

Weak Hopf bornological algebras

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Abstract: We introduce the notion of a weak Hopf bornological algebra, which simultaneously generalizes the classical frameworks of weak Hopf algebras and Hopf bornological algebras. This extension aims to unify algebraic structures with topological boundedness structures, thereby providing a natural setting for the treatment of Hopf-type objects arising in locally convex spaces and groupoid algebras. First, we present the definition and basic properties of weakly bounded Hopf algebras and prove some basic results analogous to those in weak Hopf algebras, such as the uniqueness of the antipode and its anti-homomorphic properties with respect to the multiplication and comultiplication. Finally, we mainly study the weak Hopf bornological algebra structure on finite Lie groupoids. Several basic examples are explicitly constructed, and their comultiplications, antipodes, and counits are computed in detail. These results indicate that the weak boundedness condition can be naturally derived from the composition law of finite Lie groupoids. Overall, this theoretical framework offers a new perspective for studying Hopf-type algebras and provides a novel approach for investigating symmetry and duality in bounded and analytic contexts.

Key words: (weak) Hopf bornological algebra; weak Hopf algebra; Lie groupoid; locally convex space

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1 Preliminaries

Definition 1 A bornological vector space^[1-5] is a vector space V over a field k , together with $B(V)$, a collection of subsets of V (called bornology), satisfying the following conditions:

- (1) $V = \bigcup_{Z \in B(V)} Z$;
- (2) $B_1 \subseteq B_2$ and $B_2 \in B(V)$ implies that $B_1 \in B(V)$;
- (3) $B_1, B_2 \in B(V)$ implies that $B_1 \cup B_2 \in B(V)$;
- (4) $B_1, B_2 \in B(V)$ implies that $B_1 + B_2 \in B(V)$;
- (5) For any complex number l , $Z \in B(V)$ implies that $lZ \in B(V)$ and $\bigcup_{|l| \leq 1} lZ \in B(V)$.

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These conditions can be viewed as an abstract reformulation of the properties of bounded subsets in a locally convex vector space.

Lemma 1 Let X be a topological space and f and g be continuous functions on X with compact support. Then, the closure of set $U = \{(x, y) | f(x) \neq 0, g(y) \neq 0\}$ equals $\text{supp}(f) \times \text{supp}(g)$.

Proof Take $(x, y) \in \text{supp}(f) \times \text{supp}(g)$ and choose any opens $V \times W$ of $X \times X$, where V, W are open sets of X . We note that $(V \times W) \cap U = (V \cap \{a \in X | f(a) \neq 0\}) \times (W \cap \{b \in X | g(b) \neq 0\})$.

Since $(x, y) \in \text{supp}(f) \times \text{supp}(g)$, $(V \times W) \cap U$ is nonempty. Thus, $\overline{U}$ contains $\text{supp}(f) \times \text{supp}(g)$, and $\overline{U}$ is contained in $\text{supp}(f) \times \text{supp}(g)$.

This completes the proof.

Definition 2^[6] A weak bialgebra (WBA) $(A; m, \mu; \epsilon, \Delta)$ is defined by the following properties:

(i) A is a finite-dimensional associative algebra over a field k with multiplication $m: A \otimes A \rightarrow A$ and unit $\mu: k \rightarrow A$, which are k -linear maps;

(ii) A is a coalgebra over k with comultiplication $\Delta: A \rightarrow A \otimes A$ and counit $\epsilon: A \rightarrow k$, which are k -linear maps;

(iii) The algebra and coalgebra structures obey the compatibility conditions for $a, b, c \in A$:

$$\Delta(ab) = \Delta(a)\Delta(b) \quad (1)$$

$$\epsilon(ab_{(1)})\epsilon(b_{(2)}c) = \epsilon(abc) = \epsilon(ab_{(2)})\epsilon(b_{(1)}c) \quad (2)$$

$$\begin{aligned} 1_{(1)} \otimes 1_{(2)} 1_{(1')} \otimes 1_{(2')} &= 1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = \\ 1_{(1)} \otimes 1_{(1')} 1_{(2)} \otimes 1_{(2')} & \end{aligned} \quad (3)$$

where $ab \equiv m(a, b)$, $1: = \mu(1)$. We used Sweedler's notation for iterated coproducts, omitting summation indices and a summation symbol.

Definition 3^[6-8] A weak Hopf algebra (WBH) $(A; m, \mu; \epsilon, \Delta; S)$ is a WBA used together with the following property.

There exists a k -linear map $S: A \rightarrow A$, called the antipode, that satisfies the following for $a \in A$:

$$a_{(1)}S(a_{(2)}) = \epsilon(1_{(1)}a)1_{(2)} \quad (4)$$

$$S(a_{(1)})a_{(2)} = 1_{(1)}\epsilon(a1_{(2)}) \quad (5)$$

$$S(a) = S(a_{(1)})a_{(2)}S(a_{(3)}) \quad (6)$$

In what follows, we discuss several fundamental properties of WBHs^[3]. We recall some of the notations and propositions used in this article.

For all $a \in A$, we have

$$\begin{aligned}\square^L(a) &:= \epsilon(1_{(1)}a)1_{(2)} = a_{(1)}S(a_{(2)}) \\ \square^R(a) &:= 1_{(1)}\epsilon(a1_{(2)}) = S(a_{(1)})a_{(2)}\end{aligned}\quad (7)$$

$$\begin{aligned}\tilde{\square}^L(a) &:= \epsilon(a1_{(1)})1_{(2)} = S^{-1}(\square^R(a)) \\ \tilde{\square}^R(a) &:= 1_{(1)}\epsilon(1_{(2)}a) = S^{-1}(\square^L(a))\end{aligned}\quad (8)$$

Using Eq. (3), we have the following identities:

$$\begin{aligned}\square^R(a_{(1)}) \otimes a_{(2)} &= 1_{(1)} \otimes a1_{(2)} \\ \tilde{\square}^R(a_{(1)}) \otimes a_{(2)} &= 1_{(1)} \otimes 1_{(2)}a\end{aligned}\quad (9)$$

$$\begin{aligned}a_{(1)} \otimes \square^L(a_{(2)}) &= 1_{(1)}a \otimes 1_{(2)} \\ a_{(1)} \otimes \tilde{\square}^L(a_{(2)}) &= a1_{(1)} \otimes 1_{(2)}\end{aligned}\quad (10)$$

$$\begin{aligned}\square^R \circ S &= \square^R \circ \square^L = S \circ \square^L \\ \square^L \circ S &= \square^L \circ \square^R = S \circ \square^R\end{aligned}\quad (11)$$

Proposition 1 If Eq. (3) holds, then Eq. (2) is equivalent to Eq. (12) for $a, b \in A$, as follows:

$$\epsilon(ab) = \epsilon(a1_{(2)})\epsilon(1_{(1)}b) = \epsilon(a1_{(1)})\epsilon(1_{(2)}b) \quad (12)$$

Proof One direction is obvious. For the other direction, we have

$$\epsilon(abc) = \epsilon(ab1_{(1)})\epsilon(1_{(2)}c) = \epsilon(ab_{(1)})\epsilon(\tilde{\square}^L(b_{(2)})c)$$

and

$$\epsilon(ab_{(1)})\epsilon(b_{(2)}1_{(1)})\epsilon(1_{(2)}c) = \epsilon(ab_{(1)})\epsilon(b_{(2)}c)$$

Similarly,

$$\begin{aligned}\epsilon(abc) &= \epsilon(ab1_{(2)})\epsilon(1_{(1)}c) = \epsilon(ab_{(2)})\epsilon(\square^R(b_{(1)})c) = \\ &= \epsilon(ab_{(2)})\epsilon(b_{(1)}1_{(2)})\epsilon(1_{(1)}c) = \epsilon(ab_{(2)})\epsilon(b_{(1)}c)\end{aligned}$$

Proposition 2 If Eq. (3) holds, Eq. (6) is equivalent to the following equation:

$$S(a) = 1_{(1)}S(a1_{(2)}) = S(1_{(1)}a)1_{(2)}$$

Proof From Eqs. (9) and (10), we note that

$$S(a_{(1)})a_{(2)}S(a_{(3)}) = \square^R(a_{(1)})S(a_{(2)}) = 1_{(1)}S(a1_{(2)})$$

and

$$S(a_{(1)})a_{(2)}S(a_{(3)}) = S(1_{(1)}a)1_{(2)}$$

This finishes the proof.

Proposition 3 Let $\epsilon: A \rightarrow k$ be a linear map and in two cases, suppose that we have $\square^L(a) = \epsilon(1_{(1)}a)1_{(2)}$ and $\square^R(a) = 1_{(1)}\epsilon(a1_{(2)})$, such that

$$a_{(1)} \otimes \square^L(a_{(2)}) = 1_{(1)}a \otimes 1_{(2)}$$

and

$$\square^R(a_{(1)}) \otimes a_{(2)} = 1_{(1)} \otimes a1_{(2)}$$

In the first case, if $\epsilon(1_{(2)}) = 1$, then $a_{(1)}\epsilon(a_{(2)}) = a$. And in the second case, if $\epsilon(1_{(1)})1_{(2)} = 1$, then $\epsilon(a_{(1)})a_{(2)} = a$.

Therefore, we have an alternative definition of WBH as follows.

Lemma 2 A is a weak Hopf algebra if and only if the following conditions hold.

(i) A is a finite-dimensional associative algebra over a field k with multiplication $m: A \otimes A \rightarrow A$ and unit $\mu:$

$k \rightarrow A$, which are k -linear maps.

(ii) A is a coalgebra over k with comultiplication $\Delta: A \rightarrow A \otimes A$, counit $\epsilon: A \rightarrow k$, which are k -linear maps.

(iii) The algebra and coalgebra structures obey the compatibility conditions:

$$\Delta(ab) = \Delta(a)\Delta(b) \quad a, b \in A$$

$$\epsilon(ab) = \epsilon(a1_{(1)})\epsilon(1_{(2)}b) = \epsilon(a1_{(2)})\epsilon(1_{(1)}b)$$

$$\begin{aligned}1_{(1)} \otimes 1_{(2)}1_{(1')} \otimes 1_{(2')} &= 1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = \\ &= 1_{(1)} \otimes 1_{(1')}1_{(2)} \otimes 1_{(2')}\end{aligned}$$

(iv) There exists a k -linear map $S: A \rightarrow A$ satisfying

$$a_{(1)}S(a_{(2)}) = \epsilon(1_{(1)}a)1_{(2)}$$

$$S(a_{(1)})a_{(2)} = 1_{(1)}\epsilon(a1_{(2)})$$

$$S(a) = 1_{(1)}S(a1_{(2)}) = S(1_{(1)}a)1_{(2)} \quad a \in A$$

2 Definitions and Examples

Definition 4 Let V be a k -space and $f: V \rightarrow V$ be a linear k -map. Then, f is called a weak bijection if there exists a linear k -map $g: V \rightarrow V$ such that the following identities hold:

$$f \circ g \circ f = f, \quad g \circ f \circ g = g$$

In this case, we call g a weak inverse of f .

Example 1 (1) Any bijection is a weak bijection^[4,9-14].

(2) If A is a WBH, then the linear maps $T_1, T_2: A \otimes A \rightarrow A \otimes A$ are defined by^[9-11]

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b)$$

$$T_2(a \otimes b) = (a \otimes 1)\Delta(b) \quad (13)$$

It is easy to check whether T_1 and T_2 are weak bijections using

$$R_1(a \otimes b) = ((1 \otimes S)\Delta(a))(1 \otimes b)$$

and

$$R_2(a \otimes b) = (a \otimes 1)((S \otimes 1)\Delta(b))$$

(3) If A is a WBH with a bijective antipode S , the linear maps $T_3, T_4: A \otimes A \rightarrow A \otimes A$, defined by

$$T_3(a \otimes b) = \Delta(a)(b \otimes 1)$$

$$T_4(a \otimes b) = (1 \otimes a)\Delta(b) \quad (14)$$

are weak bijections.

Let $R_3, R_4: A \otimes A \rightarrow A \otimes A$ be defined by

$$R_3(a \otimes b) = ((\iota \otimes S^{-1})\Delta^{\text{cop}}(b))(1 \otimes a)$$

$$R_4(a \otimes b) = (b \otimes 1)((S^{-1} \otimes \iota)\Delta^{\text{cop}}(a))$$

It is easy to obtain the following results:

$$T_3R_3T_3 = T_3, \quad R_3T_3R_3 = R_3$$

$$T_4R_4T_4 = T_4, \quad R_4T_4R_4 = R_4$$

$$T_3R_3(a \otimes b) = \Delta(1)(a \otimes b)$$

$$T_4R_4(a \otimes b) = (a \otimes b)\Delta(1)$$

The following proposition and definition are straightforward.

Proposition 4 Let A be a WBH. Then, we obtain

- (1) $mR_1T_1 = m$ and $mR_2T_2 = m$;
- (2) $mR_3T_3 = m$ and $mR_4T_4 = m$.

Next, we will use the notation.

For bornological spaces M, N, Q and $P = m \hat{\otimes} n$: $M \hat{\otimes} N \rightarrow M \hat{\otimes} N$, we set

$$P_{(1,2)} = P \hat{\otimes} \iota: M \hat{\otimes} N \hat{\otimes} Q \rightarrow M \hat{\otimes} N \hat{\otimes} Q$$

$$P_{(2,3)} = \iota \hat{\otimes} P: Q \hat{\otimes} M \hat{\otimes} N \rightarrow Q \hat{\otimes} M \hat{\otimes} N$$

and

$$P_{(1,3)} = m \hat{\otimes} \iota \hat{\otimes} n: M \hat{\otimes} Q \hat{\otimes} N \rightarrow M \hat{\otimes} Q \hat{\otimes} N$$

Definition 5 Let A be an algebra over $\mathbf{C}$ (the field of complex numbers) with or without an identity, and Δ be a comultiplication on A . We call A a weak Hopf bornological algebra (WBHA) if the linear maps T_1, T_2 : $A \hat{\otimes} A \rightarrow A \hat{\otimes} A$, defined by

$$T_1(a \hat{\otimes} b) = \Delta(a)(1 \hat{\otimes} b), T_2(a \hat{\otimes} b) = (a \hat{\otimes} 1)\Delta(b)$$

are weak bijections such that

- (i) $(T_1R_1, T_2R_2) \in M(A \hat{\otimes} A)$;
- (ii) $mR_1T_1 = m$ and $mR_2T_2 = m$;
- (iii) for all $a \in A$, $\Delta(a)$ is stable under T_1R_1 and T_2R_2 ;
- (iv) R_1T_1 and R_2T_2 are A -bimodules, where $A \hat{\otimes} A$ has a natural module structure induced by multiplication;
- (v) $(\Delta \hat{\otimes} \iota)T_1R_1(a \hat{\otimes} b) = (\iota \hat{\otimes} T_1R_1)(\Delta(a) \hat{\otimes} b) = (T_1R_1 \hat{\otimes} \iota)(\iota \hat{\otimes} T_1R_1)(\Delta(a) \hat{\otimes} b)$;
- (vi) $(\iota \hat{\otimes} \Delta)T_2R_2(a \hat{\otimes} b) = (T_2R_2 \hat{\otimes} \iota)(a \hat{\otimes} \Delta(b)) = (\iota \hat{\otimes} T_2R_2)(T_2R_2 \hat{\otimes} \iota)(a \hat{\otimes} \Delta(b))$;
- (vii) $(\iota \hat{\otimes} R_1T_1)(\Delta \hat{\otimes} \iota) = (\Delta \hat{\otimes} \iota)R_1T_1$ and $(R_2T_2 \hat{\otimes} \iota)(\iota \hat{\otimes} \Delta) = (\iota \hat{\otimes} \Delta)R_2T_2$;
- (viii) $(1 \hat{\otimes} \sigma)(\Delta \hat{\otimes} 1)R_2T_2 = (R_2T_2 \hat{\otimes} 1)(1 \hat{\otimes} \sigma)(\Delta \hat{\otimes} 1)$;
- (ix) $(R_2T_2 \hat{\otimes} 1)(T_2R_2)_{(1,3)} = (T_2R_2)_{(1,3)}(R_2T_2 \hat{\otimes} 1)$.

Furthermore, we call A regular (or weak regular multiplier Hopf algebra) if the following conditions hold:

- (1) $\sigma\Delta$ is a comultiplication, where σ is the flipping map: $A \hat{\otimes} A \rightarrow A \hat{\otimes} A$;
- (2) $(A, \sigma\Delta)$ is a weak Hopf bornological algebra, where the two weak bijection maps T_3 and T_4 are defined by $T_3(a \hat{\otimes} b) = \Delta(a)(b \hat{\otimes} 1)$ and $T_4(a \hat{\otimes} b) = (1 \hat{\otimes} a)\Delta(b)$ with weak inverse maps R_3 and R_4 , respectively, such that $T_1R_1 = T_3R_3$ and $T_2R_2 = T_4R_4$;
- (3) The antipode S satisfies

$$\sigma(S \hat{\otimes} S) \circ T_1R_1 = T_2R_2 \circ \sigma(S \hat{\otimes} S)$$

Definition 6 If A is a $*$ -algebra, we call A a weak Hopf bornological $*$ -algebra if comultiplication Δ is a $*$ -homomorphism, turning A into a WBHA.

Example 2 Let G be any Lie groupoid. Let G_o denote a set of objects of G and G_m denote a set of morphisms. The identity morphism on $x \in G_o$ will also be denoted by x . The source and target maps will be denoted

by s and t , respectively, i. e., for $\alpha: x \rightarrow y$ in G_m , we have $s(\alpha) = x$ and $t(\alpha) = y$. The groupoid algebra is the direct product $kG = \bigoplus_{\alpha \in G_m} ku_\alpha$, with multiplication defined by the rule $u_\alpha u_\beta = u_{\alpha\beta}$ if $s(\alpha) = t(\beta)$, and $u_\alpha u_\beta = 0$ if $s(\alpha) \neq t(\beta)$.

Define comultiplication $\Delta: kG \rightarrow M(kG \hat{\otimes} kG)$ by the formula $\Delta(u_\alpha) = u_\alpha \hat{\otimes} u_\alpha$, $\forall \alpha \in G_m$. It is obvious that Δ is a comultiplication of kG . The identity ι of kG is defined by $\iota = \bigoplus_{x \in G_o} u_x$. Therefore, we have $\Delta(\iota) =$

$$\bigoplus_{x \in G_o} u_x \hat{\otimes} u_x.$$

The map T_1 is defined by

$$T_1(u_\alpha \hat{\otimes} u_\beta) = \begin{cases} u_\alpha \hat{\otimes} u_{\alpha\beta} & s(x) = t(y) \\ 0 & \text{others} \end{cases}$$

which is a weak bijection, and its weak inverse R_1 is given by

$$R_1(u_\alpha \hat{\otimes} u_\beta) = \begin{cases} u_\alpha \hat{\otimes} u_{\alpha^{-1}\beta} & t(\alpha) = t(\beta) \\ 0 & t(\alpha) \neq t(\beta) \end{cases}$$

Furthermore, it is straightforward to check that $T_1R_1 = \Delta(\iota)$ and $mR_1T_1 = m$. Similarly, the map T_2 is defined by

$$T_2(u_\alpha \hat{\otimes} u_\beta) = \begin{cases} u_{\alpha\beta} \hat{\otimes} u_\beta & s(\alpha) = t(\beta) \\ 0 & s(\alpha) \neq t(\beta) \end{cases}$$

which is a weak bijection, and its weak inverse R_2 is given by

$$R_2(u_\alpha \hat{\otimes} u_\beta) = \begin{cases} u_{\alpha\beta^{-1}} \hat{\otimes} u_\beta & s(\alpha) = s(\beta) \\ 0 & s(\alpha) \neq s(\beta) \end{cases}$$

It is straightforward to check that $T_2R_2 = \Delta(\iota)$ and $mR_2T_2 = m$.

Example 3 Let G be any finite Lie groupoid. A notation similar to that in Example 2 is used. Let A be the algebra of complex finitely supported functions on G_m . In this case, A consists of all complex functions on G_m . A has a base $\{v_\alpha\}_{\alpha \in G_m}$, and its multiplication is given by $v_\alpha v_\beta = \delta_{\alpha,\beta} v_\alpha$. The identity ι of A is given by $\iota = \sum_{\alpha \in G_m} v_\alpha$.

If we define $\Delta: A \rightarrow A \hat{\otimes} A$ using the form $\sum_{t(\alpha)=t(\beta)} v_\beta \hat{\otimes} v_{\beta^{-1}\alpha}$, then Δ is a comultiplication on A and $\Delta(\iota) = \sum_{\alpha, \beta \in G_m; t(\alpha)=t(\beta)} v_\beta \hat{\otimes} v_{\beta^{-1}\alpha}$.

The map T_1 is a weak bijection, and its weak inverse R_1 is given by $(R_1f)(\alpha, \beta) = f(\alpha\beta^{-1}, \beta)$ when $f \in A \hat{\otimes} A$ and $\alpha, \beta \in G_m$. Similarly, the map T_2 is a weak bijection, and its weak inverse R_2 is given by $(R_2f)(\alpha, \beta) = f(\alpha, \alpha^{-1}\beta)$.

It is straightforward to check that $T_1R_1 = \Delta(\iota)$, $T_2R_2 = \Delta(\iota)$, $mR_1T_1 = m$ and $mR_2T_2 = m$.

Example 4 Suppose $G \Rightarrow M$ is a Lie groupoid with a

source s and target map t . We write $C_c^\infty(G)$ for the bornological algebra of smooth functions on G with pointwise multiplication. We define a bounded linear map Δ on $C_c^\infty(G)$ as follows:

$$\Delta: C_c^\infty(G) \rightarrow M(C_c^\infty(G \times G))$$

by

$$\Delta(f)(x, y) = \begin{cases} f(xy) & s(x) = t(y) \\ 0 & \text{others} \end{cases}$$

for any $x, y \in G$ and $f \in C_c^\infty(G)$. Then, we have bounded linear maps $T_i: C_c^\infty(G) \hat{\otimes} C_c^\infty(G) \rightarrow M(C_c^\infty(G \times G))$, with $i = 1, 2$ for Δ given by

$$T_1(f \hat{\otimes} g)(x, y) = T_3(f \hat{\otimes} g)(x, y) = \begin{cases} f(xy)g(y) & s(x) = t(y) \\ 0 & \text{others} \end{cases}$$

$$T_2(f \hat{\otimes} g)(x, y) = T_4(f \hat{\otimes} g)(x, y) = \begin{cases} f(x)g(xy) & s(x) = t(y) \\ 0 & \text{others} \end{cases}$$

for any $x, y \in G$ and $f, g \in C_c^\infty(G)$.

For any $f, g \in C_c^\infty(G)$, obviously, Δ is a homomorphism, i. e., $\Delta(fg) = \Delta(f)\Delta(g)$. We now check whether $T_1(f \hat{\otimes} g)$ is a smooth function on $G \times G$ with compact support.

Because the source and target maps s, t are smooth submersions, we note that the inverse map $i: G \rightarrow G$ is a diffeomorphism by the global constant rank theorem.

Now, we state that the support of $T_1(f \hat{\otimes} g)$ is compact. For this, we write U as $\{(p, q) \in G \times G \mid \Delta(f) \cdot (1 \otimes g)(p, q) \neq 0\}$. Note that the support of $T_1(f \hat{\otimes} g)$ is the closure of U .

For any $(p, q) \in U$, if $T_1(f \hat{\otimes} g)(p, q) \neq 0$, then $\Delta(f)(p, q)g(q) \neq 0$. Thus, $q \in \text{supp}(g)$. Meanwhile, $\Delta(f)(p, q) \neq 0$ tells us that pq is defined, and $f(pq) \neq 0$, i. e., $pq \in \text{supp}(f)$. Since $p = (pq)q^{-1}$, we know that $p \in m(\text{supp}(f) \times i^{-1}(\text{supp}(g)))$, namely K , where $m: G \times G$, the multiplication of G , is a smooth function. As mentioned before, i is a diffeomorphism, so $i^{-1}(\text{supp}(g))$ is compact. By the continuity of m , we know that U is contained in $K \times \text{supp}(g)$, a compact set of $G \times G$, and $G \times G$ is a Hausdorff dimension; therefore, $K \times \text{supp}(g)$ is closed. Thus, $\bar{U}$ is a closed subset of a compact set K . This tells us that the support of $T_1(f \hat{\otimes} g)$ is compact.

Similarly, $T_2(f \hat{\otimes} g)$ is a smooth function on $G \times G$ with compact support. Therefore, using Example 2, we have

$$T_i(C_c^\infty(G) \hat{\otimes} C_c^\infty(G)) \subseteq C_c^\infty(G) \otimes C_c^\infty(G) \quad i = 1, 2$$

Next, we check whether Δ is coassociative. For this, we only check whether

$$(T_2 \hat{\otimes} \text{id})(\text{id} \hat{\otimes} T_1) = (\text{id} \hat{\otimes} T_1)(T_2 \hat{\otimes} \text{id})$$

based on the definition of comultiplication.

For any $x, y, z \in G$ and $f, g, h \in C_c^\infty(G)$, we compute

$$\begin{aligned} & [((T_2 \hat{\otimes} \text{id})(\text{id} \hat{\otimes} T_1))(f \hat{\otimes} g \hat{\otimes} h)](x, y, z) = \\ & [(T_2 \hat{\otimes} \text{id})(f \hat{\otimes} T_1(g \hat{\otimes} h))](x, y, z) = \\ & [T_2(f \hat{\otimes} T_1^1(g \hat{\otimes} h)) \hat{\otimes} T_1^2(g \hat{\otimes} h)](x, y, z) = \\ & f(x)T_1^1(g \hat{\otimes} h)(xy)T_1^2(g \hat{\otimes} h)(z) = \\ & f(x)T_1(g \hat{\otimes} h)(xy, z) = f(x)g(xyz)h(z) = \\ & [((\text{id} \hat{\otimes} T_1)(T_2 \hat{\otimes} \text{id}))(f \hat{\otimes} g \hat{\otimes} h)](x, y, z) \end{aligned}$$

We define a bounded linear map

$$\epsilon: C_c^\infty(G) \rightarrow \mathbf{C}, f \mapsto \sum_{e \in G_0} f(e)$$

where $G_0 \subseteq M$ is the set of unit elements of a groupoid G .

For any $x \in G$ and $f, g \in C_c^\infty(G)$, we have

$$\begin{aligned} & [(\epsilon \hat{\otimes} \iota)T_1(f \hat{\otimes} g)](x) = \epsilon(T_1^1(f \hat{\otimes} g))T_1^2(f \hat{\otimes} g)(x) = \\ & \sum_{e \in G_0} T_1^1(f \hat{\otimes} g)(e)T_1^2(f \hat{\otimes} g)(x) = \\ & \sum_{e \in G_0} T_1(f \hat{\otimes} g)(e, x) = \end{aligned}$$

$$\begin{cases} \sum_{e \in G_0} f(ex)g(x) & s(e) = t(x) \\ 0 & \text{others} \end{cases} = (fg)(x)$$

where $T_1(f \hat{\otimes} g) = \sum T_1^1(f \hat{\otimes} g) \hat{\otimes} T_1^2(f \hat{\otimes} g)$ in the second equation above.

We define the idempotent element $\Delta(1) = E \in M(C_c^\infty(G \times G))$ as a function on $G \times G$ by

$$E(x, y) = \begin{cases} 1 & s(x) = t(y) \\ 0 & \text{others} \end{cases}$$

for any $x, y \in G$. It is trivial to see that E is an idempotent in $M(C_c^\infty(G \times G))$. An antipode is given by $S: C_c^\infty(G) \rightarrow C_c^\infty(G)$ by $S(f)(x) = f(x^{-1})$ for any $x \in G$.

3 Conclusions

Our work generalizes the classical notion of weak Hopf algebras and Hopf bornological algebras. The findings of the study can be applied to the cases of the classical notion of a Lie groupoid $G \Rightarrow M$ expressed as a bornological algebraic Lie groupoid $C_c^\infty(G)$ and a Lie groupoid algebra CG . Our results can be used to consider more interesting objects when focusing on the topologies of some assumed bornologies, such as C^* algebras and locally convex spaces.

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弱有界型 Hopf 代数

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摘要: 本文引入了弱有界型 Hopf 代数的概念, 该对象同时推广了弱 Hopf 代数与有界型 Hopf 代数的经典框架。此推广旨在将代数结构与拓扑有界型结构统一起来, 以更自然地处理出现在局部凸空间与群胚代数中的 Hopf 型结构。本文首先给出了弱有界型 Hopf 代数的定义与基本性质, 证明了若干与一般弱 Hopf 代数类似的重要结果, 如反对极的唯一性及其对乘法、余乘法的反同态性。最后, 研究了有限李群胚上的弱有界型 Hopf 代数结构, 构造了若干基本例子, 并计算了它们的余乘、对极与余单位。结果表明, 弱有界条件可由有限李群胚的复合律自然导出。该理论框架为 Hopf 型代数的研究提供了新的视角, 也为有界型与解析情形下的对称性与对偶性研究开辟了新途径。

关键词: (弱)有界型 Hopf 代数; 弱 Hopf 代数; 李群胚; 局部凸空间

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