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Centrally clean elements and central Drazin inverses in a ring

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Abstract: Element a in ring R is called centrally clean if it is the sum of central idempotent e and unit u. Moreover, $a=e+u$ is called a centrally clean decomposition of a and R is called a centrally clean ring if every element of R is centrally clean. First, some characterizations of centrally clean elements are given. Furthermore, some properties of centrally clean rings, as well as the necessary and sufficient conditions for R to be a centrally clean ring are investigated. Centrally clean rings are closely related to the central Drazin inverses. Then, in terms of centrally clean decomposition, the necessary and sufficient conditions for the existence of central Drazin inverses are presented. Moreover, the central cleanness of special rings, such as corner rings, the ring of formal power series over ring R, and a direct product $\prod R_\alpha$ of ring R_α, is analyzed. Furthermore, the central group invertibility of combinations of two central idempotents in the algebra over a field is investigated. Finally, as an application, an example that lists all invertible, central group invertible, group invertible, central Drazin invertible elements, and centrally clean elements of the group ring Z_2S_3 is given.

Key words: centrally clean element; centrally clean ring; central Drazin inverse; central group inverse

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In the research on ring theory, the cleanness of a ring is a basic but important topic. Clean rings originated from the study of exchange rings, which play an important role in the cancellation of modules. The interesting characterizations and properties of clean rings have motivated many scholars to conduct further investigations. The concept of clean rings was first introduced by Nicholson[1] in 1977. Subsequently, Nicholson et al.[2] proved that the linear transformation of a countable vector space over a division ring is clean. In 1999, Nicholson[3] introduced a strongly clean ring and presented some equivalent characterizations of strongly clean elements and rings. In 2001, Han et al.[4] investigated the cleanness of group rings, the ring of formal power series over a ring, and a direct product of rings. In 2011, Hiremath et al.[5] presented some characterizations of strongly clean rings. More details concerning the cleanness of the rings can be found in Refs. [6 - 12].

Throughout the paper, R denotes an associative ring with unity 1. The center of R is denoted by $C(R)=\{x\in R: ax=xa \text{ for all } a\in R\}$. The element $e\in R$ is considered idempotent if $e^2=e$. In contrast, the element $e\in R$ is considered central idempotent if $e^2=e$ and $e\in C(R)$. The symbols $E(R)$, $\mathrm{CE}(R)$, $U(R)$, and $J(R)$ denote the sets of all idempotents, central idempotents, invertible elements, and Jacobson radicals of R, respectively. Recall that the element $a\in R$ is considered clean if $u\in U(R)$ and $e\in E(R)$ exist such that $a=u+e$. The element $a\in R$ is considered strongly clean if $u\in U(R)$ and $e\in E(R)$ exist such that $a=u+e$ and $ue=eu$. In this case, $a=e+u$ is considered a strongly clean decomposition.

Drazin[13] introduced the concept of pseudo-inverse (usually called Drazin inverse) in rings and semigroups. The element $a\in R$ is considered a Drazin invertible if $x\in R$ and the nonnegative integer k exist such that $xax=x$, $ax=xa$, $xa^{k+1}=a^k$. Such x is unique if it exists and is considered the Drazin inverse of a. The smallest nonnegative integer k satisfying the previously presented equations is called the Drazin index of a. If $k=1$, then x is considered the group inverse of a.

Further research showed that there is a close connection between clean rings and Drazin inverses. For example, Zhu et al.[14] proved that $a\in R$ is a Drazin invertible if and only if $u\in U(R)$, $e\in E(R)$ and the positive integer n exist such that $a^n=u+e$ is a strongly clean decomposition and $a^nR\cap eR=0$. Moreover, many scholars investigated the Drazin invertibility of combinations of idempotents. For instance, Liu et al.[15] analyzed this topic in complex matrices, i. e., Drazin invertibility of $aP+bQ+cPQ+dQP+ePQP+fQPQ+gQPQP$ for idempotent complex matrices P and Q under the conditions $(PQ)^2=(QP)^2$. More details concerning Drazin inverses can be found in Refs. [16 - 26].

In 2019, to analyze the commutative properties of Drazin inverses (see Example 2. 8 in Ref. [27]), Wu et al.[28] introduced the concept of central Drazin inverses.

Definition 1[28] Element $a\in R$ is considered a central Drazin invertible if $x\in R$ and the nonnegative integer k exist such that $xax=x$, $xa\in C(R)$, $xa^{k+1}=a^k$. Such x is unique if it exists and is considered the central Drazin in-

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verse of a, denoted by a^c. The smallest nonnegative integer k satisfying the previously presented equations is still the Drazin index of a, denoted by ind(a). If $k=1$, then x is called the central group inverse of a, denoted by $a^{©}$.

In Ref. [28], a centrally clean element and a centrally clean ring are also introduced.

Definition 2[28] Let $a \in R$. If $u \in U(R)$ and $e \in \mathrm{CE}(R)$ exist such that $a = u + e$, then a is considered centrally clean. In this case, $a = u + e$ is considered a centrally clean decomposition of a. Thus, centrally clean is clean. If every element of R is centrally clean, then R is considered a centrally clean ring.

Moreover, $a \in R$ is a central Drazin invertible if and only if $u \in U(R)$, $e \in \mathrm{CE}(R)$ and the positive integer n exist such that $a = u + e$ is a centrally clean decomposition, and $a^n R \cap eR = 0$, or equivalently, $u \in U(R)$, $e \in \mathrm{CE}(R)$ and the positive integer n exist such that $a = u + e$ is a centrally clean decomposition and ae is nilpotent. Subsequently, Zhao et al.[29] investigated the one-sided central Drazin inverses.

Motivated by the previous studies, we investigated centrally clean elements and central Drazin inverses in R. We first give an example and characterizations of centrally clean elements. Then, we analyze the properties of centrally clean rings and provide some equivalent characterizations. Moreover, we present the necessary and sufficient conditions for the existence of central Drazin inverses in terms of centrally clean decompositions. In addition, we investigate the central group invertibility of combinations of two central idempotents. Finally, we calculate all invertible, central group invertible, group invertible, central Drazin invertible elements, and centrally clean elements of the group ring $Z_2 S_3$.

1 Characterization of Centrally Clean Elements

First, we provide an example of centrally clean elements.

Example 1

1) Units are centrally clean.

2) The elements in $J(R)$ are centrally clean.

3) Nilpotent elements are centrally clean.

4) Central idempotents are centrally clean.

Proof 1) and 3) are obvious.

2) Let $x \in J(R)$. Notably, $J(R) = \{x \in R\colon 1 - ax$ is left invertible for any $a \in R\}$ and $J(R) = \{x \in R\colon 1 - xa$ is right invertible for any $a \in R\}$.

Then, we take $a = 1$, and it follows that $1 - x \in U(R)$. Hence, x is centrally clean.

4) Let $e \in \mathrm{CE}(R)$. Given that $(2e-1)^2 = 1$ and $(1-e)^2 = 1 - e$, it follows that $2e - 1 \in U(R)$ and $1 - e \in \mathrm{CE}(R)$. Then, $e = (2e-1) + (1-e)$. Hence, e is centrally clean.

Let $a \in R$. Then, we use Ra and aR to denote the left and right ideals generated by a, respectively. We use $l(a)$ and $r(a)$ to denote the left and right annihilators of a, respectively. That is,

$$Ra = \{ra\colon\ r \in R\},\ aR = \{ar\colon\ r \in R\}$$

$$l(a) = \{x \in R\colon\ xa = 0\},\ r(a) = \{x \in R\colon\ ax = 0\}$$

Nicholson[3] proved that if $e \in E(R)$ and $a \in eRe$ is strongly clean, then $a \in R$ is also strongly clean. Moreover, he provided some characterizations of strongly clean elements. Then, we investigate the relevant characterizations of centrally clean elements.

Lemma 1[3] Let $a \in R$ and $e \in E(R)$ with $ea = ae$. Then, the following conditions are equivalent:

1) $ae \in U(eRe)$.

2) $e \in Ra$ and $l(a) \subseteq l(e)$.

3) $e \in aR$ and $r(a) \subseteq r(e)$.

Theorem 1 Let $a \in R$. Then, the following conditions are equivalent:

1) a is centrally clean.

2) $e \in \mathrm{CE}(R)$ exists such that $l(a) \subseteq R(1-e) \subseteq R(1-a)$ and $l(1-a) \subseteq Re \subseteq Ra$.

3) $e \in \mathrm{CE}(R)$ exists such that $r(a) \subseteq (1-e)R \subseteq (1-a)R$ and $l(1-a) \subseteq eR \subseteq aR$.

4) $e \in \mathrm{CE}(R)$ exists such that $ea \in U(eR)$ and $(1-e)(1-a) \in U((1-e)R)$.

5) $e \in \mathrm{CE}(R)$ exists such that ea is centrally clean in eR and $(1-e)(1-a)$ is centrally clean in $(1-e)R$.

6) $e \in \mathrm{CE}(R)$ exists such that ea is centrally clean in eR and $(1-e)a$ is centrally clean in $(1-e)R$.

7) The decomposition $1 = e_1 + e_2 + \cdots + e_n$ exists, where n is a positive integer, e is a centrally orthogonal idempotent, and $e_i a$ is centrally clean in $e_i R$ for each positive integer i.

Proof 1)$\Rightarrow$2). Given that a is centrally clean, we can suppose that $a = (1-e) + u$, where $e \in \mathrm{CE}(R)$ and $u \in U(R)$. If $ra = 0$, then $r(1-e) + ru = 0$, and it follows that $r = ruu^{-1} = [-r(1-e)]u^{-1} \in R(1-e)$, i. e., $l(a) \subseteq R(1-e)$. Moreover, from $ae = [(1-e) + u]e = ue$, we derive $e = u^{-1}ae = u^{-1}ea \in Ra$, i. e., $Re \subseteq Ra$.

Rewrite $a = (1-e) + u$ as $1 - a = e + (-u)$. Then, by a similar argument, we can obtain $l(1-a) \subseteq Re$ and $R(1-e) \subseteq R(1-a)$.

2)$\Rightarrow$4). From $e \in Re \subseteq Ra$ and $l(a) \subseteq R(1-e) = l(e)$, we can obtain $ea \in U(eR)$ based on Lemma 1. Similarly, we can derive $(1-e)(1-a) \in U((1-e)R)$.

1)$\Rightarrow$3)$\Rightarrow$4) are similar to 1)$\Rightarrow$2)$\Rightarrow$4).

4)$\Rightarrow$5). This is obvious from 1) of Example 1.

5)$\Rightarrow$6). Given that $(1-e)(1-a)$ is centrally clean in $(1-e)R$, it follows that $(1-e)a = (1-e) - (1-e)(1-a)$ is also centrally clean in $(1-e)R$.

6)$\Rightarrow$7) Write $e_1 = e$ and $e_2 = 1 - e$. Then, $e_1 e_2 = 0$ and $1 = e_1 + e_2$. Moreover, $e_i a$ is centrally clean in $e_i R$

for each i.

7) $\Rightarrow$1). For each positive integer i, $e_i a$ is centrally clean in $e_i R$. Then, we suppose that $e_i a = f_i + u_i$, where $f_i \in \mathrm{CE}(e_i R)$ and $u_i \in U(e_i R)$, and it follows that $v_i \in e_i R$ exists such that $v_i u_i = u_i v_i = e_i$. Given that e_i is a centrally orthogonal idempotent, we derive $f = \sum f_i \in \mathrm{CE}(R)$, $u = \sum u_i \in U(R)$, and $u^{-1} = \sum v_i$. Hence, $a = \sum e_i a = \sum (f_i + u_i) = \sum f_i + \sum u_i = f + u$. Therefore, a is centrally clean.

Proposition 1 Let $a \in R$. Then, the following conditions are equivalent:

1) a is centrally clean.

2) $v \in U(R)$ and $f \in \mathrm{CE}(R)$ exist such that $f = fva$ and $1 - f = -(1-f)v(1-a)$.

3) $u \in U(R)$ and $f \in \mathrm{CE}(R)$ exist such that $f = fua$ and $1 - f = (1-f)u(1-a)$.

4) $f \in \mathrm{CE}(R)$ and $x, y \in R$ exist such that $f = fxa$, $1 - f = (1-f)y(1-a)$, and $fx - (1-f)y \in U(R)$.

Proof 1)$\Rightarrow$2). Let $a = u + e$, where $u \in U(R)$ and $e \in \mathrm{CE}(R)$. Write $f = 1 - e$ and $v = u^{-1}$. Then,

$$f = (1-e)u^{-1}a = fva$$

and

$$1 - f = -eu^{-1}(1-a) = -(1-f)v(1-a)$$

2)$\Rightarrow$1). Write $e = 1 - f$. Then, $v(a-e) = [fv + (1-f)v](a - 1 + f) = fva - fv + fv - (1-f)v(1-a) = f + 1 - f = 1$. Hence, $a - e = v^{-1}$, i.e., a is centrally clean.

2)$\Rightarrow$3). Write $u = (2f-1)v$. Then, $f = fva = fua$ and $1 - f = -(1-f)v(1-a) = (1-f)u(1-a)$.

3)$\Rightarrow$2) is similar to 2)$\Rightarrow$3).

2)$\Rightarrow$4). Write $x = v$ and $y = -v$. Then, $f = fxa$ and $1 - f = (1-f)y(1-a)$.

4)$\Rightarrow$2). Write $v = fx - (1-f)y \in U(R)$. Then, $fx = fv$ and $-(1-f)y = (1-f)v$, and it follows that $f = fxa = fva$ and $1 - f = (1-f)y(1-a) = -(1-f)v(1-a)$.

2 Characterizations of Centrally Clean Rings

Recall that if $R/J(R)$ is a division ring, then R is considered a local ring.

Proposition 2 Every local ring is a centrally clean ring.

Proof Let $a \in R$. If $a \in J(R)$, then a is centrally clean based on 2) of Example 1. If $a \notin J(R)$, then $a + J(R) \in U(R/J(R))$. Hence, $x + J(R) \in R/J(R)$ exists such that

$$(a + J(R))(x + J(R)) = 1 + J(R)$$

and it follows that $ax + J(R) = 1 + J(R)$, i.e., $ax - 1 \in J(R)$. Therefore, $ax = 1 - (1 - ax) \in U(R)$. Then, a is right invertible in R. Similarly, we can deduce that a is left invertible in R. It follows that $a \in U(R)$, and hence, it is centrally clean.

In 2004, Nicholson et al.[12] proved that if $R \neq 0$, then $R[x]$ is not clean. Then, it is obvious that $R[x]$ is not centrally clean when $R \neq 0$.

Proposition 3 The following conditions are equivalent:

1) $2 \in U(R)$, and R is centrally clean.

2) For any $a \in R$, $u \in U(R)$ and $x \in C(R)$ exist, with $x^2 = 1$, such that $a = u + x$.

Proof 1)$\Rightarrow$2). Let $a \in R$. Given that R is centrally clean, $u \in U(R)$ and $e \in \mathrm{CE}(R)$ exist such that $\frac{1+a}{2} = e + u$, and it follows that $a = (2e-1) + 2u$. From $2 \in U(R)$, we derive $2u \in U(R)$. Moreover, $(2e-1)^2 = 1$ and $2e - 1 \in C(R)$.

2)$\Rightarrow$1). Hypothetically, we have $1 = x + v$, where $v \in U(R)$, $x^2 = 1$ and $x \in C(R)$. Then, $(1-v)^2 = x^2 = 1$, and it follows that $v^2 = 2v$. Hence, from $v \in U(R)$, we derive $2 = v \in U(R)$. Let $a \in R$. Then, $2a - 1 = y + w$, where $w \in U(R)$, $y^2 = 1$ and $y \in C(R)$. Therefore, $a = \frac{y+1}{2} + \frac{w}{2}$ is centrally clean.

Then, we provide some characterizations of centrally clean rings.

Theorem 2 The following conditions are equivalent:

1) R is centrally clean.

2) Every element $x \in R$ can be written as $x = u - e$, where $u \in U(R)$ and $e \in \mathrm{CE}(R)$.

3) Every element $x \in R$ can be written as $x = u + e$, where $u \in U(R) \cup 0$ and $e \in \mathrm{CE}(R)$.

4) Every element $x \in R$ can be written as $x = u - e$, where $u \in U(R) \cup 0$ and $e \in \mathrm{CE}(R)$.

Proof 1)$\Rightarrow$2). Let $x \in R$. Then, $-x = e + v$, and it follows that $x = -v - e$, $u = -v \in U(R)$ and $e \in \mathrm{CE}(R)$.

2)$\Rightarrow$3) and 3)$\Rightarrow$4) are similar to 1)$\Rightarrow$2).

4)$\Rightarrow$1). Let $x \in R$. Then, we derive $-x = u - e$ based on the assumption, where $u \in U(R) \cup 0$ and $e \in \mathrm{CE}(R)$. Hence, $x = (-u) + e$. The case when $u = 0$ follows from 4) of Example 1.

Recall that if $e \in E(R)$ exists such that $e \in aR$ and $1 - e \in (1-a)R$ for any $a \in R$, then R is an exchange ring. Moreover, if every idempotent of R is central, then R is called abelian.

Theorem 3 The following conditions are equivalent:

1) R is centrally clean.

2) R is an exchange and abelian.

3) R is clean and abelian.

4) For any $a \in R$, $e \in \mathrm{CE}(R)$ exists such that $e \in aR$ and $1 - e \in (1-a)R$.

Proof 1)$\Rightarrow$3). It suffices to prove that every idempotent of R is central. Let $e \in E(R)$. Then, we derive $e = f + u$, where $f \in \mathrm{CE}(R)$ and $u \in U(R)$, and it follows that if $f + u = (f+u)^2 = f + 2fu + u^2$, then $1 = u + 2f$.

Hence, we obtain $e = f + u = f + 1 - 2f = 1 - f \in \mathrm{CE}(R)$.

3)$\Rightarrow$4). Given that clean rings are exchange rings, it follows that, for any $a \in R$, $e \in E(R)$ exists such that $e \in aR$ and $1 - e \in (1 - a)R$. Given that R is abelian, we derive $e \in \mathrm{CE}(R)$.

4)$\Rightarrow$2). This is enough to show that R is abelian. Let $f \in E(R)$. Then, according to the assumption, $e \in \mathrm{CE}(R)$ exists such that $e \in fR$ and $1 - e \in (1 - f)R$. Hence, we obtain $fe = e$ and $(1 - f)(1 - e) = 1 - e$. Then, $f = e \in \mathrm{CE}(R)$. Therefore, R is abelian.

2)$\Rightarrow$1). Let $x \in R$. Given that R is an exchange ring, $e \in E(R)$ exists such that $e \in xR$ and $1 - e \in (1 - x)R$. Let $e = xa'$, where $a' \in R$. Then, $e = e^2 = xa'xa'$. Write $a = a'xa'$, and it follows that $e = xa$ and $ae = a'xa'xa' = a'xa' = a$. Then, $axa = a$. Given that R is abelian, we derive $ax = axax = xa(ax) = xaxa = xa$. By a similar argument, we can obtain $(1 - e) = (1 - x)b$, $b(1 - e) = b$, and $(1 - x)b = b(1 - x)$. Furthermore, we can obtain $[x - (1 - e)](a - b) = xa - xb - (1 - e)a + (1 - e)b = e + (1 - x)b = 1$ and $(a - b)[x - (1 - e)] = 1$. That is, $[x - (1 - e)]^{-1} = a - b$. Then, $x = x - (1 - e) + (1 - e)$. Hence, R is centrally clean.

Proposition 4 Let $p \in \mathrm{CE}(R)$. Then, $a \in pR$ is centrally clean in R if and only if a is centrally clean in pR.

Proof The necessity is clearly stated in Theorem 1. Conversely, assume $a \in pR$ is centrally clean in R. Then, $e \in \mathrm{CE}(R)$ and $u \in U(R)$ exist such that $a = e + u$, and it follows that $pa = pe + pu$, $pe \in \mathrm{CE}(pR)$, and $pu \in U(pR)$. From $pa = a$, we derive $a = pe + pu$. Hence, a is centrally clean in pR.

Corollary 1 Let $p \in \mathrm{CE}(R)$. If R is a centrally clean ring, then so is pR.

Han et al.[4] proved that when $e \in E(R)$, if eRe and $(1 - e)R(1 - e)$ are clean rings, then so is R. Here, we consider the case of $e \in \mathrm{CE}(R)$.

Corollary 2 Let $e \in \mathrm{CE}(R)$. If eR and $(1 - e)R$ are centrally clean, then so is R.

Proof This is clearly stated in Theorem 1.

Han et al.[4] also investigated the cleanness of group rings, the ring of formal power series over a ring, and a direct product of rings. Then, we analyze the relevant results of $R[[x]]$ and $\prod R_\alpha$.

Proposition 5 The ring $R[[x]]$ is centrally clean if and only if R is centrally clean.

Proof Let $f = a + bx + cx^2 + \cdots \in R[[x]]$. Given that R is centrally clean, we can suppose that $a = u + e$, where $e \in \mathrm{CE}(R)$ and $u \in U(R)$. Then, $f = e + (u + bx + cx^2 + \cdots)$, $e \in \mathrm{CE}(R[[x]])$, and $u + bx + cx^2 + \cdots \in U(R[[x]])$. Therefore, $R[[x]]$ is centrally clean.

Conversely, we know that $R[[x]]/(x)$ is centrally clean because $R[[x]]$ is centrally clean. Hence, $R \cong R[[x]]/(x)$ is centrally clean.

Lemma 2 Let R, S be two rings and φ: $R \to S$ be a surjective ring homomorphism. If R is centrally clean, then so is S.

Proof It is obvious.

Proposition 6 A direct product $R = \prod R_\alpha$ is centrally clean if and only if R_α is centrally clean.

Proof Given that π_α: $\prod R_\alpha \to R_\alpha$ is a surjective ring homomorphism, it follows that R_α is centrally clean based on Lemma 2.

Conversely, suppose that R_α is centrally clean. Let $x = (x_\alpha) \in \prod R_\alpha$. Then, for each α, we derive $x_\alpha = u_\alpha + e_\alpha$, where $u_\alpha \in U(R_\alpha)$ and $e_\alpha \in \mathrm{CE}(R_\alpha)$. Hence, we obtain $x = e + u$, $u = (u_\alpha) \in U(\prod R_\alpha)$ and $e = (e_\alpha) \in \mathrm{CE}(\prod R_\alpha)$, and it follows that $R = \prod R_\alpha$ is centrally clean.

Let L be a two-sided ideal of R. We suppose that the idempotents can be lifted modulo L if, given that $x \in E(R/L)$, $e \in E(R)$ exists such that $e - x \in L$. Similarly, we can define the concept that the central idempotents can be lifted modulo L if $e \in \mathrm{CE}(R)$ exists such that $e - x \in L$ for $x \in \mathrm{CE}(R/L)$.

Proposition 7 R is centrally clean if and only if $R/J(R)$ is centrally clean, and the central idempotent can be lifted modulo $J(R)$.

Proof Based on Lemma 2, we confirm that the factor ring of a centrally clean ring is centrally clean. Then, $R/J(R)$ is centrally clean. Given that a centrally clean ring is exchange, it follows that the idempotents can be lifted modulo $J(R)$. Based on Theorem 3, we determine that the idempotents of R are central. Then, the sufficiency is proven.

Conversely, let $x \in R$ and $\bar{x} = \bar{e} + \bar{u}$, where $\bar{e} \in \mathrm{CE}(R/J(R))$ and $\bar{u} \in U(R/J(R))$, which indicates that $\bar{v} \in R/J(R)$ exists such that $\bar{u}\bar{v} = \bar{v}\bar{u} = \bar{1}$. Then, $r_1, r_2 \in J(R)$ exist such that $uv = 1 + r_1$ and $vu = 1 + r_2$. Hence, $u \in U(R)$. Given that the central idempotents can be lifted modulo $J(R)$, we suppose that $p \in \mathrm{CE}(R)$ and $p - e \in J(R)$, and it follows that $r \in J(R)$ exists such that $x = p + u + r = p + u(1 + u^{-1}r)$. Given that $u^{-1}r \in J(R)$, we derive $1 + u^{-1}r \in U(R)$. Hence, R is centrally clean.

3 Characterizations of Central Drazin Inverses

In this section, we mainly provide some characterizations for the existence of central Drazin inverses.

Theorem 4 Let $a \in R$. Then, the following conditions are equivalent:

1) a is central Drazin invertible.

2) $u \in U(R)$, $e \in \mathrm{CE}(R)$, and the positive integer m exist such that $a^m = eu$ and $au = ua$.

3) $v \in U(R)$ and $f \in \mathrm{CE}(R)$ exist such that $a = f + v$ and $af \in R^{\mathrm{nil}}$.

4) $p \in \mathrm{CE}(R)$ exists such that $ap \in U(pR)$ and $a(1 - p) \in R^{\mathrm{nil}}$.

Proof 1)$\Rightarrow$2). Write $e = aa^c$. Then, $e \in \mathrm{CE}(R)$. Given that a is central Drazin invertible, and it follows that the positive integer m exists such that $a^m = a^m aa^c =$

$a^m e$. Write $u = a^m + (1-e)$. Then, $[a^m + (1-e)][(a^c)^m e + (1-e)] = a^m (a^c)^m e + a^m (1-e) + (1-e)(a^c)^m e + (1-e)^2 = e + 1 - e = 1$. Hence, we have $u \in U(R)$ and $u^{-1} = (a^c)^m e + (1-e)$, and it follows that $a^m = a^m e = [u-(1-e)]e = eu$ and $au = a[a^m + (1-e)] = a^{m+1} + a(1-e) = a^{m+1} + (1-e)a = [a^m + (1-e)]a = ua$.

2)$\Rightarrow$3). Write $f = 1-e$. Then, $f \in \mathrm{CE}(R)$. Given that $(a^m - f)(u^{-1}e - f) = 1$, we derive $a^m - f \in U(R)$. Then, $(a-f)(a^{m-1} + a^{m-2}f + \cdots + af + f) = a^m - f \in U(R)$. Hence, we obtain $v = a - f \in U(R)$ and $(af)^m = a^m f = eu(1-e) = 0$, i.e., $af \in R^{\mathrm{nil}}$.

3)$\Rightarrow$4). Write $p = 1-f$. Then, $p \in \mathrm{CE}(R)$, $ap = pa = pv \in U(pR)$ and $a(1-p) = af \in R^{\mathrm{nil}}$.

4)$\Rightarrow$1). Based on this assumption, it follows that $w \in U(pR)$ exists such that $apw = pwa = p$. From $pw = w$, we derive $aw = wa = p \in C(R)$, $waw = pw = w$, and $a - a^2 w = a(1-aw) = a(1-p) \in R^{\mathrm{nil}}$. Hence, a is central Drazin invertible.

Zhu et al.[14] showed that a is Drazin invertible if and only if $u \in U(R)$, $e \in E(R)$, and the positive integer m exist such that $a^m = u + e$ is strongly clean decomposition and $a^m R \cap eR = 0$. Here, we investigate the relevant results of central Drazin inverses.

Theorem 5 Let $a \in R$. Then, the following conditions are equivalent:

1) a is central Drazin invertible.

2) $u \in U(R)$, $e \in \mathrm{CE}(R)$, and the positive integer n exist such that $a^n = u + e$ and $a^n R \cap eR = 0$.

3) $u \in U(R)$, $e \in \mathrm{CE}(R)$, and the positive integer n exist such that $a^n = u - e$ and $a^n R \cap eR = 0$.

Proof 1)$\Rightarrow$2). Given that a is central Drazin invertible, we derive $u = a^n - 1 + aa^c \in U(R)$ for any positive integer $n > \mathrm{ind}(a)$. Write $e = 1 - aa^c$. Then, $a^n = u + e$ is the centrally clean decomposition. Let $x \in a^n R \cap eR$. Then, $y, z \in R$ exist such that $x = a^n y = ez = ea^n y = 0$. Hence, $a^n R \cap eR = 0$.

2)$\Rightarrow$1). From $e \in \mathrm{CE}(R)$, it follows that the positive integer m exists such that:

$$(a^n e)^m = (a^n)^m e = e(a^n)^m \in a^n R \cap eR = 0$$

i.e., $a^n e \in R^{\mathrm{nil}}$. Let m be the nilpotent index of $a^n e$. Then, $(a^n)^m = u^m (1-e)$. In fact,

$$\begin{aligned}
&(a^n)^m = (u+e)^m = \\
&\quad u^m + \tbinom{m}{1} u^{m-1} e + \tbinom{m}{2} u^{m-2} e + \cdots + \tbinom{m}{m-1} ue + e = \\
&\quad u^m + (u^m e - u^m e) + \tbinom{m}{1} u^{m-1} e + \tbinom{m}{2} u^{m-2} e + \\
&\quad \cdots + \tbinom{m}{m-1} ue + e = \\
&\quad u^m (1-e) + (u^m e + \tbinom{m}{1} u^{m-1} e + \tbinom{m}{2} u^{m-2} e + \\
&\quad \cdots + \tbinom{m}{m-1} ue + e) = \\
&\quad u^m (1-e) + (u^m + \tbinom{m}{1} u^{m-1} + \tbinom{m}{2} u^{m-2} + \\
&\quad \cdots + \tbinom{m}{m-1} u + e) e = \\
&\quad u^m (1-e) + (a^n)^m e = \\
&\quad u^m (1-e) + (a^n e)^m = u^m (1-e)
\end{aligned}$$

Hence, $(a^n)^m$ is central group invertible derived by Theorem 3.6 in Ref. [28]. Then, a is central Drazin invertible derived by Theorem 3.3 in Ref. [28].

1)$\Leftrightarrow$3). This is similar to the proof of 1)$\Leftrightarrow$2).

From Theorem 5, we derive the following corollary.

Proposition 8 Let $a \in R$. Then, the following conditions are equivalent:

1) a is central Drazin invertible.

2) $e \in \mathrm{CE}(R)$ and the positive integer n exist such that $a^n e = 0$ and $a^n - e \in U(R)$.

3) $e \in \mathrm{CE}(R)$ and the positive integer n exist such that $a^n e = 0$ and $a^n + e \in U(R)$.

Proof 1)$\Rightarrow$2). Given that a is central Drazin invertible, we derive $u = a^n - 1 + aa^c \in U(R)$ for any positive integer $n > \mathrm{ind}(a)$. Write $e = 1 - aa^c$. Then, $a^n e = 0$ and $a^n - e \in U(R)$.

2)$\Rightarrow$1). Let $x \in a^n R \cap eR$. Then, $y, z \in R$ exist such that $x = a^n y = ez = ea^n y = a^n ey = 0$. Hence, $a^n R \cap eR = 0$. According to Theorem 5, the proof is completed.

1)$\Leftrightarrow$3). This is similar to the proof of 1)$\Leftrightarrow$2).

For the central group inverses, we also obtain the following relevant results.

Proposition 9 Let $a \in R$. Then, the following conditions are equivalent:

1) a is central group invertible.

2) $u \in U(R)$ and $e \in \mathrm{CE}(R)$ exist such that $a = u + e$ and $aR \cap eR = 0$.

3) $v \in U(R)$ and $f \in \mathrm{CE}(R)$ exist such that $f = fva$, $1 - f = (1-f)v(1-a)$, and $af = a$.

Proof 1)$\Leftrightarrow$2). This is given in Corollary 4.6 in Ref. [28].

2)$\Rightarrow$3). Write $f = 1-e$ and $v = u^{-1}(1-2e)$. Then, $v \in U(R)$ and $f \in \mathrm{CE}(R)$, and it follows that $fva = (1-e)u^{-1}(1-2e)(u+e) = (1-e)(1-2e) = 1-e = f$ and $a(1-f) = (u+e)e \in aR \cap eR = 0$. Hence, $af = a$ and

$$\begin{aligned}
(1-f)v(1-a) &= eu^{-1}(1-2e)(1-e-u) = \\
&eu^{-1}(1-2e)(-u) = e = 1-f
\end{aligned}$$

3)$\Rightarrow$2). Write $u = v^{-1}(2f-1)$ and $e = 1-f$. Given that $1-f = (1-f)v(1-a) = (1-f)v - (1-f)va = (1-f)v - va + fva = (1-f)v - va + f$, it follows that $a = v^{-1}(1-f)v + v^{-1}(2f-1) = 1 - f + v^{-1}(2f-1) = e + u$. Let $x \in aR \cap eR$. Then, $r, t \in R$ exist such that $x = ar = et$, and it follows that $fr = fvar = fv(et) = fv(1-f)t = 0$, i.e., $r = (1-f)r$. Then, $x = ar = a(1-f)r = (a - af)r = 0$. Therefore, $aR \cap eR = 0$.

4 Central Group Invertibility of Combinations of Two Central Idempotents

Motivated by the study conducted by Liu et al.[15], we investigate the central group invertibility of combinations of two central idempotents in this section.

In this section, F denotes a field and A denotes the algebra over F.

Theorem 6 Let $p, q \in A$ be the central idempotent and $a = d_1 p + d_2 q + d_3 pq$, where $d_i \in F$, $i = 1, 2, 3$. Then, a is central group invertible, and

$$a^{©} = d_1^{\dagger} p + d_2^{\dagger} q + (d^{\dagger} - d_1^{\dagger} - d_2^{\dagger}) pq$$

where

$$d = d_1 + d_2 + d_3, \quad d^{\dagger} = \begin{cases} \dfrac{1}{d} & d \neq 0 \\ 0 & d = 0 \end{cases}$$

Proof Let $x = d_1^{\dagger} p + d_2^{\dagger} q + (d^{\dagger} - d_1^{\dagger} - d_2^{\dagger}) pq$, which suffices to prove that x is the central group inverse of a.

Given that $p, q \in \mathrm{CE}(R)$, we derive $xa \in C(R)$. By computation, it follows that

$$\begin{aligned} ax = {} & (d_1 p + d_2 q + d_3 pq)[d_1^{\dagger} p + d_2^{\dagger} q + (d^{\dagger} - d_1^{\dagger} - d_2^{\dagger}) pq] = \\ & d_1 d_1^{\dagger} p + d_2 d_2^{\dagger} q + [d_1(d_2^{\dagger} + d^{\dagger} - d_1^{\dagger} - d_2^{\dagger}) + \\ & d_2(d_1^{\dagger} + d^{\dagger} - d_1^{\dagger} - d_2^{\dagger}) + d_3(d_1^{\dagger} + d_2^{\dagger} + d^{\dagger} - d_1^{\dagger} - d_2^{\dagger})] pq = \\ & d_1 d_1^{\dagger} p + d_2 d_2^{\dagger} q + [d_1(d^{\dagger} - d_1^{\dagger}) + d_2(d^{\dagger} - d_2^{\dagger}) + d_3 d^{\dagger}] pq = \\ & d_1 d_1^{\dagger} p + d_2 d_2^{\dagger} q + (dd^{\dagger} - d_1 d_1^{\dagger} - d_2 d_2^{\dagger}) pq \end{aligned}$$

Then, we obtain

$$\begin{aligned} xax = {} & [d_1^{\dagger} p + d_2^{\dagger} q + (d^{\dagger} - d_1^{\dagger} - d_2^{\dagger}) pq][d_1 d_1^{\dagger} p + d_2 d_2^{\dagger} q + \\ & (dd^{\dagger} - d_1 d_1^{\dagger} - d_2 d_2^{\dagger}) pq] = \\ & d_1^{\dagger} d_1 d_1^{\dagger} p + d_2^{\dagger} d_2 d_2^{\dagger} q + [d_1^{\dagger}(d_2 d_2^{\dagger} + dd^{\dagger} - d_1 d_1^{\dagger} - d_2 d_2^{\dagger}) + \\ & d_2^{\dagger}(d_1 d_1^{\dagger} + dd^{\dagger} - d_1 d_1^{\dagger} - d_2 d_2^{\dagger}) + (d^{\dagger} - d_1^{\dagger} - d_2^{\dagger}) dd^{\dagger}] pq = \\ & d_1^{\dagger} p + d_2^{\dagger} q + (d^{\dagger} - d_1^{\dagger} - d_2^{\dagger}) pq = x \end{aligned}$$

and

$$\begin{aligned} a^2 x = aax = {} & (d_1 p + d_2 q + d_3 pq)[d_1 d_1^{\dagger} p + d_2 d_2^{\dagger} q + \\ & (dd^{\dagger} - d_1 d_1^{\dagger} - d_2 d_2^{\dagger}) pq] = \\ & d_1 d_1 d_1^{\dagger} p + d_2 d_2 d_2^{\dagger} q + [d_1(dd^{\dagger} - d_1 d_1^{\dagger}) + \\ & d_2(dd^{\dagger} - d_2 d_2^{\dagger}) + d_3 dd^{\dagger}] pq = \\ & d_1 d_1 d_1^{\dagger} p + d_2 d_2 d_2^{\dagger} q + \\ & [d_1 dd^{\dagger} - d_1 + d_2 dd^{\dagger} - d_2 + d_3 dd^{\dagger}] pq = \\ & d_1 d_1 d_1^{\dagger} p + d_2 d_2 d_2^{\dagger} q + d_3 pq = a \end{aligned}$$

Hence, x is the central group inverse of a, and $a^{©} = d_1^{\dagger} p + d_2^{\dagger} q + (d^{\dagger} - d_1^{\dagger} - d_2^{\dagger}) pq$, where

$$d = d_1 + d_2 + d_3, \quad d^{\dagger} = \begin{cases} \dfrac{1}{d} & d \neq 0 \\ 0 & d = 0 \end{cases}$$

Let $d_1 = 1$, $d_2 = 1$, $d_3 = 0$. Then, we obtain the following results according to Theorem 6.

Corollary 3 Let $2 \in U(R)$ and $p, q \in A$ be the central idempotents. Then, $p + q$ is central group invertible, and

$$(p + q)^{©} = p + q - \frac{3}{2} pq$$

If $pq = p$, then we obtain the following results according to Theorem 6. That is, we take $d_3 = 0$ in Theorem 6.

Corollary 4 Let $p, q \in A$ be the central idempotent and $pq = p$. Then, $d_1 p + d_2 q$ is central group invertible, and

$$(d_1 p + d_2 q)^{©} = (d^{\dagger} - d_2^{\dagger}) p + d_2^{\dagger} q$$

If $pq = q$, then we obtain the following results according to Theorem 6.

Corollary 5 Let $p, q \in A$ be the central idempotent and $pq = q$. Then, $d_1 p + d_2 q$ is central group invertible, and

$$(d_1 p + d_2 q)^{©} = d_1^{\dagger} p + (d^{\dagger} - d_1^{\dagger}) q$$

5 An Example

In this section, we present all invertible, central group invertible, group invertible, central Drazin invertible elements, and centrally clean elements of $Z_2 S_3$. For convenience, we write $g_1 = (1)$, $g_2 = (12)$, $g_3 = (13)$, $g_4 = (23)$, $g_5 = (123)$, $g_6 = (132)$, and $e = \sum_{i=1}^{6} g_i$.

By computation, we obtain the following results:

$$\begin{aligned} E(Z_2S_3) = \{ & 0, g_1, g_5 + g_6, g_1 + g_5 + g_6, g_2 + g_3 + g_5, g_2 + \\ & g_3 + g_6, g_2 + g_4 + g_5, g_2 + g_4 + g_6, g_3 + g_4 + g_5, g_3 + \\ & g_4 + g_6, g_1 + g_2 + g_3 + g_5, g_1 + g_2 + g_3 + g_6, g_1 + g_2 + \\ & g_4 + g_5, g_1 + g_2 + g_4 + g_6, g_1 + g_3 + g_4 + g_5, g_1 + g_3 + \\ & g_4 + g_6 \} \end{aligned}$$

$$\begin{aligned} C(Z_2S_3) = \{ & 0, g_1, g_5 + g_6, g_1 + g_5 + g_6, g_2 + g_3 + g_4, \\ & g_1 + g_2 + g_3 + g_4, e + g_1, e \} \end{aligned}$$

$$\mathrm{CE}(Z_2S_3) = \{0, g_1, g_5 + g_6, g_1 + g_5 + g_6\}$$

Example 2 All invertible, central group invertible, group invertible, central Drazin invertible elements, and centrally clean elements of Z_2S_3 are listed as follows.

For convenience, we use $\mathrm{CG}(Z_2S_3)$, $G(Z_2S_3)$, $\mathrm{CD}(Z_2S_3)$, and $\mathrm{CC}(Z_2S_3)$ to denote the sets of all central group invertible, group invertible, central Drazin invertible elements, and centrally clean elements of Z_2S_3, respectively.

$$\begin{aligned} U(Z_2S_3) = \{ & g_1, g_2, g_3, g_4, g_5, g_6, e + g_1, e + g_2, e + g_3, \\ & e + g_4, e + g_5, e + g_6 \} \end{aligned}$$

$$\begin{aligned} \mathrm{CG}(Z_2S_3) = \{ & 0, U(Z_2S_3), g_2 + g_3, g_2 + g_4, g_3 + g_4, \\ & g_5 + g_6, g_1 + g_5, g_1 + g_6 \} \end{aligned}$$

$$\begin{aligned} G(Z_2S_3) = \{ & \mathrm{CG}(Z_2S_3), g_1 + g_2 + g_5, g_1 + g_2 + g_6, \\ & g_1 + g_3 + g_5, g_1 + g_3 + g_6, g_1 + g_4 + g_5, g_1 + g_4 + g_6, \\ & g_2 + g_3 + g_5, g_2 + g_3 + g_6, g_2 + g_4 + g_5, g_2 + g_4 + g_6, \\ & g_3 + g_4 + g_5, g_3 + g_4 + g_6, e + g_4 + g_6, e + g_4 + g_5, \\ & e + g_3 + g_6, e + g_3 + g_5, e + g_2 + g_6, e + g_2 + g_5, e \} \end{aligned}$$

$$\begin{aligned} \mathrm{CD}(Z_2S_3) = \{ & \mathrm{CG}(Z_2S_3), g_1 + g_2, g_1 + g_3, g_1 + g_4, \\ & g_1 + g_2 + g_3, g_1 + g_2 + g_4, g_1 + g_3 + g_4, g_2 + g_5 + g_6, \\ & g_3 + g_5 + g_6, g_4 + g_5 + g_6, e + g_3 + g_4, e + g_2 + g_4, \\ & e + g_2 + g_3, e + g_1 + g_4, e + g_1 + g_3, e + g_1 + g_2, e \} \end{aligned}$$

$$\begin{aligned}\mathrm{CC}(Z_2S_3) = \{&0, U(Z_2S_3), g_1+g_2, g_1+g_3, g_1+g_4,\\ &g_1+g_5, g_1+g_6, g_5+g_6, g_2+g_3, g_2+g_4, g_3+g_4,\\ &g_1+g_2+g_3, g_1+g_2+g_4, g_1+g_3+g_4, g_2+g_3+g_4,\\ &g_1+g_5+g_6, g_2+g_5+g_6, g_3+g_5+g_6, g_4+g_5+g_6,\\ &g_2+g_3+g_4, e+g_1+g_6, e+g_1+g_5, e+g_1+g_4,\\ &e+g_1+g_3, e+g_1+g_2, e+g_3+g_4, e+g_2+g_4,\\ &e+g_2+g_3, e+g_6, e+g_5, e\}\end{aligned}$$

Proof Given that Z_2S_3 is finite, it follows that Z_2S_3 is strongly π-regular. Hence, the elements in Z_2S_3 are Drazin invertible.

Then, we calculate the units of Z_2S_3.

Let $\alpha = x_1g_1 + x_2g_2 + x_3g_3 + x_4g_4 + x_5g_5 + x_6g_6$ and $\beta = y_1g_1 + y_2g_2 + y_3g_3 + y_4g_4 + y_5g_5 + y_6g_6$. From $\alpha\beta = g_1$, we can obtain

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_6 & x_5 \\ x_2 & x_1 & x_6 & x_5 & x_3 & x_4 \\ x_3 & x_5 & x_1 & x_6 & x_4 & x_2 \\ x_1 & x_2 & x_3 & x_4 & x_6 & x_5 \\ x_2 & x_1 & x_6 & x_5 & x_3 & x_4 \\ x_3 & x_5 & x_1 & x_6 & x_4 & x_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

which has a unique solution.

Denote $\boldsymbol{A} = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_1 & x_6 \\ x_3 & x_5 & x_1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} x_4 & x_6 & x_5 \\ x_5 & x_3 & x_4 \\ x_6 & x_4 & x_2 \end{bmatrix}$.

Then, we have $\begin{vmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{A} \end{vmatrix} = |\boldsymbol{A}+\boldsymbol{B}|^2 \neq 0$. Notably, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$. Hence, we can obtain

$$\begin{vmatrix} x_1+x_4 & x_2+x_6 & x_3+x_5 \\ x_2+x_5 & x_1+x_3 & x_4+x_6 \\ x_3+x_6 & x_4+x_5 & x_1+x_2 \end{vmatrix} =$$

$$(x_1+x_2+x_3+x_4+x_5+x_6)\begin{vmatrix} 1 & 1 & 1 \\ x_2+x_5 & x_1+x_3 & x_4+x_6 \\ x_3+x_6 & x_4+x_5 & x_1+x_2 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & x_1+x_3+x_2+x_5 & x_4+x_6+x_2+x_5 \\ 0 & x_4+x_5+x_3+x_6 & x_1+x_2+x_3+x_6 \end{vmatrix} =$$

$$\begin{vmatrix} 1+x_4+x_6 & 1+x_1+x_3 \\ 1+x_1+x_2 & 1+x_4+x_5 \end{vmatrix} =$$

$$(1+x_4+x_6)(1+x_4+x_5) + (1+x_1+x_3)(1+x_1+x_2)$$

Then, from $|\boldsymbol{A}+\boldsymbol{B}| \neq 0$, it follows that $x_i = 0$ for certain $i \in \{1, 2, \cdots, 6\}$ and the others are 1, or $x_i = 1$ for certain $i \in \{1, 2, \cdots, 6\}$ and the others are 0. Hence, we have

$$\begin{aligned}U(Z_2S_3) = \{&g_1, g_2, g_3, g_4, g_5, g_6, e+g_1, e+g_2, e+g_3,\\ &e+g_4, e+g_5, e+g_6\}\end{aligned}$$

Therefore, based on Theorem 3.6 in Ref. [28], Proposition 8.24 in Ref. [30], Theorem 4.5 in Ref. [28], and the definition of centrally clean elements, we can present the sets of $\mathrm{CG}(Z_2S_3)$, $G(Z_2S_3)$, $\mathrm{CD}(Z_2S_3)$, and $\mathrm{CC}(Z_2S_3)$, respectively.

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环中的中心 clean 元和中心 Drazin 逆

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摘要: 如果环 R 中元素 a 是一个中心幂等元 e 和一个可逆元 u 的和,那么称 a 是中心 clean 元,并且称 $a=u+e$ 为 a 的一个中心 clean 分解. 如果环 R 中所有元素都是中心 clean 元,则称环 R 是中心 clean 环. 首先,给出了中心 clean 元的等价刻画,并进一步研究了中心 clean 环的一些性质以及环 R 是中心 clean 环的充分必要条件. 中心 clean 环与中心 Drazin 逆有着紧密的联系. 接着,从中心 clean 分解的角度给出了中心 Drazin 逆存在的充分必要条件,并研究了角环、幂级数环和环 R_α 的笛卡尔积 $\prod R_\alpha$ 等特殊环的中心 clean 性. 此外,还研究了一般域上代数中的 2 个中心幂等元组合的中心群可逆性. 最后,作为一个应用,分别计算出了群环 Z_2S_3 中所有的可逆元、中心群可逆元、群可逆元、中心 Drazin 可逆元以及中心 clean 元.

关键词: 中心 clean 元; 中心 clean 环; 中心 Drazin 逆; 中心群逆

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