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A note on the Moore-Penrose inverse of a companion matrix

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Abstract: Let R be an associative ring with unity 1. The existence of the Moore-Penrose inverses of block matrices over R is investigated and the sufficient and necessary conditions for such existence are obtained. Furthermore, the representation of the Moore-Penrose inverse of $\boldsymbol{M}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{A}\\ \boldsymbol{C} & \boldsymbol{B}\end{bmatrix}$ is given under the condition of $\boldsymbol{EBF}=\boldsymbol{0}$, where $\boldsymbol{E}=\boldsymbol{I}-\boldsymbol{CC}^{\dagger}$ and $\boldsymbol{F}=\boldsymbol{I}-\boldsymbol{A}^{\dagger}\boldsymbol{A}$. This result generalizes the representation of the Moore-Penrose inverse of the companion matrix $\boldsymbol{M}=\begin{bmatrix}\boldsymbol{0} & a\\ \boldsymbol{I}_n & \boldsymbol{b}\end{bmatrix}$ due to Pedro Patrício. As for applications, some examples are given to illustrate the obtained results.

Key words: companion matrix; Moore-Penrose inverse; ring

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Let R be an associative ring with unity 1 and involution $*$, which is an anti-automorphism of degree 2 in R: $(x^*)^*=x$, $(x+y)^*=x^*+y^*$ and $(xy)^*=y^*x^*$ for all $x, y\in R$.

An element $a\in R$ is said to be regular if there is an element a^- of R such that $aa^-a=a$, or equivalently $axa=a$ is a ring consistent equation. In this case, a^- is called a $\{1\}$-inverse of a. Let $\boldsymbol{M}_{m\times n}(R)$ denote the ring of all $m\times n$ matrices over R. We denote $\boldsymbol{M}_{n\times n}(R)$, $\boldsymbol{M}_{n\times 1}(R)$ and $\boldsymbol{M}_{1\times n}(R)$ by $\boldsymbol{M}_n(R)$, $\boldsymbol{R}^n$ and $\boldsymbol{R}^{(n)}$, respectively. A matrix $\boldsymbol{A}$ is said to be Moore-Penrose invertible with respect to $*$, if there is $\boldsymbol{A}^{\dagger}$ such that

$$\boldsymbol{AA}^{\dagger}\boldsymbol{A}=\boldsymbol{A},\ \boldsymbol{A}^{\dagger}\boldsymbol{AA}^{\dagger}=\boldsymbol{A}^{\dagger},(\boldsymbol{AA}^{\dagger})^*=\boldsymbol{AA}^{\dagger},(\boldsymbol{A}^{\dagger}\boldsymbol{A})^*=\boldsymbol{A}^{\dagger}\boldsymbol{A}$$

It is well known that the Moore-Penrose inverse is unique if it exists.

The existence and representations of the Moore-Penrose inverse (MP-inverse) of matrices over various settings have been considered by several scholars[1-4]. Recently, Hartwig and Patrício[5] obtained new expressions for the MP-inverse of the matrix $\begin{bmatrix}a & 0\\ b & d\end{bmatrix}$ over a $*$-regular ring. Zhu et al.[6] investigated the MP-inverse of $\boldsymbol{M}=\begin{bmatrix}a & c\\ b & d\end{bmatrix}$ over a $*$-regular ring satisfying SC_2. It is well-known that R is a $*$-regular ring if and only if all the elements in R are MP-invertible, and that $\boldsymbol{M}_2(R)$ is a $*$-regular ring if and only if R is a regular $*$-ring satisfying SC_2[5]. That is to say, in this case, there is always the MP-inverse of $\boldsymbol{A}\in\boldsymbol{M}_2(R)$ over a $*$-regular ring. In Ref.[7], the conditions for the existence of the MP-inverse of the $(n+1)\times(n+1)$ companion matrix in the form $\boldsymbol{M}=\begin{bmatrix}\boldsymbol{0} & a\\ \boldsymbol{I}_n & \boldsymbol{b}\end{bmatrix}$ over an arbitrary ring are considered and the formulae of $\boldsymbol{M}^{\dagger}$ is established. This paper is to present some equivalent conditions concerning the existence of MP-inverse of block matrices over an arbitrary ring. In what follows, we use the symbols $\boldsymbol{R}(\boldsymbol{A})=\{\boldsymbol{Ax}\mid \boldsymbol{x}\in\boldsymbol{R}^n\}$ and $\boldsymbol{R}_r(\boldsymbol{A})=\{\boldsymbol{xA}\mid \boldsymbol{x}\in\boldsymbol{R}^{(n)}\}$ to denote the range of $\boldsymbol{A}$ and the row range of $\boldsymbol{A}$, respectively.

1 Main Results

Lemma 1[7] Let $a\in R$ be a regular element, and a^- be a regular inverse of a. The following conditions are equivalent:

1) a is Moore-Penrose invertible;

2) $u=aa^*+1-aa^-$ is a unit. Moreover, $(a^{\dagger})^*=u^{-1}a$.

Lemma 2 Suppose that $\boldsymbol{A}\in\boldsymbol{M}_n(R)$ and $\boldsymbol{A}$ is regular, then the following are equivalent:

1) $\boldsymbol{A}$ is Moore-Penrose invertible;

2) $\boldsymbol{R}(\boldsymbol{A})=\boldsymbol{R}(\boldsymbol{AA}^*)$ and $\boldsymbol{R}_r(\boldsymbol{A})=\boldsymbol{R}_r(\boldsymbol{A}^*\boldsymbol{A})$;

3) Matrix equations $\boldsymbol{A}=\boldsymbol{AA}^*\boldsymbol{X}$ and $\boldsymbol{YA}^*\boldsymbol{A}=\boldsymbol{A}$ have solutions over $\boldsymbol{M}_n(R)$.

Furthermore, if $\boldsymbol{A}$ is Moore-Penrose invertible, then $\boldsymbol{A}^{\dagger}=\boldsymbol{A}^*\boldsymbol{XY}^*(\boldsymbol{AA}^-)^*$, where $\boldsymbol{X}$, $\boldsymbol{Y}$ are the corresponding solution sets of the matrix equations in 3).

Proof 2)⇔3) It is clear. Thus we only need to prove 1)⇔2).

2)⇒1). By hypothesis, there exist $\boldsymbol{X}$, $\boldsymbol{Y}$ such that $\boldsymbol{A}=\boldsymbol{AA}^*\boldsymbol{X}$ and $\boldsymbol{YA}^*\boldsymbol{A}=\boldsymbol{A}$. Then $\boldsymbol{A}=\boldsymbol{A}(\boldsymbol{A}^*\boldsymbol{AY}^*)\boldsymbol{X}$ and $\boldsymbol{A}=\boldsymbol{Y}(\boldsymbol{X}^*\boldsymbol{AA}^*)\boldsymbol{A}$. This implies that $\boldsymbol{R}(\boldsymbol{A})=\boldsymbol{R}(\boldsymbol{AA}^*\boldsymbol{A})$ and $\boldsymbol{R}_r(\boldsymbol{A})=\boldsymbol{R}_r(\boldsymbol{AA}^*\boldsymbol{A})$. Thus, $\boldsymbol{A}$ is Moore-Penrose invertible (see Ref.[8]).

1)⇒2). It is well known that $\boldsymbol{A}^{\dagger}$ exists if and only if $\boldsymbol{R}(\boldsymbol{A})=\boldsymbol{R}(\boldsymbol{AA}^*\boldsymbol{A})$ and $\boldsymbol{R}_r(\boldsymbol{A})=\boldsymbol{R}_r(\boldsymbol{AA}^*\boldsymbol{A})$. So we

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obtain $\boldsymbol{R}(\boldsymbol{A}) = \boldsymbol{R}(\boldsymbol{AA}^*)$ and $\boldsymbol{R}_r(\boldsymbol{A}) = \boldsymbol{R}_r(\boldsymbol{A}^*\boldsymbol{A})$.

By Ref. [8], we have that $\boldsymbol{v} = \boldsymbol{AA}^* + \boldsymbol{I} - \boldsymbol{AA}^-$ is a unit of $\boldsymbol{M}_n(R)$. Also, $\boldsymbol{v}^{-1} = \boldsymbol{AA}^-(\boldsymbol{YX}^*)\boldsymbol{AA}^- + \boldsymbol{I} - \boldsymbol{AA}^-(\boldsymbol{YX}^*)\boldsymbol{AA}^*$. It is simple to check that $\boldsymbol{A}^\dagger = \boldsymbol{A}^*\boldsymbol{XY}^*(\boldsymbol{AA}^-)^*$.

Proposition 1 Suppose that $\boldsymbol{M} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{B} & \boldsymbol{BCA} \end{bmatrix}$ such that $\boldsymbol{A}, \boldsymbol{B}$ are regular. Then $\boldsymbol{M}$ is Moore-Penrose invertible if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ are both Moore-Penrose invertible.

Proof We give the decomposition of $\boldsymbol{M}$ as follows:

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{CA} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} = \boldsymbol{NQ} \tag{1}$$

and

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{BC} & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} = \boldsymbol{PN} \tag{2}$$

By Ref. [9], it is simple to determine that $\boldsymbol{M}$ is regular. According to Eqs. (1) and (2), it follows that

$$\boldsymbol{M}^*\boldsymbol{M} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ (\boldsymbol{CA})^* & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{B}^*\boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}^*\boldsymbol{A} \end{bmatrix}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{CA} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}$$

and

$$\boldsymbol{MM}^* = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{BC} & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{AA}^* & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{BB}^* \end{bmatrix}\begin{bmatrix} \boldsymbol{I} & (\boldsymbol{BC})^* \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}$$

By Lemma 2, $\boldsymbol{M}$ is Moore-Penrose invertible if and only if $\boldsymbol{MM}^*\boldsymbol{X} = \boldsymbol{M}$ and $\boldsymbol{YM}^*\boldsymbol{M} = \boldsymbol{M}$ are consistent.

Let $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{I} & -(\boldsymbol{BC})^* \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \\ \boldsymbol{x}_3 & \boldsymbol{x}_4 \end{bmatrix}$.

From $\begin{bmatrix} \boldsymbol{AA}^* & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{BB}^* \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \\ \boldsymbol{x}_3 & \boldsymbol{x}_4 \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix}$, we have

$$\boldsymbol{AA}^*\boldsymbol{x}_2 = \boldsymbol{A} \tag{3}$$

$$\boldsymbol{BB}^*\boldsymbol{x}_4 = \boldsymbol{B} \tag{4}$$

Similarly, set $\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{y}_1 & \boldsymbol{y}_2 \\ \boldsymbol{y}_3 & \boldsymbol{y}_4 \end{bmatrix}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -(\boldsymbol{CA})^* & \boldsymbol{I} \end{bmatrix}$.

From $\begin{bmatrix} \boldsymbol{y}_1 & \boldsymbol{y}_2 \\ \boldsymbol{y}_3 & \boldsymbol{y}_4 \end{bmatrix}\begin{bmatrix} \boldsymbol{B}^*\boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}^*\boldsymbol{A} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix}$, it follows that

$$\boldsymbol{y}_2\boldsymbol{A}^*\boldsymbol{A} = \boldsymbol{A} \tag{5}$$

$$\boldsymbol{y}_3\boldsymbol{B}^*\boldsymbol{B} = \boldsymbol{B} \tag{6}$$

In view of Eqs. (3) to (6), one can see that $\boldsymbol{A}, \boldsymbol{B}$ are MP-invertible if and only if $\boldsymbol{MM}^*\boldsymbol{X} = \boldsymbol{M}$ and $\boldsymbol{YM}^*\boldsymbol{M} = \boldsymbol{M}$ are consistent, as desired.

In the following, we will characterize the MP-invertibility of $\boldsymbol{M} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{C} & \boldsymbol{B} \end{bmatrix}$ under the condition of $\boldsymbol{EBF} = \boldsymbol{0}$, where $\boldsymbol{E} = \boldsymbol{I} - \boldsymbol{CC}^\dagger$ and $\boldsymbol{F} = \boldsymbol{I} - \boldsymbol{A}^\dagger\boldsymbol{A}$.

Theorem 1 Suppose that $\boldsymbol{M} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{C} & \boldsymbol{B} \end{bmatrix}$ such that $\boldsymbol{A}^\dagger$, $\boldsymbol{C}^\dagger$ exist and $\boldsymbol{EBF} = \boldsymbol{0}$. Then $\boldsymbol{M}^\dagger$ exists if and only if $\boldsymbol{u} = \boldsymbol{I} + \boldsymbol{C}^\dagger\boldsymbol{BF}(\boldsymbol{C}^\dagger\boldsymbol{B})^*$ and $\boldsymbol{v} = \boldsymbol{I} + (\boldsymbol{BA}^\dagger)^*\boldsymbol{EBA}^\dagger$ are invertible. If $\boldsymbol{M}$ is Moore-Penrose invertible, then

$$\boldsymbol{M}^\dagger = \begin{bmatrix} \boldsymbol{M}_1 & \boldsymbol{M}_2 \\ \boldsymbol{M}_3 & \boldsymbol{M}_4 \end{bmatrix}$$

where

$$\boldsymbol{M}_1 = -\boldsymbol{u}^{-1}\boldsymbol{C}^\dagger\boldsymbol{BA}^\dagger\boldsymbol{v}^{-1}$$
$$\boldsymbol{M}_2 = \boldsymbol{u}^{-1}\boldsymbol{C}^\dagger[\boldsymbol{I} - \boldsymbol{BA}^\dagger\boldsymbol{v}^{-1}(\boldsymbol{BA}^\dagger)^*\boldsymbol{E}]$$
$$\boldsymbol{M}_3 = \boldsymbol{TA}^\dagger\boldsymbol{v}^{-1}$$
$$\boldsymbol{M}_4 = \boldsymbol{TA}^\dagger\boldsymbol{v}^{-1}(\boldsymbol{BA}^\dagger)^*\boldsymbol{E} + \boldsymbol{F}(\boldsymbol{C}^\dagger\boldsymbol{B})^*\boldsymbol{u}^{-1}\boldsymbol{C}^\dagger$$
$$\boldsymbol{T} = \boldsymbol{I} - \boldsymbol{F}(\boldsymbol{C}^\dagger\boldsymbol{B})^*\boldsymbol{u}^{-1}\boldsymbol{C}^\dagger\boldsymbol{B}$$

Proof Note that $\boldsymbol{EBF} = \boldsymbol{0}$ and $\boldsymbol{M}$ is regular according to Ref. [9]. We give the decomposition of $\boldsymbol{M}$ as follows:

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{EBA}^\dagger & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{C}^\dagger\boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}$$

Let $\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{EBA}^\dagger & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{y}_1 & \boldsymbol{y}_2 \\ \boldsymbol{y}_3 & \boldsymbol{y}_4 \end{bmatrix}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -(\boldsymbol{C}^\dagger\boldsymbol{B})^* & \boldsymbol{I} \end{bmatrix}$.

From $\boldsymbol{YM}^*\boldsymbol{M} = \boldsymbol{M}$, we have

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_1 & \boldsymbol{y}_2 \\ \boldsymbol{y}_3 & \boldsymbol{y}_4 \end{bmatrix}\begin{bmatrix} \boldsymbol{C}^*\boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}^*\boldsymbol{A} + (\boldsymbol{BA}^\dagger\boldsymbol{A})^*\boldsymbol{EBA}^\dagger\boldsymbol{A} \end{bmatrix} \tag{7}$$

So, we obtain

$$\boldsymbol{y}_1\boldsymbol{C}^*\boldsymbol{C} = \boldsymbol{0} \tag{8}$$

$$\boldsymbol{y}_2[\boldsymbol{A}^*\boldsymbol{A} + (\boldsymbol{BA}^\dagger\boldsymbol{A})^*\boldsymbol{EBA}^\dagger\boldsymbol{A}] = \boldsymbol{A} \tag{9}$$

$$\boldsymbol{y}_3\boldsymbol{C}^*\boldsymbol{C} = \boldsymbol{C} \tag{10}$$

$$\boldsymbol{y}_4[\boldsymbol{A}^*\boldsymbol{A} + (\boldsymbol{BA}^\dagger\boldsymbol{A})^*\boldsymbol{EBA}^\dagger\boldsymbol{A}] = \boldsymbol{0} \tag{11}$$

It is clear that Eqs. (8), (10) and (11) are always consistent. Also, $\boldsymbol{y}_2[\boldsymbol{A}^*\boldsymbol{A} + (\boldsymbol{BA}^\dagger\boldsymbol{A})^*\boldsymbol{EBA}^\dagger\boldsymbol{A}] = \boldsymbol{A}$ is equivalent to $\boldsymbol{y}_2\boldsymbol{A}^*[\boldsymbol{I} + (\boldsymbol{BA}^\dagger)^*\boldsymbol{EBA}^\dagger] = \boldsymbol{AA}^\dagger$. Set $\boldsymbol{v} = \boldsymbol{I} + (\boldsymbol{BA}^\dagger)^*\boldsymbol{EBA}^\dagger$. It is simple to check that $\boldsymbol{vAA}^\dagger = \boldsymbol{AA}^\dagger\boldsymbol{v}$. If $\boldsymbol{v}$ is invertible, then Eq. (9) is consistent, and $\boldsymbol{y}_2 = \boldsymbol{v}^{-1}(\boldsymbol{A}^\dagger)^*$ is a solution.

Conversely, suppose that Eq. (9) is consistent. From $\boldsymbol{y}_2\boldsymbol{A}^*\boldsymbol{AA}^\dagger\boldsymbol{vAA}^\dagger = \boldsymbol{AA}^\dagger$ and $\boldsymbol{v}^* = \boldsymbol{v}$, we obtain that $\boldsymbol{R}(\boldsymbol{AA}^\dagger) = \boldsymbol{R}(\boldsymbol{AA}^\dagger\boldsymbol{vAA}^\dagger)$ and $\boldsymbol{R}_r(\boldsymbol{AA}^\dagger) = \boldsymbol{R}_r(\boldsymbol{AA}^\dagger\boldsymbol{vAA}^\dagger)$. By Ref. [8], we find that $\boldsymbol{AA}^\dagger\boldsymbol{v} + \boldsymbol{I} - \boldsymbol{AA}^\dagger$ is invertible. Note that $\boldsymbol{AA}^\dagger\boldsymbol{v} + \boldsymbol{I} - \boldsymbol{AA}^\dagger = \boldsymbol{v}$. Thus, $\boldsymbol{v}$ is invertible.

Similarly, we give the decomposition of $\boldsymbol{M}$ as follows:

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{BA}^\dagger & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{C}^\dagger\boldsymbol{BF} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{12}$$

which leads to

$$MM^* = \begin{bmatrix} I & 0 \\ BA^\dagger & I \end{bmatrix} \begin{bmatrix} AA^* & 0 \\ 0 & CC^* + BFB^* \end{bmatrix} \begin{bmatrix} I & (BA^\dagger)^* \\ 0 & I \end{bmatrix} \tag{13}$$

Let $X = \begin{bmatrix} I & -(BA^\dagger)^* \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} I & C^\dagger BF \\ 0 & I \end{bmatrix}$.

From $MM^* X = M$, it follows

$$\begin{bmatrix} AA^* & 0 \\ 0 & CC^* + BFB^* \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 0 & A \\ C & 0 \end{bmatrix} \tag{14}$$

Therefore,

$$AA^* x_1 = 0 \tag{15}$$

$$AA^* x_2 = A \tag{16}$$

$$(CC^* + BFB^*) x_3 = C \tag{17}$$

$$(CC^* + BFB^*) x_4 = 0 \tag{18}$$

It is simple to determine that Eqs. (15), (16) and (18) are consistent by Lemma 1. That implies that $M^\dagger$ exists if and only if Eqs. (9) and (17) are both consistent. By $EBF = 0$, we can find that Eq. (17) is equivalent to

$$[I + C^\dagger BF(C^\dagger B)^*] C^* x_3 = C^\dagger C \tag{19}$$

Set $u = I + C^\dagger BF(C^\dagger B)^*$. Thus, $C^\dagger Cu = uC^\dagger C$.

If u is invertible, then Eq. (19) is consistent, and $x_3 = (C^\dagger)^* u^{-1}$ is a solution.

Conversely, assume that Eq. (19) is consistent.

From $C^\dagger CuC^\dagger CC^* x_3 = C^\dagger C$ and $u^* = u$, we obtain $R(C^\dagger C) = R(C^\dagger CuC^\dagger C)$ and $R_r(C^\dagger C) = R_r(C^\dagger CuC^\dagger C)$. By Ref. [8], we find that $C^\dagger Cu + I - C^\dagger C$ is invertible. Note that $C^\dagger Cu + I - C^\dagger C = u$. Hence, it follows that u is invertible.

By direct computation, it is simple to find that

$$X = \begin{bmatrix} -(u^{-1}C^\dagger BA^\dagger)^* & (A^\dagger)^* - (u^{-1}C^\dagger BA^\dagger)^* C^\dagger BF \\ (u^{-1}C^\dagger)^* & (u^{-1}C^\dagger)^* C^\dagger BF \end{bmatrix}$$

and

$$Y = \begin{bmatrix} -(C^\dagger BA^\dagger v^{-1})^* & (A^\dagger v^{-1})^* \\ (C^\dagger)^* - EBA^\dagger (C^\dagger BA^\dagger v^{-1})^* & EBA^\dagger (A^\dagger v^{-1})^* \end{bmatrix}$$

are the solutions of $MM^* X = M$ and $YM^* M = M$. Using Lemma 2, $M^\dagger = M^* XY^* (MM^-)^*$. A direct computation yields $M^\dagger = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$, where

$$M_1 = -u^{-1} C^\dagger BA^\dagger v^{-1}$$
$$M_2 = u^{-1} C^\dagger [I - BA^\dagger v^{-1} (BA^\dagger)^* E]$$
$$M_3 = TA^\dagger v^{-1}$$
$$M_4 = TA^\dagger v^{-1} (BA^\dagger)^* E + F(C^\dagger B)^* u^{-1} C^\dagger$$
$$T = I - F(C^\dagger B)^* u^{-1} C^\dagger B$$

Corollary 1 Suppose that $M = \begin{bmatrix} 0 & A \\ C & B \end{bmatrix}$ such that $A^\dagger$, $C^\dagger$ exist and $B = CC^\dagger B$. Then $M^\dagger$ exists if and only if $u = I + C^\dagger B(I - A^\dagger A)(C^\dagger B)^*$ is invertible.

If M is Moore-Penrose invertible, then $M^\dagger = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$, where

$$M_1 = -u^{-1} C^\dagger BA^\dagger, \ M_2 = u^{-1} C^\dagger, \ M_3 = TA^\dagger$$
$$M_4 = (I - A^\dagger A)(C^\dagger B)^* u^{-1} C^\dagger$$
$$T = I - (I - A^\dagger A)(C^\dagger B)^* u^{-1} C^\dagger B$$

Corollary 2 Let $M = \begin{bmatrix} 0 & A \\ C & B \end{bmatrix}$ with $A^\dagger$, $C^\dagger$ exist and $B = BA^\dagger A$. Then $M^\dagger$ exists if and only if $v = I + (BA^\dagger)^* (I - CC^\dagger) BA^\dagger$ is invertible.

If M is Moore-Penrose invertible, then $M^\dagger = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$, where $M_1 = -C^\dagger BA^\dagger v^{-1}$, $M_2 = C^\dagger [I - BA^\dagger v^{-1} (BA^\dagger)^* E]$, $M_3 = A^\dagger v^{-1}$, $M_4 = A^\dagger v^{-1} (BA^\dagger)^* E$.

Note that $I + B^* (I - A^\dagger A) B$ is invertible if and only if $I + BB^* (I - A^\dagger A)$ is also invertible. Then we have the following result which generalizes the relative results of Ref. [7].

Corollary 3 Given $a \in R$ such that $a^\dagger$ exists and $b = \{b_1, b_2, \ldots, b_n\}^T$, then the following are equivalent:

1) The companion matrix $M = \begin{bmatrix} 0 & a \\ I_n & b \end{bmatrix}$ is Moore-Penrose invertible.

2) $1 + (1 - a^\dagger a) b^* b (1 - a^\dagger a)$ is a unit of R.

3) $1 + b^* b (1 - a^\dagger a)$ is a unit of R.

4) $1 + (1 - a^\dagger a) b^* b$ is a unit of R.

2 Examples

Example 1 Suppose that $S = Z_{12}$ and R is the matrix ring over S with transposition as the involution, and set $C = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$, $A = \begin{bmatrix} -4 & -4 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 0 \\ -4 & 0 \end{bmatrix}$ and $M = \begin{bmatrix} 0 & A \\ C & B \end{bmatrix}$. By a direct computation, we have $C^\dagger = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$, $A^\dagger = \begin{bmatrix} -8 & 0 \\ -8 & 0 \end{bmatrix}$ and $CC^\dagger B = B$. It is simple to find that $u = I_2$, then $M^\dagger$ exists, and using Theorem 4 or Corollary 5, we obtain $M^\dagger = \begin{bmatrix} -4 & 0 & 4 & 4 \\ -4 & 0 & 4 & 4 \\ -8 & 0 & 0 & 0 \\ 0 & 0 & -8 & -8 \end{bmatrix}$.

Example 2 Consider the complex matrix ring R with transposition as the involution, and set $M = \begin{bmatrix} 0 & A \\ C & B \end{bmatrix}$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1-i & i \\ 1-i & i \end{bmatrix}$.

It is simple to find that $CC^* X = C$ and $YC^* C = C$, where $X = \begin{bmatrix} -1 & 0 \\ \frac{1}{5}(6+3i) & -\frac{1}{5}(2+i) \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 1+\frac{1}{2}i \\ 0 & -\frac{1}{2}i \end{bmatrix}$. Clearly, $C^2 = C$ is regular. Then by Lemma 2, $C^\dagger = C^* XY^* (CC^-)^*$, then we have $C^\dagger = \frac{1}{10}\begin{bmatrix} 1+3i & 1+3i \\ -2-i & -2-i \end{bmatrix}$. By a direct computation, we have $EBF = 0$ and $u = I_2$, then $M^\dagger$ exists. Using Theorem 1, we can obtain

$$M^\dagger = \begin{bmatrix} 2+6i & 0 & \frac{1}{10}+\frac{3}{10}i & \frac{1}{10}+\frac{3}{10}i \\ -4-2i & 0 & -\frac{1}{5}-\frac{1}{10}i & -\frac{1}{5}-\frac{1}{10}i \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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关于友矩阵的 Moore-Penrose 逆的一个注记

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摘要: 假设 R 是一个有单位元 1 的结合环. 探讨了 R 上分块矩阵 Moore-Penrose 逆的存在性,得到了环上分块矩阵的 Moore-Penrose 逆存在性的充要条件. 进而,在 $EBF = 0$ 条件下,其中 $E = I - CC^\dagger$ 和 $F = I - A^\dagger A$,给出了 Moore-Penrose 逆的表达式 $M = \begin{bmatrix} 0 & A \\ C & B \end{bmatrix}$. 此结果推广了 Pedro Patrício 关于友矩阵 $M = \begin{bmatrix} 0 & a \\ I_n & b \end{bmatrix}$ 的 Moore-Penrose 逆表达式. 作为应用,给出一些例子验证了所得到的结果.

关键词: 友矩阵; Moore-Penrose 逆; 环

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