Journal of Southeast University (English Edition) Vol. 40, No. 2, pp. 203 – 209 June 2024 ISSN 1003—7985

Numerical solution method for fundamental frequency and mode shape of Euler-Bernoulli beam based on Monte Carlo method

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Abstract: To address the challenge of solving free vibration problems in beams with uniform cross-sections, beams with variable cross-sections, and Euler-Bernoulli beams with concentrated masses, an innovative method combining the Rayleigh method and the Monte Carlo method is introduced. This dual-method strategy offers a novel solution by first discretizing the continuous beam structure model, followed by employing the Monte Carlo method to determine the vibration modes of the beam structure. Subsequently, these identified vibration modes are integrated into the Rayleigh method to calculate the fundamental frequency and vibration modes. The process involves a meticulous comparison with the minimum value obtained during calculations to ensure the satisfaction of the convergence condition. The results show that this combined method achieves a maximum error of 10% or less in predicting the fundamental frequency across different calculation models. This accuracy level is well within acceptable engineering requirements. The control parameters for accuracy and time can be easily adjusted to meet various needs. The method, which is simple in theory and widely applicable, enables the quick and precise determination of fundamental frequencies and vibration modes for diverse beam structures.

Key words: Euler-Bernoulli beam; fundamental frequency; Monte Carlo method; numerical solution

DOI: 10.3969/j.issn.1003 – 7985.2024.02.011

Beams play a pivotal role in various engineering disciplines, such as civil engineering and mechanical engineering. Understanding their vibration characteristics holds both theoretical and practical significance. The Euler-Bernoulli theory, a cornerstone in the study of beam mechanics, effectively captures the mechanical behavior of elongated beams undergoing small deformations and exhibiting minimal shearing forces. This simplicity has facilitated the analytical or semi-analytical exploration of free vibration challenges for partial isotropic Euler-Bernoulli beam models[1–4]. However, when it comes to models with variable cross-sections, the complexity increases, necessitating the adoption of numerical approaches for investigation. These include finite element method (FEM)[5–7], finite difference method (FDM)[8–9], differential transformation element method (DTEM)[10], and other numerical analysis methods[11–13].

Banerjee et al.[14] derived the exact solution for the free vibration of Euler-Bernoulli conical beams under general boundary conditions using Bessel functions. Lee et al.[15] introduced a transfer matrix method to determine the exact solution of the free vibration of Euler-Bernoulli beams with variable cross-sections. Further contributions include Zheng et al.[16], who leveraging the Euler-Bernoulli beam method presented an analytical calculation method to assess the mechanical response of cement concrete pavement structures, considering the shear slip effect at interlaminar interfaces. Similarly, Kang et al.[17] examined the static bending behavior of axially functionally graded Euler-Bernoulli microbeams subjected to concentrated and distributed loads. Niu et al.[18] modeled beams as proportional damping systems to derive analytic expressions based on the first-order sensitivity of elemental modal strain energy. Despite the merits of these theoretical approaches, their applicability often remains confined to specific model types. Addressing this limitation, this paper introduces a novel methodology based on the Euler-Bernoulli beam theory. This method uniquely accommodates general boundary conditions and accounts for the effects of arbitrary variable cross-sections and concentrated masses. Integrating Rayleigh's method with Monte Carlo sampling facilitates the examination of the first-order mode expansion of beam structures. The validity of this method is confirmed through comparisons with theoretical solutions and finite element examples.

1 Theoretical Method

1.1 Monte Carlo method for determining the mode function

This section focuses on resolving the subsequent funda-

Received 2023-11-30, **Revised** 2024-03-11.

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Foundation item: The National Natural Science Foundation of China (No.51778035).

Citation: Zhu Lei, Zhang Jianxun, Sun Hailin. Numerical solution method for fundamental frequency and mode shape of Euler-Bernoulli beam based on Monte Carlo method[J]. Journal of Southeast University (English Edition), 2024, 40(2): 203 – 209. DOI: 10.3969/j.issn.1003 – 7985.2024.02.011.

mental frequency using the Rayleigh method. However, before applying this method, it is imperative to first obtain the vibration mode function of the structure. Utilizing the Monte Carlo method, we assume the expression form of the vibration mode function as follows:

$$\varphi = \sum_{i=1}^{n} a_i f_i(x) \tag{1}$$

where φ represents the modal function of the beam structure; n signifies the number of sections into which the beam is divided; a is a randomly chosen number derived from the Monte Carlo method; f symbolizes the assumed modal function.

It is important to note that in the initial assumption of the vibration mode function, adherence to the deformation compatibility condition is not required. Nevertheless, this approach mandates adjustments to the vibration mode function to ensure compliance with the boundary conditions.

The simply supported boundary is as follows:

$$\varphi \mid_{x=x_i} = 0; \ \varphi = (x - x_i) \sum_{i=1}^{n} a_i f_i(x) \tag{2}$$

The clamping boundary is as follows:

$$\varphi \mid_{x=x_i} = 0; \ \varphi' \mid_{x=x_i} = 0; \ \varphi = (x - x_i)^2 \sum_{i=1}^{n} a_i f_i(x) \tag{3}$$

Through extensive calculations and experimentation, it has been determined that adopting a 4th-degree normalized power series as the vibration mode function yields excellent simulation results in most beam component models.

1.2 Model discretization

This study begins with the discretization of the original beam model by segmenting it into several equal-length portions, each possessing uniform cross-sections. This process involves deriving the properties, such as line density and bending stiffness, from the characteristic values at the intermediate sections of each beam segment. Furthermore, the positions of all node coordinates along the beam length are determined.

1.3 Rayleigh method to solve the frequency

The Rayleigh method is employed to calculate the frequency of the beam structure while considering mass concentration effects. This approach calculates the maximum kinetic energy and maximum strain energy of the discrete beam structure model.

$$T_{\max} = \frac{1}{2}\omega^2 \sum_{i=1}^{n} m_i \Phi_{xi} L_i + \sum \omega^2 M_i \Phi_{xi}^2 \tag{4}$$

$$V_{\max} = \frac{1}{2} \sum_{i=1}^{n} EI_i (\Phi''_{xi})^2 L_i \tag{5}$$

where $T_{\max}$ represents the system's maximum potential energy; ω represents the structure's fundamental frequency; m_i is the linear density of the i-th beam section; Φ_{xi} represents the characteristic value of the mode function at the cross-section of the i-th beam section; L_i represents the length of the i-th beam section; M_i denotes the i-th concentrated mass; $V_{\max}$ is the maximum kinetic energy of the system; E is the elastic modulus of the material; I_i represents the moment of inertia of the i-th beam section.

The structure's fundamental frequency ω can be determined from Eqs. (4) and (5) above as follows:

$$\omega^2 = \frac{\sum_{i=1}^{n} m_i \Phi_{xi} L_i + \sum M_i \Phi_{xi}^2}{\sum_{i=1}^{n} EI_i (\Phi''_{xi})^2 L_i} \tag{6}$$

1.4 Convergence criteria

Determining the fundamental frequency of a structure using the Monte Carlo method can be influenced by the chosen function form, which might not always meet the desired accuracy requirements. Consequently, convergence criteria must be established. Leveraging the principles of the Rayleigh method, which typically yields results greater than the actual fundamental frequency, we implement the following convergence criterion through MATLAB programming:

1) Initialize the upper limit of the fundamental frequency as infinite and define the convergence limit as the percentage difference between the fundamental frequency derived from the Monte Carlo method and the lower limit of the fundamental frequency, divided by the absolute value of the lower limit.

2) Calculate the fundamental frequency for the first time, update the lower limit of the fundamental frequency, and record the mode shape.

3) Recalculate and determine the system's fundamental frequency. Compare this new value with the previously established lower limit of the fundamental frequency. If the new result is lower than the current lower limit, adopt this new figure as the updated lower limit of the fundamental frequency, record the corresponding mode shape, and proceed with another recalculation. If the new frequency exceeds the lower limit but remains within the threshold of the lower limit multiplied by (1 + convergence limit), accept this lower limit as the definitive fundamental frequency and its associated mode shape as the fundamental mode shape. In scenarios not covered by the above conditions, initiate another calculation cycle.

2 Theoretical Calculation

This section aims to validate our method by examining three distinct beam structure models: isotropic beams with constant cross-sections, beams with variable cross-sections, and multi-segment beams with concentrated masses. These models are derived from practical engineering examples. To establish the reliability of our approach, we juxtapose the calculated values from our method against those obtained from theoretical analytical solutions and finite element analysis results.

2.1 Uniform straight beam with equal section

In this section, a rectangular beam with a uniform cross-section measuring 0.1 m × 0.1 m and a span length of 1 m is examined. This beam is constructed from steel, characterized by an elastic modulus (E) of 206 GPa and a density (ρ) of 7 850 kg/m^3. Two boundary conditions are considered: one where the beam is simply supported at both ends (S-S) and another where it is clamped at one end while free at the other (C-F).

To assess the accuracy of our method, we compare the research results presented in this with theoretical solutions and finite element analyses. For the finite element analysis, we utilize the ANSYS Workbench platform, employing a beam element model divided into 50 segments, with Beam188 selected as the element type. A schematic diagram of the finite element model is shown in Fig. 1.

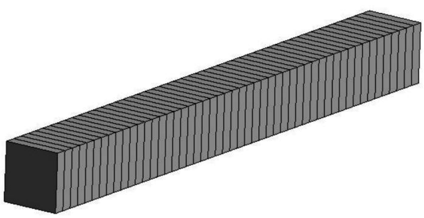

Fig. 1 Finite element model of equal section beam

In the Monte Carlo method solution program, a convergence limit of 1% is used. The original beam model is discretized into 100 equally long microelement segments along the beam length. Theoretical solutions are referenced from prior research[19]. The fundamental frequencies of simply supported beams and cantilever beams have been identified as 228.88 and 81.53 Hz, respectively, while the FEM yields values of 225.15 and 80.90 Hz, respectively. The comparable results are listed in Tables 1 and 2.

For the simply supported beam model, the maximum deviation from the theoretical solution was found to be 4.93%, while the maximum divergence from the finite element solution reached 6.67%. When considering the average outcomes of 10 calculations, these errors were reduced to 1.84% against the theoretical benchmark and 3.52% in relation to the finite element analysis. Regarding the cantilever beam model, the maximum error relative to the theoretical solution was 1.94%, with a slightly lesser error of 1.74% compared to the finite element solution. The error between the average value of 10 calculations and the theoretical solution was 0.95%, while

Table 1 Comparison of calculation results of simply supported beams of equal cross-section

Calculation times	Monte Carlo solution	Error with theoretical solution/%	Error with finite element solution/%	Cycles
1	240.17	4.93	6.67	280
2	239.24	4.53	6.26	256
3	229.97	0.48	2.14	354
4	230.16	0.56	2.23	323
5	232.02	1.37	3.05	329
6	230.43	0.68	2.35	319
7	229.36	0.21	1.87	364
8	236.00	3.11	4.82	314
9	229.34	0.20	1.86	274
10	234.15	2.30	4.00	230
Average	233.08	1.84	3.52	304
Variable coefficient	0.018	0.976	0.519	0.142

Table 2 Comparison of calculation results of cantilever beams with equal cross-sections

Calculation times	Monte Carlo solution	Error with theoretical solution/%	Error with finite element solution/%	Cycles
1	80.31	−1.50	−0.73	7 511
2	81.81	0.34	1.12	1 935
3	80.06	−1.80	−1.04	11 703
4	80.95	−0.71	0.06	9 824
5	80.26	−1.56	−0.79	17 831
6	81.03	−0.61	0.16	6 884
7	82.31	0.96	1.74	814
8	79.95	−1.94	−1.17	22 828
9	80.36	−1.44	−0.67	17 250
10	80.54	−1.21	−0.44	8 473
Average	80.75	−0.95	−0.18	10 505
Variable coefficient	0.010	−1.006	−5.440	0.670

the error with the finite element solution was 0.18%. These calculation errors confirm that our method satisfies the stringent precision requirements typical of engineering applications. An examination of the coefficients of variation for the fundamental frequencies (0.018 for simply supported beams and 0.010 for cantilever beams) reveals a commendable stability in frequency calculation. Although the relative error values are minimal, rendering their variation coefficients less informative, it is noteworthy that the variation coefficient concerning the number of iterations exhibits considerable variability, with values of 0.142 for simply supported beams and 0.670 for cantilever beams. This indicates significant fluctuations in the total computational effort required. Despite these variations, the method efficiency is highlighted by the remarkably short computational time, with each iteration concluding in less than 10 s. Furthermore, the practice of conducting repeated calculations has been proven to significantly improve accuracy.

2.2 Variable cross-section beam

In this analysis, we explore a rectangular beam with a variable cross-section that spans 3 m. This beam exhibits a linear variation in height along its axis, transitioning from a cross-section dimension of 0.2 m × 0.2 m at one end to a flat cross-section of 0.2 m × 0 m at the other end. The material properties are consistent with those used in our previous example of a constant cross-section beam. The boundary conditions for this model are as follows: one end of the beam, where the rectangular cross-section is located, is fixed, while the opposite end, featuring the flat cross-section, is left free.

Regarding the modeling process in ANSYS Workbench, solid elements with a cell size of 0.03 m are used. The selected element cell type is Solid186, a high-order 3D 20-node hexahedral solid element known for its computational accuracy. Given the simplicity of the model discussed in this paper, the choice of this particular element type does not significantly affect the overall computational time. A schematic diagram of the finite element model is shown in Fig. 2.

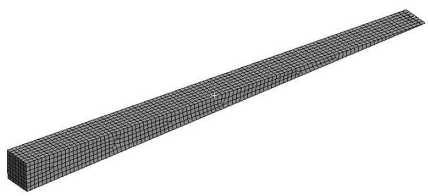

Fig. 2 Finite element model of variable cross-section beam

Within the framework of the Monte Carlo method solution program, we set a convergence limit of 1%. The original beam model is discretized into 100 equally long microelement segments along the beam length. The theoretical solution, referenced from previous research[14], indicates a fundamental frequency of 27.39 Hz. According to our FEM, the fundamental frequency is 27.38 Hz. The results are listed in Table 3.

Table 3 Comparison of calculation results of variable cross-section cantilever beams

Calculation times	Monte Carlo solution	Error with theoretical solution/%	Error with finite element solution/%	Cycles
1	27.31	-0.29	-0.26	54
2	27.84	1.64	1.68	23
3	27.30	-0.33	-0.29	24
4	27.36	-0.11	-0.07	20
5	27.38	-0.04	0.00	29
6	27.31	-0.29	-0.26	30
7	27.73	1.24	1.28	16
8	27.58	0.69	0.73	20
9	27.37	-0.07	-0.04	48
10	27.59	0.73	0.77	59
Average	27.47	0.32	0.35	32
Variable coefficient	0.007	2.237	2.011	0.481

In the case of the variable section beam model, the maximum error from the theoretical solution is 1.64%, while the maximum error compared to the finite element solution is 1.68%. Furthermore, the average error across 10 calculations, when compared to the theoretical solution, is 0.32%, and the error relative to the finite element analysis is 0.35%. Given the minimal relative error and number of iterations, the variation coefficient is not informative. These results underscore the robust applicability and reliability of the method presented in this paper for analyzing variable section beam models.

2.3 NREL 5-MW wind turbine tower

The US National Renewable Energy Laboratory (NREL) 5-MW wind turbine model is a cornerstone in various academic studies, representing a standard three-blade, upwind horizontal-axis wind turbine[20]. This model's design parameters, as outlined by NREL, offer a framework for detailed analyses.

In this section, we continue to leverage the ANSYS Workbench platform for modeling purposes. In the finite element model, the tower is represented as an isosceles beam structure with uniform variations in diameter and wall thickness from the base to the top. For the sake of simplicity, the mass of the rotor and the hub is concentrated at a single point, neglecting the influence of their mass distribution. The hub, located 2.34 m above the tower and 90 m above ground level, is positioned eccentrically, 5 m ahead of the tower. To accurately replicate the dynamic characteristics of the actual structure, the tower is modeled using beam elements, segmented into 50 equal-length sections and a beam element type Beam188. The resulting finite element vibration mode shape is shown in Fig. 3.

Fig. 3 NREL 5-MW wind turbine tower vibration shape

The Monte Carlo method solution program simplifies the process by ignoring the eccentricity of the tower top mass and setting a convergence limit of 1%. The original tower model is discretized into 100 equal-length microelement segments along the vertical direction. The fundamental frequency is 0.324 Hz, and the fundamental frequencies obtained by FEM are 0.320 Hz. The results are listed in Table 4.

Table 4 Comparison of NREL 5-MW wind tower results

Calculation times	Monte Carlo solution	Error with theoretical solution/%	Error with finite element solution/%	Cycles
1	0.335	3.40	4.69	619
2	0.339	4.63	5.94	113
3	0.337	4.01	5.31	182
4	0.336	3.70	5.00	99
5	0.334	3.09	4.38	166
6	0.337	4.01	5.31	120
7	0.336	3.70	5.00	148
8	0.332	2.47	3.75	1 579
9	0.332	2.47	3.75	411
10	0.335	3.40	4.69	462
Average	0.335	3.49	4.78	389
Variable coefficient	0.007	0.196	0.145	1.165

The maximum error in the NREL 5-MW tower model compared to the reference value is 4.63% and 5.94% against the finite element solution. Furthermore, averaging the outcomes of 10 computations yields errors of 3.49% relative to the theoretical predictions and 4.78% when compared to FEM results. These figures, presented in Table 4, highlight the method's robust stability in frequency calculation despite the significant variability in the number of iterations required, as indicated by a variation coefficient of 1.165. This variability suggests significant fluctuations in the total computational effort demanded by this method. Nonetheless, the method's high efficiency ensures that the overall time cost remains similar. Such findings underscore the method's strong applicability to actual engineering models with complex geometry.

3 Discussion

3.1 Influence of convergence limit

The convergence limit primarily impacts both the computational time and the accuracy of the program. Based on the convergence judgment criteria outlined in this paper, the likelihood of result distortion during the calculation process is extremely low. However, if the convergence limit is set too loosely, it might lead to situations where the calculations meet the convergence criteria but still significantly deviate from the actual fundamental frequency of the structure. Such incorrect results are more likely to arise in the early stages of calculations, leading to substantial discrepancies in the results. By either conducting multiple operations or increasing the stringency of the convergence criteria, the likelihood of these distortions can be greatly minimized.

On the other hand, there is a marginal effect between the convergence limit and the computational accuracy. Setting excessively strict convergence limits will significantly increase the number of iterations necessary for the program to complete its calculations. This not only increases the computational time but also yields minimal improvements in accuracy. In practice, setting a convergence limit of 1% has been found to effectively balance the requirements of computational accuracy and time.

3.2 Influence of the number of segments divided in beams

The division of the beam into sections significantly influences the accuracy of determining the fundamental frequency, ω_B, using the Rayleigh method. This method's reliance on approximating the derivative of discrete data poses inherent limitations during the programming process in MATLAB. Specifically, accurately solving the second-order derivative of the beam mode function at the most marginal sections of the beam ends can be challenging. A small number of section divisions can magnify this impact and distort the beam mode's function. Increasing the number of sections helps mitigate the error, offering a practical solution without substantially raising the computational cost. In terms of the program's execution time, the program requires minimal time to complete a single calculation. Dividing the beam into 100 sections is generally sufficient to meet most analytical requirements. However, increasing the number of sections by an order of magnitude has a minimal impact on the program's execution time.

3.3 Influence of parameters selected by the Monte Carlo method

In this paper, we operate under the assumption that the modal function is expressed as a normalized power series, with the highest order term being 4. The coefficients defining this power series are restricted to fall within the interval $[-1,1]$, ensuring that they remain constrained. This approach to introducing boundary conditions through power functions ensures that the engineering accuracy requirements are met while maintaining computational efficiency. When compared to other common methods of fitting functions, such as those utilizing the Fourier series, the power series approach consistently demonstrates superior efficiency.

4 Conclusions

1) In this study, we introduced a numerical calculation method to accurately determine the fundamental frequency of beam structures. This innovative approach combines the Rayleigh method with the Monte Carlo method.

2) The method outlined is straightforward and user-friendly, boasting broad applicability. It offers rapid and precise calculations of the natural frequency of beams, whether they have constant or variable cross-sections.

Notably, it achieves this while maintaining the maximum error rate below 10%, a level of accuracy that meets the stringent requirements of engineering applications.

3) This study delves into a detailed examination of the results across various cases, shedding light on how parameters such as the convergence limit and the number of segments influence both the accuracy and the computational time required. Furthermore, it provides extensive insights into universal parameters, outlining their value ranges and how they are represented in function form.

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基于蒙特卡洛法的 Euler-Bernoulli 梁基频和振型求解方法

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摘要: 将 Rayleigh 法和蒙特卡洛法相结合，在 Euler-Bernoulli 梁理论假设下求解了均匀梁、变截面梁和附带集中质量的变截面梁自由振动问题. 对原本连续的梁结构模型进行离散化处理，利用蒙特卡洛法给出梁结构的假设振型. 将假设得到的梁结构振型函数代入 Rayleigh 法，多次计算过程中，将历次基频所得值与计算所得最小值进行比较，根据其相对误差判断是否满足收敛条件，进而求得基频及对应的振型. 结果表明，不同计算模型中基频最大误差不超过 10% ，能够满足工程需求，且精度和时间的控制参数调整灵活，使用者可根据自身需要自行调节. 该方法理论简明，适用范围广泛，能够快速准确地求解诸多类型的梁结构基频和振型.

关键词: Euler-Bernoulli 梁；基频；蒙特卡洛法；数值解

中图分类号: O302