlinear evolution equations, we need to consider the corresponding initial and boundary value conditions. The three main types of problems are the initial, periodic boundary, and initial-boundary value problems. Currently, many studies on solving nonlinear evolution equations using the finite difference method have been conducted. Some of them analyzed the initial value problems. For this class of problems, difference schemes were conveniently established by adding homogeneous boundary conditions to the boundary in the practical computation, which is a Dirichlet boundary value problem. If the highest order of the spatial derivative of the equation for the space variable x is two, then the addition of homogeneous boundary conditions will not affect the construction and analysis of difference schemes, such as Burgers' equation:

$$u_t + uu_x = \nu u_{xx}$$

where ν is a positive constant. The difference scheme has the same form at all inner grid points. If the highest order of the spatial derivatives is over two, then the difference will be significant because of the presence of derivative boundary conditions, such as the KdV equation:

$$u_t + \gamma uu_x + u_{xxx} = 0$$

where γ is a constant whose boundary conditions[17] are

$$u(0,t) = u(L,t) = u_x(L,t) = 0 \qquad t \geqslant 0$$

The boundary condition is asymmetric, which means that the difference scheme will also be asymmetric. Therefore, for the initial-boundary value problem of the KdV equation, the construction and analysis of difference schemes need to be more detailed. Studies of the periodic boundary problems of the KdV equation using the finite

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difference method have also been conducted. However, because of the existence of derivative boundary conditions, similarly generalizing the method for the periodic boundary value problems to the initial-boundary value problems of the KdV equation is difficult.

Recently, we established two finite difference schemes for the KdV equation with the initial-boundary value conditions in Ref. [5]. One was a nonlinear difference scheme and the other was a linearized difference scheme. The nonlinear difference scheme was proven to be unconditionally convergent, whereas the linearized difference scheme was conditionally convergent. The accuracy of these two finite difference schemes was the first order in space. Subsequently, in Ref. [6], we established a nonlinear difference scheme and proved that its spatial convergence order was two. In the numerical examples, we solved the nonlinear difference scheme using the Newton iterative method, which increased the computational cost at each time level. To improve computational efficiency and keep the convergence order unchanged, we consider establishing a linearized difference scheme for solving the initial-boundary problem of the KdV equation.

In this study, we construct a three-level linearized difference scheme for the following problem:

$$u_t + \gamma u u_x + u_{xxx} = 0 \qquad 0 < x < L,\ 0 < t \leqslant T \tag{1a}$$

$$u(x,0) = \varphi(x) \qquad 0 < x < L \tag{1b}$$

$$u(0,t) = u(L,t) = u_x(L,t) = 0 \qquad 0 \leqslant t \leqslant T \tag{1c}$$

where $\varphi(0) = \varphi(L) = \varphi'(L) = 0$, γ is a constant. We also establish the difference scheme and illustrate the truncation errors in detail. Then, we will present its conservation law, prove its unique solvability and conditional convergence, and provide some numerical simulations to verify our theoretical results and compare them with those of the nonlinear difference scheme in Ref. [6].

1 Difference Scheme

In this section, we will use the method of reduction of order to establish the difference scheme for Problem (1) and illustrate the truncation errors in detail.

1.1 Notation

Before presenting the difference scheme, we introduce the notations used.

We take two positive integers m and n. Then, we let $h = L/m$, $x_j = jh$, $0 \leqslant j \leqslant m$; $\tau = T/n$, $t_k = k\tau$, $0 \leqslant k \leqslant n$; $\Omega_h = \{x_j \mid 0 \leqslant j \leqslant m\}$. $\Omega_\tau = \{t_k \mid 0 \leqslant k \leqslant n\}$. Moreover, we let

$$U_h = \{v \mid v = \{v_j\}_{j=0}^{m} \text{ be the grid function of } \Omega_h\},$$

$$V_h = \{v \mid v \in U_h \text{ and } v_0 = v_m = 0\}$$

For any $u, v \in U_h$, we introduce the following notations:

$$\delta_x v_{j+1/2} = \frac{1}{h}(v_{j+1} - v_j)$$

$$\delta_x^2 v_j = \frac{1}{h^2}(v_{j+1} - 2v_j + v_{j-1})$$

$$\Delta_x v_j = \frac{1}{2h}(v_{j+1} - v_{j-1})$$

$$\delta_x^3 v_{j+1/2} = \frac{1}{h^3}(v_{j+2} - 3v_{j+1} + 3v_j - v_{j-1})$$

$$\psi(u,v)_j = \frac{1}{3}[u_j \Delta_x v_j + \Delta_x (uv)_j]$$

$$(u,v) = h\left(\frac{1}{2}u_0 v_0 + \sum_{j=1}^{m-1} u_j v_j + \frac{1}{2}u_m v_m\right)$$

$$(\delta_x u, \delta_x v) = h\sum_{j=1}^{m} (\delta_x u_{j-1/2})(\delta_x v_{j-1/2})$$

$$\| u \| = \sqrt{(u,u)}, \qquad | u |_1 = \sqrt{(\delta_x u, \delta_x u)}$$

Then, we let

$$S_\tau = \{w \mid w = \{w^k\}_{k=0}^{n} \text{ be the grid function of } \Omega_\tau\}$$

For any $w \in S_\tau$, we introduce the following notations:

$$w^{k+1/2} = \frac{1}{2}(w^k + w^{k+1}), \quad w^{\bar{k}} = \frac{1}{2}(w^{k+1} + w^{k-1})$$

$$\delta_t w^{k+1/2} = \frac{1}{\tau}(w^{k+1} - w^k), \quad \Delta_t w^k = \frac{1}{2\tau}(w^{k+1} - w^{k-1})$$

It is easy to know that

$$\delta_x^3 v_{j+1/2} = \delta_x^2 (\delta_x v_{j+1/2})$$

$$\Delta_t w^k = \frac{1}{2}(\delta_t w^{k+1/2} + \delta_t w^{k-1/2})$$

1.2 Derivation of the difference scheme

We construct the difference scheme using the method of reduction of order.

Let

$$v = u_x \qquad 0 \leqslant x \leqslant L,\ 0 \leqslant t \leqslant T$$

Then Problem (1) is equal to the following problem of coupled equations:

$$u_t + \gamma u u_x + v_{xx} = 0 \qquad 0 < x < L,\ 0 < t \leqslant T \tag{2a}$$

$$v = u_x \qquad 0 < x < L,\ 0 < t \leqslant T \tag{2b}$$

$$u(x,0) = \varphi(x) \qquad 0 \leqslant x \leqslant L \tag{2c}$$

$$u(0,t) = u(L,t) = 0 \qquad 0 < t \leqslant T \tag{2d}$$

$$v(L,t) = 0 \qquad 0 \leqslant t \leqslant T \tag{2e}$$

Then, we denote

$$U_j^k = u(x_j, t_k), \quad V_j^k = v(x_j, t_k) \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n$$

Considering Eq. (2a) at points $(x_j, t_{1/2})$ and (x_j, t_k) and using the Taylor expansion, we obtain

$$\delta_t U_j^{1/2} + \gamma\psi(U^0, U^{1/2})_j + \delta_x^2 V_j^{1/2} = P_j^0 \quad 1 \leqslant j \leqslant m-1 \tag{3}$$

and

$$\Delta_t U_j^k + \gamma\psi(U^k, \bar{U}^k)_j + \delta_x^2 V_j^k = P_j^k \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1 \tag{4}$$

The constant $c_1 > 0$ exists, such that

$$|P_j^0| \leqslant c_1(\tau + h^2), \quad |P_j^k| \leqslant c_1(\tau^2 + h^2) \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1 \tag{5}$$

We consider Eq. (2b) at points $(x_{j+1/2}, t_k)$ and use the Taylor expansion to obtain

$$V_{j+1/2}^k = \delta_x U_{j+1/2}^k + Q_j^k \quad 0 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n \tag{6}$$

For the discrete error Q_j^k in Eq. (6), we obtain the following results.

Lemma 1[6] We denote

$$S_j^k = \sum_{l=j+1}^{m-1} (-1)^{l-1-j} \delta_x Q_{l-1/2}^k \quad 0 \leqslant j \leqslant m-2,\ 0 \leqslant k \leqslant n$$

The constant $c_2 > 0$ exists, such that

$$|Q_j^k| \leqslant c_2 h^2 \quad 0 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n$$

$$|S_{m-2}^k| \leqslant c_2 h^3 \quad 0 \leqslant k \leqslant n$$

$$|S_j^k| \leqslant c_2 h^2 \quad 0 \leqslant j \leqslant m-2,\ 0 \leqslant k \leqslant n$$

$$|\delta_x S_{j+1/2}^k| \leqslant c_2 h^2 \quad 0 \leqslant j \leqslant m-3,\ 0 \leqslant k \leqslant n$$

Considering the initial and boundary value conditions expressed in Eqs. (2c) to (2e), we obtain

$$U_j^0 = \varphi(x_j) \quad 0 \leqslant j \leqslant m \tag{7}$$

$$U_0^k = 0, \quad U_m^k = 0 \quad 1 \leqslant k \leqslant n \tag{8}$$

$$V_m^k = 0 \quad 0 \leqslant k \leqslant n \tag{9}$$

By omitting the small terms in Eqs. (3), (4), and (6) and combining them with Eqs. (7) to (9), we construct a three-level linearized difference scheme for Problem (2), as follows:

$$\delta_t u_j^{1/2} + \gamma\psi(u^0, u^{1/2})_j + \delta_x^2 v_j^{1/2} = 0 \quad 1 \leqslant j \leqslant m-1 \tag{10a}$$

$$\Delta_t u_j^k + \gamma\psi(u^k, \bar{u}^k)_j + \delta_x^2 v_j^k = 0 \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1 \tag{10b}$$

$$v_{j+1/2}^k = \delta_x u_{j+1/2}^k \quad 0 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n \tag{10c}$$

$$u_j^0 = \varphi(x_j) \quad 0 \leqslant j \leqslant m \tag{10d}$$

$$u_0^k = 0, \quad u_m^k = 0 \quad 1 \leqslant k \leqslant n \tag{10e}$$

$$v_m^k = 0 \quad 0 \leqslant k \leqslant n \tag{10f}$$

1.3 Calculation of the difference scheme

For ease of calculation, we separate the variables for Eq. (10).

Theorem 1 The difference scheme expressed in Eq. (10) is equivalent to the following system of equations:

$$\delta_t u_{j+1/2}^{1/2} + \frac{\gamma}{2}[\psi(u^0, u^{1/2})_j + \psi(u^0, u^{1/2})_{j+1}] + \delta_x^3 u_{j+1/2}^{1/2} = 0 \quad 1 \leqslant j \leqslant m-2 \tag{11a}$$

$$\delta_t u_{m-1}^{1/2} + \gamma\psi(u^0, u^{1/2})_{m-1} + \frac{2}{h^2}(\delta_x u_{m-3/2}^{1/2} - 3\delta_x u_{m-1/2}^{1/2}) = 0 \tag{11b}$$

$$\Delta_t u_{j+1/2}^k + \frac{\gamma}{2}[\psi(u^k, \bar{u}^k)_j + \psi(u^k, \bar{u}^k)_{j+1}] + \delta_x^3 \bar{u}_{j+1/2}^k = 0 \quad 1 \leqslant j \leqslant m-2,\ 1 \leqslant k \leqslant n-1 \tag{11c}$$

$$\Delta_t u_{m-1}^k + \gamma\psi(u^k, \bar{u}^k)_{m-1} + \frac{2}{h^2}(\delta_x \bar{u}_{m-3/2}^k - 3\delta_x \bar{u}_{m-1/2}^k) = 0 \quad 1 \leqslant k \leqslant n-1 \tag{11d}$$

$$u_j^0 = \varphi(x_j) \quad 0 \leqslant j \leqslant m \tag{11e}$$

$$u_0^k = u_m^k = 0 \quad 1 \leqslant k \leqslant n \tag{11f}$$

and

$$v_m^k = 0 \quad 0 \leqslant k \leqslant n \tag{12a}$$

$$v_j^k = 2\delta_x u_{j+1/2}^k - v_{j+1}^k \quad j = m-1, m-2, \cdots, 0;\ 0 \leqslant k \leqslant n \tag{12b}$$

Proof Ⅰ) Based on Eqs. (10) to (12), first, we determine that Eqs. (10d) and (10e) are equivalent to Eqs. (11e) and (11f), respectively, and Eq. (10f) is equivalent to Eq. (12a). Moreover, Eq. (10c) is equivalent to Eq. (12b), and Eq. (10a) is equivalent to

$$\delta_t u_{j+1/2}^{1/2} + \frac{\gamma}{2}[\psi(u^0, u^{1/2})_j + \psi(u^0, u^{1/2})_{j+1}] + \delta_x^2 \frac{v_j^{1/2} + v_{j+1}^{1/2}}{2} = 0 \quad 1 \leqslant j \leqslant m-2 \tag{13a}$$

$$\delta_t u_{m-1}^{1/2} + \gamma\psi(u^0, u^{1/2})_{m-1} + \delta_x^2 v_{m-1}^{1/2} = 0 \tag{13b}$$

Based on Eq. (10c), we derive

$$v_{j+1/2}^{1/2} = \delta_x u_{j+1/2}^{1/2} \quad 0 \leqslant j \leqslant m-1 \tag{14}$$

By substituting Eq. (14) into Eq. (13a), we derive Eq. (11a).

Similarly, based on Eq. (10f), we derive

$$v_m^{1/2} = 0$$

Then, we obtain

$$\delta_x^2 v_{m-1}^{1/2} = \frac{1}{h^2}(v_{m-2}^{1/2} - 2v_{m-1}^{1/2} + v_m^{1/2}) = \frac{2}{h^2}(v_{m-3/2}^{1/2} - 3v_{m-1/2}^{1/2})$$

By substituting the obtained equality into Eq. (13b) and using Eq. (14), we derive Eq. (11b).

Similarly, we obtain Eqs. (11c) and (11d) from Eqs. (10b), (10c), and (10f).

Ⅱ) Based on Eqs. (10) to (12), we determine that Eqs. (11e) and (11f) are equivalent to Eqs. (10d) and (10e), respectively, and Eq. (12a) is equivalent to Eq. (10f). Based on Eq. (12), we derive Eq. (10c) and

$$v_m^{1/2} = 0 \tag{15a}$$

$$v_{j+1/2}^{1/2} = \delta_x u_{j+1/2}^{1/2} \qquad 0 \leqslant j \leqslant m-1 \tag{15b}$$

Based on Eq. (15), we obtain

$$\frac{2}{h^2}(\delta_x u_{m-3/2}^{1/2} - 3\delta_x u_{m-1/2}^{1/2}) = \frac{2}{h^2}(v_{m-3/2}^{1/2} - 3v_{m-1/2}^{1/2}) = \frac{1}{h^2}(v_{m-2}^{1/2} - 2v_{m-1}^{1/2} - 3v_m^{1/2}) = \delta_x^2 v_{m-1}^{1/2}$$

By substituting the obtained equation into Eq. (11b), we derive

$$\delta_t u_{m-1}^{1/2} + \gamma\psi(u^0, u^{1/2})_{m-1} + \delta_x^2 v_{m-1}^{1/2} = 0 \tag{16}$$

which is Eq. (10a) with $j = m-1$. By substituting Eq. (15b) into Eq. (11a), we obtain

$$\delta_t u_{j+1/2}^{1/2} + \frac{\gamma}{2}[\psi(u^0, u^{1/2})_j + \psi(u^0, u^{1/2})_{j+1}] + \delta_x^2 \frac{v_j^{1/2} + v_{j+1}^{1/2}}{2} = 0 \qquad 1 \leqslant j \leqslant m-2$$

that is

$$\frac{1}{2}[\delta_t u_{j+1}^{1/2} + \gamma\psi(u^0, u^{1/2})_{j+1} + \delta_x^2 v_{j+1}^{1/2}] + \frac{1}{2}[\delta_t u_j^{1/2} + \gamma\psi(u^0, u^{1/2})_j + \delta_x^2 v_j^{1/2}] = 0 \qquad 1 \leqslant j \leqslant m-2$$

By combining the obtained equation with Eq. (16), we derive

$$\delta_t u_j^{1/2} + \gamma\psi(u^0, u^{1/2})_j + \delta_x^2 v_j^{1/2} = 0 \qquad j = m-2, m-3, \cdots, 1$$

which is Eq. (10a) with $1 \leqslant j \leqslant m-2$.

Similarly, we obtain Eq. (10b) from Eqs. (11c), (11d), and (12).

The proof is completed.

The difference scheme expressed in Eq. (11) contains only the unknown quantity $\{u_j^k\}$. Thus, calculating u from Eq. (11) is easier than that from Eq. (10). Next, we describe in detail how to solve the difference scheme expressed in Eq. (11).

We denote

$$u^k = \{u_0^k, u_1^k, \cdots, u_m^k\}$$

Based on Eq. (11e), we obtain u^0. Then, we compute u^1 using Eqs. (11a), (11b), and (11f). We let $w = u^{1/2}$. If we determine w, then we can obtain u^1 using the following expression:

$$u^1 = 2w - u^0$$

Based on Eqs. (11a), (11b), and (11f), we can obtain the system of equations for w, as follows:

$$\frac{2}{\tau}(w_{j+1/2} - u_{j+1/2}^0) + \frac{\gamma}{2}[\psi(u^0, w)_j + \psi(u^0, w)_{j+1}] + \delta_x^3 w_{j+1/2} = 0 \qquad 1 \leqslant j \leqslant m-2$$

$$\frac{2}{\tau}(w_{m-1} - u_{m-1}^0) + \gamma\psi(u^0, w)_{m-1} + \frac{2}{h^2}(\delta_x w_{m-3/2} - 3\delta_x w_{m-1/2}) = 0 \qquad w_0 = 0,\ w_m = 0$$

The value of w can be obtained by solving the aforementioned system of quatic diagonal linear equations using the double-sweep method.

Assuming that we already know u^k and u^{k-1}, we solve the value of u^{k+1}. We let $w = u^{\bar{k}}$. If we determine w, then we can obtain u^{k+1} using the following expression:

$$u^{k+1} = 2w - u^{k-1}$$

Based on Eqs. (11c), (11d), and (11f), we can obtain the system of equations for w, as follows:

$$\frac{2}{\tau}(w_{j+1/2} - u_{j+1/2}^k) + \frac{\gamma}{2}[\psi(u^k, w)_j + \psi(u^k, w)_{j+1}] + \delta_x^3 w_{j+1/2} = 0 \qquad 1 \leqslant j \leqslant m-2$$

$$\frac{2}{\tau}(w_{m-1} - u_{m-1}^k) + \gamma\psi(u^k, w)_{m-1} + \frac{2}{h^2}(\delta_x w_{m-3/2} - 3\delta_x w_{m-1/2}) = 0 \qquad w_0 = 0,\ w_m = 0$$

The value of w can be obtained by solving the aforementioned system of quadratic diagonal linear equations using the sweep method.

Furthermore, Theorem 1 illustrates that analyzing Eq. (10) is equivalent to analyzing Eq. (11).

2 Theoretical Analysis

In this section, we will analyze the conservation, unique solvability, and convergence of the difference scheme expressed in Eq. (10).

2.1 Conservation and Unique solvability

First, we provide several lemmas, which will be subsequently used.

Lemma 2[18] For $\forall u \in U_h$ and $v \in V_h$, we have

$$(\psi(u, v), v) = 0$$

Lemma 3[6] We let $v \in U_h$ and $u \in V_h$ satisfy

$$v_m = 0,\ v_{j+1/2} = \delta_x u_{j+1/2} \qquad 0 \leqslant j \leqslant m-1$$

Then, we derive

$$(\delta_x^2 v, u) = \frac{1}{2}(v_0)^2$$

For the continuous problem expressed in Eq. (1), conservation exists.

Theorem 2[5] Supposing that $u(x,t)$ is the solution to Problem (1), we denote

$$\widetilde{E}(t) = \int_0^L u^2(x,t)\,\mathrm{d}x + \int_0^t u_x^2(0,s)\,\mathrm{d}s$$

Then, we derive

$$\widetilde{E}(t) = \widetilde{E}(0) \qquad 0 < t \leqslant T$$

Similarly, the difference scheme expressed in Eq. (10) has an invariant.

Theorem 3 Supposing that $\{u_j^k, v_j^k \mid 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ is the solution to Eq. (10), we denote

$$E^k = \frac{\| u^k \|^2 + \| u^{k-1} \|^2}{2} + \tau\Big[\sum_{l=1}^{k-1}(v_0^l)^2 + \frac{1}{2}(v_0^{1/2})^2\Big] \qquad 1 \leqslant k \leqslant n$$

Then, we derive

$$E^k = \| u^0 \|^2 \qquad 1 \leqslant k \leqslant n$$

Proof Ⅰ) Taking the inner product of Eq. (10a) with $u^{1/2}$, we obtain

$$(\delta_t u^{1/2}, u^{1/2}) + \gamma(\psi(u^0, u^{1/2}), u^{1/2}) + (\delta_x^2 v^{1/2}, u^{1/2}) = 0$$

Together with Lemmas 2 and 3, we derive

$$\frac{1}{\tau}\,\frac{\| u^1 \|^2 - \| u^0 \|^2}{2} + \frac{1}{2}(v_0^{1/2})^2 = 0$$

That is,

$$\frac{\| u^1 \|^2 + \| u^0 \|^2}{2} + \frac{\tau}{2}(v_0^{1/2})^2 = \| u^0 \|^2 \qquad (17)$$

Ⅱ) Taking the inner product of Eq. (10b) with $2u^{\bar{k}}$, we obtain

$$2(\Delta_t u^k, u^{\bar{k}}) + 2\gamma(\psi(u^k, u^{\bar{k}}), u^{\bar{k}}) + 2(\delta_x^2 v^{\bar{k}}, u^{\bar{k}}) = 0 \qquad 1 \leqslant k \leqslant n-1$$

Together with Lemmas 2 and 3, we derive

$$\frac{1}{\tau}\left(\frac{\| u^{k+1} \|^2 + \| u^k \|^2}{2} - \frac{\| u^k \|^2 + \| u^{k-1} \|^2}{2}\right) + (v_0^{\bar{k}})^2 = 0 \qquad 1 \leqslant k \leqslant n-1$$

That is,

$$\frac{\| u^{k+1} \|^2 + \| u^k \|^2}{2} + \tau\sum_{l=1}^{k}(v_0^{\bar{l}})^2 = \frac{\| u^1 \|^2 + \| u^0 \|^2}{2} \qquad 1 \leqslant k \leqslant n-1$$

By adding the equality expressed in Eq. (17) to the obtained equality, we derive

$$\frac{\| u^{k+1} \|^2 + \| u^k \|^2}{2} + \tau\Big[\sum_{l=1}^{k}(v_0^{\bar{l}})^2 + \frac{1}{2}(v_0^{1/2})^2\Big] = \| u^0 \|^2 \qquad 0 \leqslant k \leqslant n-1$$

This completes the proof.

Then, we prove the unique solvability.

Theorem 4 The difference scheme expressed in Eq. (10) has a unique solution.

Proof We obtain u^0 using Eq. (10d) and v^0 using Eqs. (10c) and (10f).

Based on Eqs. (10a), (10c), (10e), and (10f), we derive u^1 and v^1. Considering the system of homogeneous equations, we obtain

$$\frac{1}{\tau}u_j^1 + \frac{\gamma}{2}\psi(u^0, u^1)_j + \frac{1}{2}\delta_x^2 v_j^1 = 0 \qquad 1 \leqslant j \leqslant m-1 \qquad (18a)$$

$$v_{j+1/2}^1 = \delta_x u_{j+1/2}^1 \qquad 0 \leqslant j \leqslant m-1 \qquad (18b)$$

$$u_0^1 = 0, \quad u_m^1 = 0 \qquad (18c)$$

$$v_m^1 = 0 \qquad (18d)$$

Taking the inner product of Eq. (18a) with u^1, we derive

$$\frac{1}{\tau}\| u^1 \|^2 + \frac{\gamma}{2}(\psi(u^0, u^1), u^1) + \frac{1}{2}(\delta_x^2 v^1, u^1) = 0$$

Together with Lemmas 2 and 3, we obtain

$$\frac{1}{\tau}\| u^1 \|^2 + \frac{1}{4}(v_0^1)^2 = 0$$

Then, we have

$$\| u^1 \| = 0$$

which follows

$$u_j^1 = 0 \qquad 0 \leqslant j \leqslant m$$

From Eq. (18b) and Eq. (18d), we can get

$$v_j^1 = 0 \qquad 0 \leqslant j \leqslant m$$

That is, Eqs. (10a), (10c), (10e) and (10f) have the unique solutions u^1 and v^1.

We suppose that u^k, u^{k-1} and v^k, v^{k-1} are known. Based on Eqs. (10b), (10c), (10e) and (10f), we determine u^{k+1} and v^{k+1}. Considering the system of homogeneous equations, we obtain

$$\frac{1}{2\tau}u_j^{k+1} + \frac{\gamma}{2}\psi(u^k, u^{k+1})_j + \frac{1}{2}\delta_x^2 v_j^{k+1} = 0 \qquad 1 \leqslant j \leqslant m-1 \qquad (19a)$$

$$v_{j+1/2}^{k+1} = \delta_x u_{j+1/2}^{k+1} \qquad 0 \leqslant j \leqslant m-1 \qquad (19b)$$

$$u_0^{k+1} = 0, \quad u_m^{k+1} = 0 \qquad (19c)$$

$$v_m^{k+1} = 0 \qquad (19d)$$

Taking an inner product of Eq. (19a) with $2u^{k+1}$, we get

$$\frac{1}{\tau}\|u^{k+1}\|^2+\gamma(\psi(u^k,u^{k+1}),u^{k+1})+(\delta_x^2 v^{k+1},u^{k+1})=0$$

Using Lemmas 2 and 3, we obtain

$$\frac{1}{\tau}\|u^{k+1}\|^2+\frac{1}{2}(v_0^{k+1})^2=0$$

Then, we have

$$\|u^{k+1}\|=0$$

which follows

$$u_j^{k+1}=0 \qquad 0\leqslant j\leqslant m$$

Based on Eqs. (19b) and (19d), we obtain

$$v_j^{k+1}=0 \qquad 0\leqslant j\leqslant m$$

That is, Eqs. (10b), Eq. (10c), Eq. (10e), and Eq. (10f) have the unique solutions u^{k+1} and v^{k+1}.

This completes the proof.

2.2 Convergence

Assuming that $\{u(x,t),v(x,t)\mid(x,t)\in[0,L]\times[0,T]\}$ is the solution to Problem (2) and $\{u_j^k,v_j^k\mid 0\leqslant j\leqslant m, 0\leqslant k\leqslant n\}$ is the solution to the difference scheme expressed in Eq. (10), we denote

$$e_j^k=u(x_j,t_k)-u_j^k,\quad f_j^k=v(x_j,t_k)-v_j^k$$
$$0\leqslant j\leqslant m,\ 0\leqslant k\leqslant n$$

By subtracting Scheme (10) from Eqs. (3), (4), and (6) to (9), we derive the following system of error equations:

$$\delta_t e_j^{1/2}+\gamma\psi(u^0,e^{1/2})_j+\delta_x^2 f_j^{1/2}=P_j^0 \qquad 1\leqslant j\leqslant m-1 \tag{20a}$$

$$\Delta_t e_j^k+\gamma[\psi(U^k,U^{\bar{k}})_j-\psi(u^k,u^{\bar{k}})_j]+\delta_x^2 f_j^{\bar{k}}=P_j^k \qquad 1\leqslant j\leqslant m-1 \tag{20b}$$

$$f_{j+1/2}^k=\delta_x e_{j+1/2}^k+Q_j^k \qquad 0\leqslant j\leqslant m-1,\ 0\leqslant k\leqslant n \tag{20c}$$

$$e_j^0=0 \qquad 0\leqslant j\leqslant m \tag{20d}$$

$$e_0^k=0,\ e_m^k=0 \qquad 1\leqslant k\leqslant n \tag{20e}$$

$$f_m^k=0 \qquad 0\leqslant k\leqslant n \tag{20f}$$

Before obtaining the convergence result, we present the following two lemmas.

Lemma 4 From the proof of Theorem 4.3 in Ref. [5], it follows that the following equality holds:

$$\begin{aligned}(\psi(U^k,U^{\bar{k}})-\psi(u^k,u^{\bar{k}}),e^{\bar{k}}) ={}& \frac{1}{12}\Big[\sum_{j=1}^{m-2}U_j^{k+1/2}(e_j^{k+1}e_{j+1}^k-e_j^k e_{j+1}^{k+1})-\\ &\sum_{j=1}^{m-2}U_j^{k-1/2}(e_j^k e_{j+1}^{k-1}-e_j^{k-1}e_{j+1}^k)\Big]+\\ &\frac{h}{3}\sum_{j=1}^{m-1}(\Delta_x U_j^{\bar{k}})e_j^k e_j^{\bar{k}}+\frac{h}{6}\sum_{j=1}^{m-2}(\delta_x U_{j+1/2}^{\bar{k}})e_{j+1}^k e_j^{\bar{k}}+\\ &\frac{1}{12}\sum_{j=1}^{m-2}[(U_i^{\bar{k}}-U_i^{k+1/2})(e_j^{k+1}e_{j+1}^k-e_j^k e_{j+1}^{k+1})+\\ &(U_j^{k-1/2}-U_j^{\bar{k}})(e_j^k e_{j+1}^{k-1}-e_j^{k-1}e_{j+1}^k)]\end{aligned}$$

Lemma 5 From the proof of Theorem 4.1 in Ref. [6], it follows that the following equality holds:

$$\begin{aligned}(\delta_x^2 f^k,e^k) ={}& \frac{1}{2}(f_0^k)^2-f_0^k Q_0^k+hf_0^k S_0^k-\\ &2h\sum_{j=0}^{m-2}S_j^k Q_j^k+2h\sum_{j=1}^{m-2}e_j^k\delta_x S_{j-1/2}^k-2e_{m-1}^k S_{m-2}^k\end{aligned}$$

We denote

$$c_0=\max_{(x,t)\in[0,L]\times[0,T]}\{|u(x,t)|,|u_x(x,t)|\}$$

Theorem 5 We let

$$\lambda=\frac{c_0|\gamma|\tau}{3h},\quad c_3=\frac{|\gamma|c_0+6}{2(1-\lambda)}$$

$$c_4=\frac{\exp\left(\dfrac{3c_3T}{2}\right)}{\dfrac{4c_1^2L+8c_2^2(1+3L)}{1-\lambda}+\dfrac{c_1^2L+2c_2^2+8c_2^2L}{c_3(1-\lambda)}}$$

If $\lambda<1$, then we have

$$\|e^k\|\leqslant c_4(\tau^2+h^2) \qquad 0\leqslant k\leqslant n$$

Proof Ⅰ) It follows from Eq. (20d) that

$$\|e^0\|=0 \tag{21}$$

Taking the inner product of Eq. (20a) with $2e^{1/2}$, we derive

$$2(\delta_t e^{1/2},e^{1/2})+2\gamma(\psi(u^0,e^{1/2}),e^{1/2})+2(\delta_x^2 f^{1/2},e^{1/2})=2(P^0,e^{1/2})$$

That is,

$$\frac{1}{\tau}\|e^1\|^2+2(\delta_x^2 f^{1/2},e^{1/2})=(P^0,e^1) \tag{22}$$

It follows from Lemma 5 that

$$\begin{aligned}(\delta_x^2 f^{1/2},e^{1/2}) ={}& \frac{1}{2}(f_0^{1/2})^2-f_0^{1/2}Q_0^{1/2}+hf_0^{1/2}S_0^{1/2}-2h\sum_{j=0}^{m-2}S_j^{1/2}Q_j^{1/2}+\\ &2h\sum_{j=1}^{m-2}e_j^{1/2}\delta_x S_{j-1/2}^{1/2}-2e_{m-1}^{1/2}S_{m-2}^{1/2}=\\ &\frac{1}{2}(f_0^{1/2})^2-f_0^{1/2}Q_0^{1/2}+hf_0^{1/2}S_0^{1/2}-2h\sum_{j=0}^{m-2}S_j^{1/2}Q_j^{1/2}+\\ &h\sum_{j=1}^{m-2}e_j^1\delta_x S_{j-1/2}^{1/2}-e_{m-1}^1 S_{m-2}^{1/2}\end{aligned}$$

By substituting the previously derived equality into Eq. (22) and combining it with Lemma 1, Lemma 5, and the truncation error expressed in (5), we obtain

$$\begin{aligned}&\frac{1}{\tau}\|e^1\|^2 + (f_0^{1/2})^2 = \\&\quad (P^0,e^1) + 2f_0^{1/2}Q_0^{1/2} - 2hf_0^{1/2}S_0^{1/2} + 4h\sum_{j=0}^{m-2} S_j^{1/2}Q_j^{1/2} - \\&\quad 2h\sum_{j=1}^{m-2} e_j^1\delta_x S_{j-1/2}^{1/2} + 2e_{m-1}^1 S_{m-2}^{1/2} \leqslant \\&\quad \frac{\tau}{2}\|P^0\|^2 + \frac{1}{2\tau}\|e^1\|^2 + \frac12 (f_0^{1/2})^2 + 2(Q_0^{1/2})^2 + \frac12 (f_0^{1/2})^2 + \\&\quad 2h^2(S_0^{1/2})^2 + 4h\sum_{j=0}^{m-2}|S_j^{1/2}|\,|Q_j^{1/2}| + \frac h2\sum_{j=1}^{m-2}(e_j^1)^2 + \\&\quad 2h\sum_{j=1}^{m-2}(\delta_x S_{j-1/2}^{1/2})^2 + \frac h2 (e_{m-1}^1)^2 + \frac 2h (S_{m-2}^{1/2})^2 \leqslant \\&\quad \frac\tau2 c_1^2 L(\tau+h^2)^2 + \frac1{2\tau}\|e^1\|^2 + \frac12\|e^1\|^2 + (f_0^{1/2})^2 + \\&\quad 2c_2^2h^4 + 2c_2^2h^6 + 4c_2^2Lh^4 + 2h\sum_{j=1}^{m-2}c_2^2h^4 + 2c_2^2h^5\end{aligned}$$

That is,

$$(1-\tau)\|e^1\|^2 \leqslant c_1^2L(\tau^2+h^2)^2 + 4c_2^2(1+3L)\tau h^4$$

When $\tau\leqslant\frac12$, we obtain

$$\begin{aligned}\|e^1\|^2 &\leqslant 2c_1^2L(\tau^2+h^2)^2 + 4c_2^2(1+3L)h^4 \leqslant \\ &\quad [2c_1^2L + 4c_2^2(1+3L)](\tau^2+h^2)^2 \qquad (23)\end{aligned}$$

Ⅱ) Taking the inner product of Eq. (20b) with $2e^{\bar k}$, we derive

$$\begin{aligned}&2(\Delta_t e^k, e^{\bar k}) + 2\gamma(\psi(U^k,U^{\bar k}) - \psi(u^k,e^{\bar k}), e^{\bar k}) + \\&\quad 2(\delta_x^2 f^{\bar k}, e^{\bar k}) = 2(P^k, e^{\bar k}) \qquad 1\leqslant k\leqslant n-1\end{aligned}$$

That is,

$$\begin{aligned}&\frac1\tau\left(\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2} - \frac{\|e^k\|^2+\|e^{k-1}\|^2}{2}\right) + 2(\delta_x^2 f^{\bar k}, e^{\bar k}) = \\&\quad 2(P^k,e^{\bar k}) - 2\gamma(\psi(U^k,U^{\bar k}) - \psi(u^k,e^{\bar k}), e^{\bar k}) \\&\quad 1\leqslant k\leqslant n-1 \qquad (24)\end{aligned}$$

It follows from Lemma 5 that

$$\begin{aligned}(\delta_x^2 f^{\bar k}, e^{\bar k}) &= \frac12 (f_0^{\bar k})^2 - f_0^{\bar k}Q_0^{\bar k} + hf_0^{\bar k}S_0^{\bar k} - 2h\sum_{j=0}^{m-2}S_j^{\bar k}Q_j^{\bar k} + \\&\quad 2h\sum_{j=1}^{m-2}e_j^{\bar k}\delta_x S_{j-1/2}^{\bar k} - 2e_{m-1}^{\bar k}S_{m-2}^{\bar k}\end{aligned}$$

By substituting the previously derived equality into Eq. (24) and combining it with Lemmas 4 and 1, we obtain

$$G^{k-1/2} = \frac{\gamma\tau}{6}\sum_{j=1}^{m-2}U_j^{k-1/2}(e_j^k e_{j+1}^{k-1} - e_j^{k-1}e_{j+1}^k)$$

Then, we derive

$$\begin{aligned}&\frac1\tau\left[\left(\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2} + G^{k+1/2}\right) - \left(\frac{\|e^k\|^2+\|e^{k-1}\|^2}{2} + G^{k-1/2}\right)\right] + (f_0^{\bar k})^2 = \\&\quad 2(P^k,e^{\bar k}) - \gamma\Bigg\{\frac{2h}{3}\sum_{j=1}^{m-1}(\Delta_x U_j^k)e_j^{\bar k}e_j^{\bar k} + \\&\quad \frac h3\sum_{j=1}^{m-2}(\delta_x U_{j+1/2}^k)e_{j+1}^{\bar k}e_j^{\bar k} + \\&\quad \frac16\sum_{j=1}^{m-2}[(U_i^k - U_i^{k+1/2})(e_j^{k+1}e_{j+1}^k - e_j^k e_{j+1}^{k+1}) + \\&\quad (U_j^{k-1/2} - U_j^k)(e_j^k e_{j+1}^{k-1} - e_j^{k-1}e_{j+1}^k)]\Bigg\} + 2f_0^{\bar k}Q_0^{\bar k} - \\&\quad 2hf_0^{\bar k}S_0^{\bar k} + 4h\sum_{j=0}^{m-2}S_j^{\bar k}Q_j^{\bar k} - 4h\sum_{j=1}^{m-2}e_j^{\bar k}\delta_x S_{j-1/2}^{\bar k} + 4e_{m-1}^{\bar k}S_{m-2}^{\bar k} \leqslant \\&\quad \|P^k\|^2 + \|e^{\bar k}\|^2 + |\gamma|\,(c_0\|e^k\|\,\|e^{\bar k}\| + \\&\quad \frac{\tau}{3h}c_0\|e^{k+1}\|\,\|e^k\| + \frac{\tau}{3h}c_0\|e^k\|\,\|e^{k-1}\|) + \\&\quad \frac12 (f_0^{\bar k})^2 + 2(Q_0^{\bar k})^2 + \frac12(f_0^{\bar k})^2 + 2h^2(S_0^{\bar k})^2 + \\&\quad 4h\sum_{j=0}^{m-2}|S_j^{\bar k}|\,|Q_j^{\bar k}| + h\sum_{j=1}^{m-2}(e_j^{\bar k})^2 + \\&\quad 4h\sum_{j=1}^{m-2}(\delta_x S_{j-1/2}^{\bar k})^2 + h(e_{m-1}^{\bar k})^2 + \frac4h (S_{m-2}^{\bar k})^2\end{aligned}$$

Using the truncation error expressed in Eq. (5), we obtain

$$\begin{aligned}&\frac1\tau\left[\left(\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2} + G^{k+1/2}\right) - \left(\frac{\|e^k\|^2+\|e^{k-1}\|^2}{2} + G^{k-1/2}\right)\right] + (f_0^{\bar k})^2 \leqslant \\&\quad c_1^2L(\tau^2+h^2)^2 + 2\|e^{\bar k}\|^2 + \frac{|\gamma|\,c_0}{2}(\|e^k\|^2 + \|e^{\bar k}\|^2) + \\&\quad \frac{c_0|\gamma|\tau}{6h}(\|e^{k+1}\|^2 + \|e^k\|^2 + \|e^k\|^2 + \|e^{k-1}\|^2) + \\&\quad (f_0^{\bar k})^2 + 2c_2^2h^4 + 2c_2^2h^6 + 4h\sum_{j=0}^{m-2}c_2^2h^4 + \\&\quad 4h\sum_{j=1}^{m-2}c_2^2h^4 + 4c_2^2h^5 \leqslant \left(\frac{|\gamma|\,c_0}{2} + 2 + \lambda\right)\cdot \\&\quad \left(\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2} + \frac{\|e^k\|^2+\|e^{k-1}\|^2}{2}\right) + (f_0^{\bar k})^2 + \\&\quad (c_1^2L + 2c_2^2 + 8c_2^2L)(\tau^2+h^2)^2 \qquad 1\leqslant k\leqslant n-1 \\&\hfill (25)\end{aligned}$$

We let

$$E^k = \frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2} + G^{k-1/2}$$

It follows that

$$|G^{k-1/2}| \leqslant \lambda\,\frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2}$$

If $\lambda<1$, then we derive

$$\begin{aligned}&(1-\lambda)\frac{\|e^k\|^2+\|e^{k-1}\|^2}{2} \leqslant E^k \leqslant \\&\quad (1+\lambda)\frac{\|e^k\|^2+\|e^{k-1}\|^2}{2} \leqslant \|e^k\|^2 + \|e^{k-1}\|^2\end{aligned}$$

It follows from inequality (25) that

$$\frac1\tau(E^{k+1} - E^k) \leqslant c_3(E^{k+1} + E^k) +$$

$$(c_1^2L+2c_2^2+8c_2^2L)(\tau^2+h^2)^2 \qquad 1\leqslant k\leqslant n-1$$

That is,

$$(1-c_3\tau)E^{k+1}\leqslant(1+c_3\tau)E^k+ (c_1^2L+2c_2^2+8c_2^2L)\tau(\tau^2+h^2)^2 \qquad 1\leqslant k\leqslant n-1$$

When $c_3\tau\leqslant\frac{1}{3}$, we derive

$$E^{k+1}\leqslant(1+3c_3\tau)E^k+\left(\frac{3c_1^2L}{2}+3c_2^2+12c_2^2L\right)\tau(\tau^2+h^2)^2 \qquad 1\leqslant k\leqslant n-1$$

Using the Gronwall inequality, we obtain

$$E^k\leqslant\exp\{3c_3(k-1)\tau\}\left[E^1+\left(\frac{c_1^2L}{2c_3}+\frac{c_2^2}{c_3}+\frac{4c_2^2L}{c_3}\right)(\tau^2+h^2)^2\right]\leqslant \exp(3c_3T)\left[\|e^1\|^2+\|e^0\|^2+\left(\frac{c_1^2L}{2c_3}+\frac{c_2^2}{c_3}+\frac{4c_2^2L}{c_3}\right)(\tau^2+h^2)^2\right] \qquad 1\leqslant k\leqslant n$$

By substituting Eqs. (21) and (23) into the previously derived inequality, we obtain

$$E^k\leqslant\exp(3c_3T)\left(2c_1^2L+4c_2^2(1+3L)+\frac{c_1^2L}{2c_3}+\frac{c_2^2}{c_3}+\frac{4c_2^2L}{c_3}\right)(\tau^2+h^2)^2 \qquad 1\leqslant k\leqslant n$$

It is easy to know that

$$\frac{\|e^k\|^2+\|e^{k-1}\|^2}{2}\leqslant\frac{c_4^2}{2}(\tau^2+h^2)^2 \qquad 1\leqslant k\leqslant n$$

Consequently,

$$\|e^k\|\leqslant c_4(\tau^2+h^2) \qquad 1\leqslant k\leqslant n$$

This completes the proof.

3 Numerical Examples

In this section, we present two numerical examples. The numerical results illustrate the efficiency of the difference scheme expressed in Eq. (11).

In Ref. [6], we presented the following two-level nonlinear difference scheme:

$$\delta_t u_{j+1/2}^{k+1/2}+\frac{\gamma}{2}[\psi(u^{k+1/2},u^{k+1/2})_j+\psi(u^{k+1/2},u^{k+1/2})_{j+1}]+\delta_x^3u_{j+1/2}^{k+1/2}=0 \qquad 1\leqslant j\leqslant m-2,\ 0\leqslant k\leqslant n-1 \tag{26a}$$

$$\delta_t u_{m-1}^{k+1/2}+\gamma\psi(u^{k+1/2},u^{k+1/2})_{m-1}+\frac{2}{h^2}(\delta_x u_{m-3/2}^{k+1/2}-3\delta_x u_{m-1/2}^{k+1/2})=0 \qquad 0\leqslant k\leqslant n-1 \tag{26b}$$

$$u_j^0=\varphi(x_j) \qquad 0\leqslant j\leqslant m \tag{26c}$$

$$u_0^k=0,\ u_m^k=0 \qquad 1\leqslant k\leqslant n \tag{26d}$$

and solved it using the Newton iterative method.

We make the numerical solution corresponding to the step size h and τ be $\{u_j^k(h,\tau)\mid 0\leqslant j\leqslant m,\ 0\leqslant k\leqslant n\}$.

We denote the error as follows:

$$E(h,\tau)=\max_{1\leqslant k\leqslant n}\sqrt{h\sum_{j=1}^{m-1}\left[u_j^k(h,\tau)-u_{2j}^k\left(\frac{h}{2},\tau\right)\right]^2}$$

$$F(h,\tau)=\left\|u^n(h,\tau)-u^{2n}\left(h,\frac{\tau}{2}\right)\right\|$$

When τ is sufficiently small, the spatial convergence order is defined as follows:

$$r_h=\log_2\left(\frac{E(2h,\tau)}{E(h,\tau)}\right)$$

When h is sufficiently small, the temporal convergence order is defined as follows:

$$r_\tau=\log_2\left(\frac{F(h,2\tau)}{F(h,\tau)}\right)$$

Example 1 In Problem (1), we take $T=1$, $L=1$, $\gamma=-6$, $\varphi(x)=x(x-1)^2(x^3-2x^2+2)$. The exact solution is unknown.

The difference scheme expressed in Eq. (11) will be employed to numerically solve this problem. The numerical accuracy of the difference scheme in space and time will be verified.

By varying step size h with the sufficiently small step size $\tau=1/12\ 800$ and varying step size τ with the sufficiently small step size $h=1/12\ 800$, the numerical errors and convergence orders for Scheme (11) are listed in Tabs. 1 and 2. From these tables, we determine that the numerical convergence orders of Scheme (11) can achieve $O(\tau^2+h^2)$, which is consistent with Theorem 5.

Tab. 1 Errors and space convergence orders of Example 1 ($\tau=1/12\ 800$)

h	Scheme (11)			Scheme (26)		
	$E(h,\tau)$	r_h	CPU time/s	$E(h,\tau)$	r_h	CPU time/s
1/80	8.250×10^{-7}		0.14	8.249×10^{-7}		0.42
1/160	2.062×10^{-7}	2.00	0.19	2.062×10^{-7}	2.00	0.44
1/320	5.156×10^{-7}	2.00	0.28	5.156×10^{-8}	2.00	0.63
1/640	1.289×10^{-8}	2.00	0.48	1.289×10^{-8}	2.00	1.06
1/1 280	3.215×10^{-9}	2.00	0.94	3.210×10^{-9}	2.01	8.40

Tab. 2 Errors and time convergence orders of Example 1 ($h=1/12\ 800$)

τ	Scheme (11)			Scheme (26)		
	$F(h,\tau)$	r_h	CPU time/s	$F(h,\tau)$	r_h	CPU time/s
1/80	1.410×10^{-2}		0.09	4.445×10^{-3}		4.60
1/160	4.363×10^{-3}	1.69	0.16	1.030×10^{-3}	2.11	9.11
1/320	1.005×10^{-3}	2.12	0.32	2.508×10^{-4}	2.04	17.35
1/640	2.451×10^{-4}	2.04	0.63	7.133×10^{-5}	1.81	32.65
1/1 280	7.068×10^{-5}	1.80	1.24	1.843×10^{-5}	1.95	63.29

Furthermore, we observe that the difference scheme expressed in Eq. (11) is more computationally efficient than the nonlinear difference scheme expressed in Eq. (26).

Fig. 1 indicates that the energy of Scheme (11) is conserved for Example 1.

We denote

$$H(h,\tau) = \max_{0\leqslant k\leqslant n} \| e^k(h,\tau) \|$$

and

$$r_h^e = \log_2\left(\frac{H(2h,\tau)}{H(h,\tau)}\right), \quad r_\tau^e = \log_2\left(\frac{H(h,2\tau)}{H(h,\tau)}\right)$$

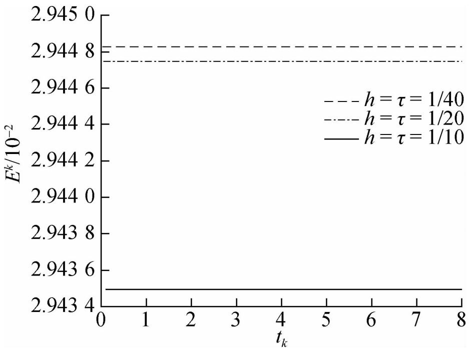

Fig. 1 Energy conservation law of Example 1

Example 2 We conduct a numerical experiment with the exact solution of a solitary wave[19], as follows:

$$u(x,t) = 4\operatorname{sech}^2(x-4t-4)$$
$$0\leqslant x\leqslant 20,\ 0\leqslant t\leqslant 1$$

The corresponding parameter is $\gamma = 3$.

Similarly, we compute the example using the difference scheme expressed in Eq. (11) and observe the convergence. First, we fix the time step size as 1/3 200 and calculate the errors and convergence orders in the space directions, as shown in Tab. 3. Then, we fix the space step size as 1/3 200 and calculate the errors and convergence orders in the time directions, as shown in Tab. 4.

Tab. 3 Errors and space convergence orders of Example 2

h	τ	$H(h,\tau)$	r_h^e
1/10	1/3 200	1.073×10^{-1}	
1/20	1/3 200	2.721×10^{-2}	1.98
1/40	1/3 200	6.855×10^{-3}	1.99
1/80	1/3 200	1.976×10^{-3}	1.80

Tab. 4 Errors and time convergence orders of Example 2

h	τ	$H(h,\tau)$	r_τ^e
1/3 200	1/10	6.082×10^{-1}	
1/3 200	1/20	1.643×10^{-1}	1.89
1/3 200	1/40	4.196×10^{-2}	1.97
1/3 200	1/80	1.057×10^{-2}	1.99

Notably, the numerical results are consistent with the theoretical analysis. In Fig. 2, the numerical and exact solution curves are plotted for $t = 0.25, 0.5, 0.75, 1.0$. Fig. 2 shows that the numerical solutions are consistent with the exact solutions.

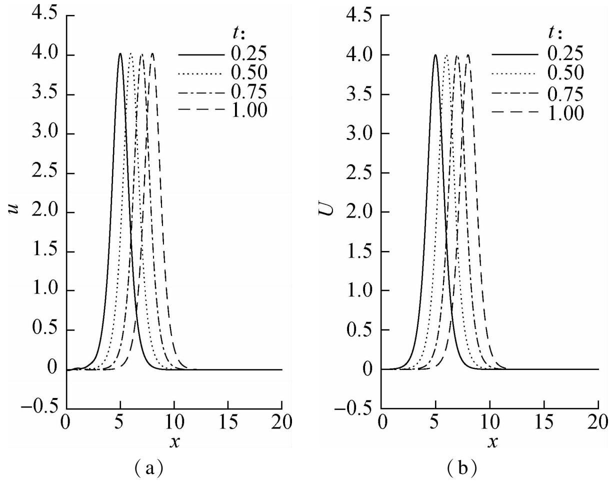

Fig. 2 Curves of Example 2 ($h=\tau=1/40$). (a) Numerical solution; (b) Exact solution

4 Conclusions

In this study, we consider the numerical solution to the initial-boundary value problem of the KdV equation using the finite difference method. With the use of the method of reduction of order, we establish a three-level linearized difference scheme and show that it is more computationally efficient than the two-level nonlinear difference scheme through numerical simulations while retaining the same convergence orders. Using the energy analysis method, we prove the conservation, unique solvability, and conditional convergence of the difference scheme.

In Theorem 5, the convergence of the difference scheme was proven under the constraint of the step size ratio. This difference scheme may be unconditionally convergent. Indeed, we found from Example 1 that the restriction of step size ratio is unnecessary for ensuring that the convergence result holds. However, we have not yet discovered a better method to prove the unconditional convergence and will continue our research in future work.

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Korteweg-de Vries 方程初边值问题的一个二阶收敛线性化差分格式

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摘要：为求解 Korteweg-de Vries 方程的初边值问题，首先利用降阶法得到一个等价的耦合非线性方程组，再对该方程组建立差分格式. 引进的新变量可以从差分格式中分离，得到仅含有原变量的差分格式，该差分格式在实际计算中，每一时间层上只需要解一个四对角的线性方程组，计算量和存储量都很小. 应用能量法对差分格式进行了理论分析，证明了差分格式是唯一可解的，且满足一个与原问题相应的能量守恒律. 在步长比满足一个限制条件下，差分格式是收敛的，时间收敛阶和空间收敛阶都为 2. 数值算例验证了差分格式的收敛阶和数值解满足能量守恒律，且步长比的限制性条件对差分格式的收敛性不是必要的. 通过与一个已知的两层非线性差分格式进行对比，所提出的差分格式在数值计算方面更有优势.

关键词：KdV 方程；线性化差分格式；守恒性；收敛性

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