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Characterizations of *m*-weak group inverses

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Abstract: To characterize m-weak group inverses, several algebraic methods are used, such as the use of idempotents, one-side principal ideals, and units. Consider an element a within a unitary ring that possesses Drazin invertibility and an involution. This paper begins by outlining the conditions necessary for the existence of the m-weak group inverse of a. Moreover, it explores the criteria under which a can be considered pseudo core invertible and weak group invertible. In the context of a weak proper $*$-ring, it is proved that a is weak group invertible if, and only if, a^D can serve as the weak group inverse of au, where u represents a specially invertible element closely associated with a^D. The paper also introduces a counterexample to illustrate that a^D cannot universally serve as the pseudo core inverse of another element. This distinction underscores the nuanced differences between pseudo core inverses and weak group inverses. Ultimately, the discussion expands to include the commuting properties of weak group inverses, extending these considerations to m-weak group inverses. Several new conditions on commuting properties of generalized inverses are given. These results show that pseudo core inverses, weak group inverses, and m-weak group inverses are not only closely linked but also have significant differences that set them apart.

Key words: m-weak group inverse; weak group inverse; Drazin inverse; commuting property

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Throughout this article, R denotes a unitary ring with an involution. The journey into this algebraic terrain began in 1958 when Drazin[1] introduced the notion of the pseudo inverse for elements in a ring. This foundational concept, now known as the Drazin inverse, has since emerged as a pivotal tool across various research fields.

Definition 1[1] Let $a \in R$. An element $x \in R$ is called the Drazin inverse of a if there exists a positive integer $k \in \mathbf{N}^+$ such that $xa^{k+1} = a^k, ax^2 = x, xa = ax$. This element x is unique and denoted by a^D. The smallest positive k for which these equations are satisfied is referred to as the Drazin index of a and denoted by $I(a)$. In particular, x is called the group inverse of a when $k = 1$.

Following the introduction of the Drazin inverse, the mathematical community has unveiled various new types of generalized inverses. These include pseudo core inverses, weak group inverses, and m-weak group inverses.

Definition 2[2-5] Let $a \in R$. An element $x \in R$ qualifies as the pseudo core inverse of a if there exists $k \in \mathbf{N}^+$ such that $xa^{k+1} = a^k, ax^2 = x, (ax)^* = ax$. Such x is unique and denoted by $a^{\{D\}}$. The minimum positive integer k that satisfies these conditions is called the pseudo core index of a. In particular, x is called the core inverse of a when $k = 1$.

Definition 3[6-8] Let $a \in R$ and $m \in \mathbf{N}$. An element $x \in R$ is designated as the m-weak group inverse of a if there exists $k \in \mathbf{N}^+$ such that $xa^{k+1} = a^k, ax^2 = x, (a^k)^* a^{m+1} x = (a^k)^* a^m$. If the m-weak group inverse of a is unique, it is denoted by a^{W_m}. The smallest positive integer k meeting these criteria is termed the m-weak group inverse index of a. In particular, the definition of the 1-weak group inverse is aligned precisely with that of the weak group inverse, with a^W signifying the unique weak group inverse of a.

In addition, the authors[8] proved that a is pseudo core invertible if, and only if, a is 0-weak group invertible. Moreover, it should be noted that if a is m-weak group invertible, then it is inherently Drazin invertible, with the m-weak group index coinciding with the Drazin index. This fact permits the use of $I(a)$ to denote the m-weak group index of a. The notations R^D, $R^{\{D\}}$, R^W, R^{W_m} are used to denote the sets of all Drazin invertible elements, pseudo core invertible elements, weak group invertible elements, and m-weak group invertible elements in R, respectively.

Li et al.[9] characterized the existence of m-weak group inverses through the means of one-side principal ideals, while Zhou et al.[10] investigated the commuting properties of weak group inverses. In this article, we aim to extend these discussions by offering further characterizations of the existence of m-weak group inverses. Additionally, we explore how the commuting properties of weak group inverses can be expanded to m-weak group inverses.

1 Existence of *m*-Weak Group Inverses

In this section, we introduce several new conditions that

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serve to characterize the m-weak group inverse for a Drazin invertible element within a ring that features an involution. Some auxiliary lemmas are presented as follows:

Lemma 1[3] Let $a \in R$. If there exist $x \in R$ and $k \in \mathbf{N}^+$ such that $xa^{k+1} = a^k, ax^2 = x$, then

1) $ax = a^m x^m$ for arbitrary positive integer m;

2) $xax = x$;

3) a is Drazin invertible, $a^D = x^{k+1}a^k$ and $I(a) \leqslant k$.

In Ref. [11], the authors denote:

$T_l(a) = \{x \in R: xa^{k+1} = a^k, ax^2 = x$ for some positive integer $k\}$.

Besides, denote $E_l(a) = \{ax: x \in T_l(a)\}$. It is obvious that each element in $E_l(a)$ is idempotent.

Lemma 2[11] Let $a \in R^D$, $k_1, k_2, \dots, k_n, s_1, s_2 \dots, s_n \in \mathbf{N}$ and $x_1, x_2, \dots, x_n \in T_l(a)$. If $s_n \neq 0$, then

$$\prod_{i=1}^{n} a^{k_i} x_i^{s_i} = a^k x_n^s$$

where $k = \sum_{i=1}^{n} k_i$, and $s = \sum_{i=1}^{n} s_i$.

Lemma 3 Let $a \in R^D$ and $S = \{nI_R \mid n \in \mathbf{Z}\}$, where I_R is the identity of R. If $k_1, k_2, \dots, k_n \in S$, $x_1, x_2 \dots, x_n \in T_l(a)$ and $\sum_{i=1}^{n} k_i = I_R$, then $\sum_{i=1}^{n} k_i x_i \in T_l(a)$.

Proof Suppose that the Drazin index of a is k. By computation, we have $\left(\sum_{i=1}^{n} k_i x_i\right)a^{k+1} = \left(\sum_{i=1}^{n} k_i\right)a^k = a^k$. It follows from Lemma 2 that $\left(\sum_{i=1}^{n} k_i x_i\right)^2 = \sum_{i=1}^{n} k_i x_i^2$. Thus, $a\left(\sum_{i=1}^{n} k_i x_i\right)^2 = \sum_{i=1}^{n} k_i x_i$.

Without confusion, we simply use 1 to stand for the identity of R. Let $a \in R^D$ and denote $U_l(a) = \{1 - ax + ay: x, y \in T_l(a)\}$. By Lemma 3, we have $x - y + a^D \in T_l(a)$ for any $x, y \in T_l(a)$. Therefore, $1 - ax + ay = 1 - a(x - y + a^D) + aa^D$. Thus, $U_l(a) = \{1 - ax + aa^D: x \in T_l(a)\}$. Besides, $1 - ax + aa^D$ is invertible with $(1 - ax + aa^D)^{-1} = 1 + ax - aa^D \in U_l(a)$. For any $x, y \in T_l(a)$, it follows $(1 - ax + aa^D)(1 - ay + aa^D) = 1 - a(x + y - a^D) + aa^D \in U_l(a)$. Thus, $U_l(a)$ is a multiplicative group.

Lemma 4[8] Let $a \in R$ and $m \in \mathbf{N}$. An element $x \in R$ is the m-weak group inverse of a if, and only if, there exists $k \in \mathbf{N}^+$ such that $xa^{k+1} = a^k, ax^2 = x, [(a^m)^* a^{m+1} \cdot x]^* = (a^m)^* a^{m+1} x$.

The following theorem can be regarded as an expansion of Theorem 3.1 in Ref. [9], and the equivalence between Conditions 1) and 10) was also shown in Ref. [9].

Theorem 1 Let $a \in R^D$, then the following conditions are equivalent:

1) $a \in R^{W_m}$;

2) There exists $e \in E_l(a)$ such that $[(a^m)^* a^m e]^* = (a^m)^* a^m e$;

3) There exists $e \in E_l(a)$ such that $(a^D)^* a^m e = (a^D)^* \cdot a^m$;

4) There exists $e = e^2 \in R$ such that $e \in aa^D R$ and $(a^D)^* a^m e = (a^D)^* a^m$;

5) For any $u \in U_l(a)$, $au \in R^{W_m}$;

6) There exists $u \in U_l(a)$ such that au is m-weak group invertible;

7) There exists $t \in aa^D R(1 - aa^D)$ such that $(a^m)^* a^{m+1} \cdot t - (a^{m+1}a^D)^* a^m$ is Hermitian;

8) For arbitrary $e \in E_l(a)$, $(a^D)^*(a^m - a^m e) \in (a^D)^* \cdot a^D R(1 - aa^D)$;

9) There exists $e \in E_l(a)$ such that $(a^D)^*(a^m - a^m e) \in (a^D)^* a^D R(1 - aa^D)$;

10) $(a^D)^* a^m \in (a^D)^* a^D R$.

Proof It follows from Lemma 2.8 in Ref. [8] and Lemma 4 that Conditions 1) to 4) are equivalent.

For 1)⇒5) suppose that a is m-weak group invertible with $I(a) \leqslant k$ and y is a m-weak group inverse of a. There exists $x \in T_l(a)$ such that $u = 1 - ax + aa^D$, so we have $au = a - a^2x + a^2a^D$. Obviously, $a - a^2x$ is nilpotent, a^2a^D is group invertible, and $(a - a^2x)a^2a^D = 0$. It follows that au is Drazin invertible with $I(a) \leqslant k + 1$ by direct calculation. Since $aa^D(au)^{k+1} = (au)^{k+1}$ and $(au)^{k+1}(a^D)^{k+1} = aa^D$, it follows that $au(au)^D R = aa^D R$. Thus, $U_l(a) = U_l(au)$ and $E_l(a) = E_l(au)$. When $m \geqslant 2$, we have

$$\begin{aligned}
(aa^D)^*(au)^m &= (aa^D)^* \left[\sum_{i=0}^{m} (a^2a^D)^i (a - a^2x)^{m-i}\right] = \\
&(aa^D)^* \left[\sum_{i=1}^{m-1} a^{i+1}a^D(a - a^2x)a^{m-i-1} + (a - a^2x)a^{m-1} + a^{m+1}a^D\right] = \\
&(aa^D)^* a^m \left[\sum_{i=1}^{m-1} a^{i+1}(a^D)^{m+1}(a - a^2x)a^{m-i-1} + 1 - x^{m-1}a^{m-1} + aa^D\right] = \\
&(aa^D)^* a^m \left[\sum_{i=1}^{m-2} (aa^D - x^{m-i-1}a^{m-i-1}) + aa^D - ax + 1 - x^{m-1}a^{m-1} + aa^D\right] = \\
&(aa^D)^* a^m \left[\sum_{i=0}^{m-2} (aa^D - x^{m-i-1}a^{m-i-1}) + aa^D - ax + 1\right]
\end{aligned}$$

Take

$$f = \sum_{i=0}^{m-2} (aa^D - x^{m-i-1}a^{m-i-1}) + aa^D$$

$$v = \sum_{i=0}^{m-2} (aa^D - x^{m-i-1}a^{m-i-1}) + aa^D - ax + 1$$

From Lemma 2, we have $x^{i+1}a^i \in T_l(a)$ and $x^ia^i = a(x^{i+1}a^i) \in E_l(a)$ for any $i \in \mathbf{N}^+$. Therefore, we get $f \in$

$E_l(a)$ and $v \in U_l(a)$ by Lemma 3. Take $g = ayv$. From Lemma 2, we have $g^2 = g$. Since $gaa^D = aa^D$ and $aa^D g = g$, we have $g \in E_l(a) = E_l(au)$. Then

$$(aa^D)^*(au)^m g = (aa^D)^*(a^m v) g =$$
$$(aa^D)^* a^m \Big[\sum_{i=0}^{m-2}(aa^D - x^{m-i-1}a^{m-i-1}) + aa^D - ax + 1\Big] ayv =$$
$$(aa^D)^* a^m ayv = (aa^D)^* a^m v = (aa^D)^*(au)^m$$

Based on the equivalence between Conditions 1) and 3), au is m-weak group invertible. This is easy to demonstrate when $m = 0$ or 1.

The transition 5)⇒6) is trivial.

We claim 6)⇒1). It is readily apparent that $U_l(a) = U_l(au)$ and $E_l(a) = E_l(au)$. Since $U_l(a)$ is a multiplicative group, it follows that $u^{-1} \in U_l(au)$. Through the proof from Conditions 1) to 5), we establish that $a = auu^{-1}$ is m-weak group invertible.

For 1)⇒7), we assume that a is m-weak group invertible and x is a m-weak group inverse of a. Given that $t = x - a^D$, we obtain the following:

$$(a^m)^* a^{m+1} t - (a^{m+1}a^D)^* a^m =$$
$$(a^m)^* a^{m+1}(x - a^D) - (a^{m+1}a^D)^* a^m =$$
$$(a^m)^* a^{m+1} x - [(a^m)^* a^{m+1} a^D + (a^{m+1}a^D)^* a^m] =$$
$$[(a^m)^* a^{m+1} x]^* - [(a^m)^* a^{m+1} a^D + (a^{m+1}a^D)^* a^m]^* =$$
$$[(a^m)^* a^{m+1} t - (a^{m+1}a^D)^* a^m]^*$$

In addition, it is easy to obtain that $t \in aa^D R(1 - aa^D)$.

For 7)⇒1), take $x = a^D + t$. It follows $x \in T_l(a)$ by direct calculation. Moreover,

$$(a^m)^* a^{m+1} x = (a^m)^* a^{m+1} a^D + (a^m)^* a^{m+1} t =$$
$$(a^m)^* a^{m+1} a^D + (a^{m+1}a^D)^* a^m +$$
$$(a^m)^* a^{m+1} t - (a^{m+1}a^D)^* a^m =$$
$$[(a^m)^* a^{m+1} a^D + (a^{m+1}a^D)^* a^m]^* +$$
$$[(a^m)^* a^{m+1} t - (a^{m+1}a^D)^* a^m]^* =$$
$$[(a^m)^* a^{m+1} x]^*$$

Therefore, a is m-weak group invertible by Lemma 4.

For 1)⇒8), we assume that a is m-weak group invertible and x is a m-weak group inverse of a. Then, we derive the following:

$$(a^D)^*(a^m - a^m e) = (a^D)^* a^m (ax - e) =$$
$$(a^D)^* a^m (a^D a^2 x - a^D a e) =$$
$$(a^D)^* a^D a^m (a^2 x - ae)$$

Since $(a^2 x - ae) aa^D = 0$ as obtained from Lemma 2, we have $(a^2 x - ae) \in R(1 - aa^D)$. Therefore, $(a^D)^*(a^m - a^m e) \in (a^D)^* a^D R(1 - aa^D)$.

The implication 8)⇒9) is clear.

We now prove 9)⇒1). There exists $t \in aa^D R(1 - aa^D)$ such that $(a^D)^*(a^m - a^m e) = (a^D)^* a^D t$. We take $f = e + (a^D)^{m+1} t$. Since $f^2 = f$, $faa^D = aa^D$ and $aa^D f = f$, we establish that $f \in E_l(a)$. Following calculations, we obtain the following:

$$(a^D)^* a^m f = (a^D)^* a^m e + (a^D)^* a^D t =$$
$$(a^D)^* a^m e + (a^D)^*(a^m - a^m e) = (a^D)^* a^m$$

Based on the equivalence between Conditions 1) and 3), a is m-weak group invertible.

Next, we prove 1)⇒10). Based on the equivalence between Conditions 1) and 3), there exists $e \in E_l(a)$ such that $(a^D)^* a^m = (a^D)^* a^m e$. Since $aa^D R = eR$, it follows that $(a^D)^* a^m = (a^D)^* a^m e = (a^D)^* a^m (aa^D) e \in (a^D)^* a^D R$.

The implication 10)⇒9) is obvious.

Remark 1 Condition 2) in Theorem 1 can be replaced by the condition that there exists $e \in E_l(a)$ satisfying either of the following conditions: $(a^m)^* a^m = e^*(a^m)^* a^m e + (1 - e)^*(a^m)^* a^m (1 - e)$ or $(a^m)^* a^m = e^*(a^m)^* a^m + (a^m)^* a^m (1 - e)$.

Take $m = 0, 1$ in Theorem 1. Two corollaries can be given immediately.

Corollary 1 Let $a \in R^D$, then the following conditions are equivalent:

1) $a \in R^{\{D\}}$;

2) There exists $e \in E_l(a)$ such that $e^* = e$;

3) There exists $e \in E_l(a)$ such that $(a^D)^* e = (a^D)^*$;

4) For any $u \in U_l(a)$, $au \in R^{\{D\}}$;

5) There exists $u \in U_l(a)$ such that au is pseudo core invertible;

6) There exists $t \in aa^D R(1 - aa^D)$ such that $at - (aa^D)^*$ is Hermitian;

7) For any $e \in E_l(a)$, $(a^D)^*(1 - e) \in (a^D)^* a^D R(1 - aa^D)$;

8) There exists $e \in E_l(a)$ such that $(a^D)^*(1 - e) \in (a^D)^* a^D R(1 - aa^D)$;

9) $(a^D)^* \in (a^D)^* a^D R$.

Corollary 2 Let $a \in R^D$, then the following conditions are equivalent:

1) $a \in R^W$;

2) There exists $e \in E_l(a)$ such that $(a^* ae)^* = a^* ae$;

3) There exists $e \in E_l(a)$ such that $(a^D)^* ae = (a^D)^* \cdot a$;

4) There exists $e = e^2 \in R$ such that $e \in aa^D R$ and $(a^D)^* ae = (a^D)^* a$;

5) For any $u \in U_l(a)$, $au \in R^W$;

6) There exists $u \in U_l(a)$ such that au is weak group invertible;

7) There exists $t \in aa^D R(1 - aa^D)$ such that $a^* a^2 t - (a^2 a^D)^* a$ is Hermitian;

8) For any $e \in E_l(a)$, $(a^D)^*(a - ae) \in (a^D)^* a^D R(1 - aa^D)$;

9) There exists $e \in E_l(a)$ such that $(a^D)^*(a - ae) \in (a^D)^* a^D R(1 - aa^D)$;

10) $(a^D)^* a \in (a^D)^* a^D R$.

Recall that R is a weak proper $*$-ring[8] if, and only if, each element in R has at most a single weak group inverse.

Corollary 3 If R is a weak proper $*$-ring, then a is weak group invertible if, and only if, there exists $u \in U_l(a)$ such that au is weak group invertible and $(au)^W = a^D$.

Proof Suppose that a is weak group invertible with $I(a) = k$. Take $u = 1 - aa^W + aa^D$. By Theorem 1, au is weak group invertible. Now, we can verify that $(au)^W = a^D$.

$$\begin{aligned} a^D(au)^{k+1} &= a^D \Big[\sum_{i=0}^{k-1} (a^2a^D)^{k+1-i}(a - a^2a^W)^i + \\ &\quad a^2a^D(a - a^2a^W)^k + (a - a^2a^W)^{k+1} \Big] = \\ &\quad a^D \Big[\sum_{i=0}^{k-1} (a^2a^D)^{k+1-i}(a - a^2a^W)^i + \\ &\quad a^2a^D(a - a^2a^W)a^{k-1} + (a - a^2a^W)a^k \Big] = \\ &\quad \sum_{i=0}^{k-1} (a^2a^D)^{k-i}(a - a^2a^W)^i + a^k - a^2a^Wa^{k-1} + \\ &\quad a^k - aa^Wa^k = \sum_{i=0}^{k-1} (a^2a^D)^{k-i}(a - a^2a^W)^i + \\ &\quad (a - a^2a^W)a^{k-1} + a^k - a(a^W)^2a^{k+1} = \\ &\quad \sum_{i=0}^{k-1} (a^2a^D)^{k-i}(a - a^2a^W)^i + (a - a^2a^W)^k + \\ &\quad a^k - a^Wa^{k+1} = \sum_{i=0}^{k} (a^2a^D)^{k-i}(a - a^2a^W)^i = (au)^k \end{aligned}$$

$$au(a^D)^2 = a(1 - aa^W + aa^D)(a^D)^2 = a^D$$

$$\begin{aligned} (au)^*(au)^2a^D &= u^*a^*a^2a^D = u^*a^*a^2a^Du^{-1}u = \\ &\quad u^*a^*a^2a^D(1 + aa^W - aa^D)u = \\ &\quad u^*a^*a^2a^Wu = (u^*a^*a^2a^Wu)^* = [(au)^*(au)^2a^D]^* \end{aligned}$$

Therefore, there exists $u \in U_l(a)$ such that au is weak group invertible and $(au)^W = a^D$. The converse is obvious by Theorem 1.

The following example shows that a^D may not be the pseudo core inverse of another element when a is Drazin invertible.

Example 1 Let $R = M_2(\mathbf{R})$, $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Take the transpose as the involution. It is obvious that $\boldsymbol{A}^D = \boldsymbol{A}$ and $\boldsymbol{A}^{\{D\}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. If there exists a matrix $\boldsymbol{B}$ satisfying $\boldsymbol{B}^{\{D\}} = \boldsymbol{A}^D$, then $\boldsymbol{B}^{\{D\}}R = \boldsymbol{A}^DR = \boldsymbol{A}\boldsymbol{A}^{\{D\}}R$. By Theorem 2.3 in Ref. [3], we have $\boldsymbol{A}^D = \boldsymbol{B}^{\{D\}} = \boldsymbol{B}^D\boldsymbol{A}\boldsymbol{A}^{\{D\}}$. However, $\boldsymbol{A}^D \notin R\boldsymbol{A}\boldsymbol{A}^{\{D\}}$. Therefore, $\boldsymbol{A}^D$ cannot be the pseudo core inverse of any matrix.

2 Commuting Properties of the *m*-Weak Group Inverses

Drazin[12] explored the commuting properties of different kinds of generalized inverses, while Zhou et al.[10] investigated the characteristics of weak group inverses. We now examine the commuting properties of m-weak group inverses. In a weak proper $*$-ring, each element can possess at most a single m-weak group inverse. It is important to note that $a \in R$ is left $*$-cancellable if $a^*ay = 0$ implies $ay = 0$ for arbitrary $y \in R$.

Proposition 1 Let R be a weak proper $*$-ring, $y \in R$ and a_1, $a_2 \in R^{W_m}$. If $a_1a_1^D$ is left $*$-cancellable and there exist $x_1, x_2 \in T_l(a_1)$ and $x_3, x_4 \in T_l(a_2)$ such that $x_1y = yx_3$ and $x_2^*a_1^my = yx_4^*a_2^m$, then $a_1^{W_m}y = ya_2^{W_m}$.

Proof From Lemmas 2 and 4, it follows that

$$\begin{aligned} (a_1^m)^*a_1^{m+1}a_1^{W_m}y &= (a_1^{m+1}a_1^{W_m})^*a_1^my = (x_2a_1^{m+2}a_1^{W_m})^*a_1^my = \\ &\quad (a_1^{m+2}a_1^{W_m})^*yx_4^*a_2^m = (a_1^{m+2}a_1^{W_m})^*y(a_2^{m+1}a_2^{W_m}x_4^{m+1})^*a_2^m = \\ &\quad (a_1^{m+2}a_1^{W_m})^*y(x_4^{m+1})^*(a_2^m)^*a_2^{m+1}a_2^{W_m} = \\ &\quad (a_1^{m+2}a_1^{W_m})^*yx_4^*a_2^{m+1}a_2^{W_m} = (a_1^m)^*a_1^{m+1}a_1^{W_m}ya_2a_2^{W_m} \end{aligned}$$

Therefore,

$$(a_1a_1^D)^*a_1a_1^Da_1^{m+1}a_1^{W_m}y = (a_1a_1^D)^*a_1a_1^Da_1^{m+1}a_1^{W_m}ya_2a_2^{W_m}$$

Since $a_1a_1^D$ is left $*$-cancellable, we show that $a_1a_1^Da_1^{m+1}a_1^{W_m}y = a_1a_1^Da_1^{m+1}a_1^{W_m}ya_2a_2^{W_m}$. That is, $a_1^{m+1}a_1^{W_m}y = a_1^{m+1}a_1^{W_m}ya_2a_2^{W_m}$. Then

$$\begin{aligned} a_1^{W_m}y &= x_1^{m+1}a_1^{m+1}a_1^{W_m}y = x_1^{m+1}a_1^{m+1}a_1^{W_m}ya_2a_2^{W_m} = \\ &\quad x_1^{m+1}a_1^{m+1}a_1^{W_m}yx_3a_2^2a_2^{W_m} = x_1^{m+1}a_1^{m+1}a_1^{W_m}x_1ya_2^2a_2^{W_m} = \\ &\quad x_1^2ya_2^2a_2^{W_m} = yx_3^2a_2^2a_2^{W_m} = ya_2^{W_m} \end{aligned}$$

Taking $a = a_1 = a_2$ in Proposition 1, we can immediately obtain the next result.

Corollary 4 Let R be a weak proper $*$-ring, $y \in R$ and $a \in R^{W_m}$. If there exists $x_1, x_2 \in T_l(a)$ satisfying $x_1y = yx_1$ and $x_2^*a^my = yx_2^*a^m$, then $a^{W_m}y = ya^{W_m}$.

Proof From the proof of Proposition 1, we have $(a^m)^*a^{m+1}a^{W_m}y = (a^m)^*a^{m+1}a^{W_m}yaa^{W_m}$. Take $x = a^{m+1}a^{W_m}y - a^{m+1}a^{W_m}yaa^{W_m}$. It is easy to obtain that $(aa^D)^*x = 0$ and $x \in aa^DR(1 - aa^D)$. Since R is a weak proper $*$-ring, we have $x = 0$. The rest of the proof is similar to that of Proposition 1.

Corollary 5 Let $a \in R^{\{D\}}$ and $y \in R$. If there exists $x_1, x_2 \in T_l(a)$ such that $x_1y = yx_1$ and $x_2^*y = yx_2^*$, then $a^{\{D\}}y = ya^{\{D\}}$ and $(a^{\{D\}})^*y = y(a^{\{D\}})^*$.

Corollary 6 Let $a \in R^{\{D\}}$ and $y \in R$. If $a^Dy = ya^D$ and $(a^D)^*y = y(a^D)^*$, then $a^{\{D\}}y = ya^{\{D\}}$ and $(a^{\{D\}})^*y = y(a^{\{D\}})^*$.

Corollary 7 Let a, $b \in R^{\{D\}}$. If $a^Db^D = b^Da^D$ and $(a^D)^*b^D = b^D(a^D)^*$, then $a^{\{D\}}b^{\{D\}} = b^{\{D\}}a^{\{D\}}$ and $(a^{\{D\}})^*b^{\{D\}} = b^{\{D\}}(a^{\{D\}})^*$.

Example 2 Let $R = M_2(\mathbf{R})$, $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Consider the transpose as the involution. It is obvious that $\boldsymbol{A}^D = \boldsymbol{A}$ and $\boldsymbol{A}^{\{D\}} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$. Considering that $\boldsymbol{B} = \boldsymbol{A}^{\{D\}}$, then $\boldsymbol{B}^D = \boldsymbol{B}^{\{D\}} = \boldsymbol{B}$. Therefore, $\boldsymbol{A}^{\{D\}}\boldsymbol{B}^{\{D\}} = \boldsymbol{B}^{\{D\}}\boldsymbol{A}^{\{D\}}$. Following computations, $\boldsymbol{B}^D\boldsymbol{A}^D \neq \boldsymbol{A}^D\boldsymbol{B}^D$, so the converses of Corollaries 6 and 7 may not be true.

The generalized invertibility of products of elements has attracted the attention of many scholars and readers can refer to Refs. [13 - 15]. Next, the commuting properties are applied to give the pseudo core inverse of a product.

Corollary 8 Let $a, b \in R^{\{D\}}$. If $ab = ba$ and $(a^D)^* b^D = b^D (a^D)^*$, then $ab \in R^{\{D\}}$, and $(ab)^{\{D\}} = a^{\{D\}} b^{\{D\}} = b^{\{D\}} a^{\{D\}}$.

Proof From Ref. [1], the condition $ab = ba$ implies $a^D b^D = b^D a^D$. This together with $(a^D)^* b^D = b^D (a^D)^*$ implies that $a^{\{D\}} b^{\{D\}} = b^{\{D\}} a^{\{D\}}$ and $a^D b^{\{D\}} = b^{\{D\}} a^D$ by Corollaries 6 and 7. Thus, $a^{\{D\}} b^{\{D\}} \in T_l(ab)$. We only need to prove $(abb^{\{D\}} a^{\{D\}})^* = abb^{\{D\}} a^{\{D\}}$. Since $a^D b = ba^D$, we have $a^D bb^{\{D\}} = bb^{\{D\}} a^D$, which implies $(a^D)^* \cdot bb^{\{D\}} = bb^{\{D\}} (a^D)^*$ and $a^{\{D\}} bb^{\{D\}} = bb^{\{D\}} a^{\{D\}}$ by Corollary 6. Similarly, $b^{\{D\}} aa^{\{D\}} = aa^{\{D\}} b^{\{D\}}$. Thus, $(abb^{\{D\}} \cdot a^{\{D\}})^* = (aa^{\{D\}} bb^{\{D\}})^* = bb^{\{D\}} aa^{\{D\}} = baa^{\{D\}} b^{\{D\}} = abb^{\{D\}} a^{\{D\}}$.

Under the assumption of Corollary 7, the following example shows that ab may not be pseudo core invertible.

Example 3 Let $R = M_3(\mathbf{Z})$, $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Consider the transpose as the involution. It is obvious that $\boldsymbol{A}^D = \boldsymbol{B}^D = \mathbf{0}$. Therefore, $\boldsymbol{A}^D \boldsymbol{B}^D = \boldsymbol{B}^D \boldsymbol{A}^D$ and $(\boldsymbol{A}^D)^* \boldsymbol{B}^D = \boldsymbol{B}^D (\boldsymbol{A}^D)^*$. However, $\boldsymbol{AB} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \notin R^D$. Therefore, $\boldsymbol{AB}$ is not pseudo core invertible.

3 Conclusions

1) In a ring featuring an involution, we present further characterizations of the existence of m-weak group inverse of an element. These are based on new conditions corresponding to idempotents, one-sided principal ideals, and units.

2) The commuting properties of m-weak group inverses are shown and applied to obtain the pseudo invertibility of a product of elements. Through several illustrative examples, we highlight the relationships and differences between Drazin inverses, pseudo core inverses and weak group inverses, which are profoundly linked.

3) Our results can be used to improve previous work on m-weak group inverses and offer fresh perspectives on the study of other generalized inverses. Given the burgeoning interest in m-weak group inverses, future research might profitably investigate their applications across various fields.

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m-弱群逆的刻画

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摘要：为了刻画 m-弱群逆，采用幂等元刻画、单边主理想刻画、可逆元刻画等代数方法. 设 a 为带对合的幺环中的 Drazin 可逆元. 首先，给出了元素 a 的 m-弱群逆的存在性刻画，得到 a 为伪核可逆元和弱群可逆元的等价条件. 其次，在弱 proper $*$-环中，证明元素 a 为弱群可逆的当且仅当 a 的 Drazin 逆为 au 的弱群逆，u 为一个与 a 的 Drazin 逆密切相关的特殊可逆元. 然后，给出一个反例说明 a 的 Drazin 逆一般不是其他元素的伪核逆，由此表明了伪核逆与弱群逆的区别. 最后，将关于弱群逆交换性的结论推广到 m-弱群逆的情形，给出了新的关于广义逆交换性的条件. 结果表明，伪核逆、弱群逆与 m-弱群逆联系密切，但又存在显著区别.

关键词：m-弱群逆；弱群逆；Drazin 逆；交换性

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